The Annals of Statistics
1989, Vol. 17, No. 3, 1124–1132

ASYMPTOTIC EXPANSIONS OF SOME MIXTURES OF THE MULTIVARIATE NORMAL DISTRIBUTION AND THEIR ERROR BOUNDS

BY YASUNORI FUJIKOSHI AND RYOICHI SHIMIZU

Hiroshima University and Institute of Statistical Mathematics

This paper deals with the distribution of $\mathbf{X} = \Sigma^{1/2}\mathbf{Z}$, where $\mathbf{Z}$: $p \times 1$ is distributed as $N_p(\mathbf{0}, I_p)$, Σ is a positive definite random matrix and $\mathbf{Z}$ and Σ are independent. Assuming that $\Sigma = I_p + BB'$, we obtain an asymptotic expansion of the distribution function of $\mathbf{X}$ and its error bound, which is useful in the situation where Σ tends to I_p. A stronger version of the expansion is also given. The results are applied to the asymptotic distribution of the MLE in a general MANOVA model.

1. Introduction. We are concerned with asymptotic expansions for the distribution of

$$\mathbf{X} = \Sigma^{1/2}\mathbf{Z}, \tag{1.1}$$

where $\mathbf{Z}$: $p \times 1$ is distributed as $N_p(\mathbf{0}, I_p)$, Σ is a positive definite random matrix and $\mathbf{Z}$ and Σ are independent. The random vector $\mathbf{X}$ in (1.1) is said to be a scale mixture of $\mathbf{Z}$ with the scale factor $\Sigma^{1/2}$. Fujikoshi (1985) has obtained an asymptotic expansion of the distribution function of X and its error bound in the case where $p = 1$ and the scale factor $\sigma \geq 1$, by expanding the conditional distribution of X given σ. The result is useful in the situation that σ tends to 1. Expanding the conditional characteristic function of X given σ. Shimizu (1987) extended Fujukoshi's result to the case of $\sigma > 0$.

In this paper we make the following assumptions.

ASSUMPTION 1. The scale factor $\Sigma^{1/2}$ has the structure

$$\Sigma = I_p + BB', \tag{1.2}$$

where B is a $p \times q$ random matrix.

Under Assumption 1 we can express $\mathbf{X}$ as

$$\mathbf{X} = \mathbf{Z} - B\mathbf{U}, \tag{1.3}$$

where $\mathbf{U}$ is distributed as $N_q(\mathbf{0}, I_q)$ and is independent of $\mathbf{Z}$ and B. We note that under Assumption 1, the two expressions (1.1) and (1.3) are equivalent. As a realization of (1.2), we deal with the case where B is determined by the following assumption.

Received April 1987; revised July 1988.

AMS 1980 *subject classifications*. Primary 62H20; secondary 62H10.

Key words and phrases. Error bound, asymptotic expansion, distribution function, scale mixture of the multivariate normal distribution, MLE, a general MANOVA model.

ASSUMPTION 2. The random matrix B in (1.2) has the structure

$$B = I_b \otimes LW^{-1/2}, \tag{1.4}$$

where the elements of L: $g \times r$ are independently distributed as $N(0,1)$, W: $r \times r$ is distributed as the Wishart distribution $W_r(I_r, n)$ and L and W are independent.

We note that under Assumptions 1 and 2, Σ tends to I_p as $n \to \infty$. In Section 2 we state the motivation of the distributional problem, which is based on the distribution of the maximum likelihood estimate under the general MANOVA model due to Potthoff and Roy (1964). In Section 3 we obtain an asymptotic expansion of the distribution function of $\mathbf{X}$ and its error bounds, which is useful in the situation where Σ tends to I_p. In Section 4 we give an uniform bound for $|P(\mathbf{X} \in A) - P(\mathbf{Z} \in A)|$, where A belongs to the set of all Borel measurable subsets in R^p. We also give some applications in Section 5 assuming Assumption 2.

2. The motivation of the distributional problem. We consider the general MANOVA model due to Potthoff and Roy (1964). Let Y be an $N \times m$ observation matrix whose rows have independent m-variate normal distribution with unknown covariance matrix Ψ and

$$E(Y) = A_1 \Xi A_2, \tag{2.1}$$

where A_1 is a known $N \times b$ matrix of rank $m \le n = N - b$, A_2 is a known $g \times m$ matrix of rank $g \le m$ and Ξ is a $b \times g$ matrix of unknown parameters. Let $\hat{\Xi}$ be the maximum likelihood estimate of Ξ and let $\mathbf{V} = (\mathbf{V}_1', \dots, \mathbf{V}_b')'$, where $V = (\mathbf{V}_1, \dots, \mathbf{V}_b)' = (A_1'A_1)^{1/2} (\hat{\Xi} - \Xi)(A_2\Psi^{-1}A_2')^{1/2}$. Then it is known [Gleser and Olkin (1970)] that the distribution of $\mathbf{V}$ is the same as that of $\mathbf{X}$ with $p = bg$, $q = br$, and $r = m - g$, under Assumptions 1 and 2. Let $X = (\mathbf{X}_1, \dots, \mathbf{X}_b)'$, $Z = (\mathbf{Z}_1, \dots, \mathbf{Z}_b)'$ and $U = (\mathbf{U}_1, \dots, \mathbf{U}_b)'$, where $\mathbf{X} = (\mathbf{X}_1', \dots, \mathbf{X}_b')'$, $\mathbf{X}_i$: $g \times 1$, $\mathbf{Z} = (\mathbf{Z}_1', \dots, \mathbf{Z}_b')'$, $\mathbf{Z}_i$: $g \times 1$, $\mathbf{U} = (\mathbf{U}_1', \dots, \mathbf{U}_b')'$ and $\mathbf{U}_i$: $r \times 1$. Then (1.3) can be expressed in a matrix notation as

$$X = Z - UW^{-1/2}L'. \tag{2.2}$$

Since the rows of X are independently distributed as $N_g(\mathbf{0}, I_g + LW^{-1}L')$, we can write the probability density function of X as

$$\begin{aligned} f(X) = E_{L,W}\Big[&(2\pi)^{-bg/2} |I_g + LW^{-1}L'|^{-b/2} \\ &\times \exp\Big\{-\tfrac{1}{2} \operatorname{tr} X'X\big(I_g + LW^{-1}L'\big)^{-1}\Big\}\Big]. \end{aligned} \tag{2.3}$$

Gleser and Olkin (1970) gave an integral expression for (2.3), which can be also expressed as

$$\begin{aligned} &(2\pi)^{-bg/2} \frac{\Gamma_g\big(\tfrac{1}{2}(n+g)\big)\Gamma_g\big(\tfrac{1}{2}(n+b+g-r)\big)}{\Gamma_g\big(\tfrac{1}{2}(n+g-r)\big)\Gamma_g\big(\tfrac{1}{2}(n+b+g)\big)} \\ &\quad \times {}_1F_1\left(\frac{1}{2}(n+b+g-r); \frac{1}{2}(n+b+g); -\frac{1}{2}X'X\right), \end{aligned} \tag{2.4}$$

where $\Gamma_g(t) = \pi^{g(g-1)/4}\prod_{j=1}^{g}\Gamma(t - \frac{1}{2}(j-1))$ and ${}_1F_1$ is the hypergeometric function of matrix argument [see, e.g., Muirhead (1982)]. Expanding the gamma functions in (2.4) and ${}_1F_1$ [see, for example, Muirhead (1982), pages 347–350], we obtain

$$f(X) = (2\pi)^{-bg/2}\exp\left(-\frac{1}{2}\operatorname{tr}X'X\right) \times\left[1 + \frac{\gamma}{2n}(\operatorname{tr}X'X - bg) + O(n^{-2})\right]. \tag{2.5}$$

A formal asymptotic expansion of the distribution function of X is obtained by a formal integration of (2.5). The expansion in the vector notation is given by

$$P(\mathbf{X} \le \mathbf{x}) = \Phi(\mathbf{x}) + \frac{\gamma}{2n}\sum_{i=1}^{p}\left[\frac{\Phi^{(2)}(x_i)}{\Phi(x_i)}\right]\Phi(\mathbf{x}) + O(n^{-2}), \tag{2.6}$$

where $\Phi(\mathbf{x}) = \prod_{i=1}^{p}\Phi(x_i)$ and $\Phi^{(j)}(x)$ denotes the jth derivative of the distribution function $\Phi(x)$ of the standard normal variable. However, we note that this approach does not guarantee the validity of the expansion. The purpose of the present paper is to derive a general result so that we can give an error bound and thereby the validity of the expansion (2.6) as well.

3. Asymptotic expansion for the distribution function of X. In this section we derive an expansion of the distribution function $F(\mathbf{x})$ of $\mathbf{X}$ around that of $\mathbf{Z}$, by extending Fujikoshi (1985) to the multivariate case. Using (1.3) we can write $F(\mathbf{x})$ as

$$F(\mathbf{x}) = E_{\mathbf{U},B}[\Phi(\mathbf{x} + B\mathbf{U})]. \tag{3.1}$$

By Taylor's theorem, we can expand $\Phi(\mathbf{x} + B\mathbf{U})$ as

$$\Phi(\mathbf{x} + B\mathbf{U}) = \Phi(\mathbf{x}) + \sum_{j=1}^{2k-1}\frac{1}{j!}(\mathbf{U}'B'\partial)^j\Phi(\mathbf{x}) + \frac{1}{(2k)!}(\mathbf{U}'B'\partial)^{2k}\Phi(\mathbf{x} + \theta B\mathbf{U}),$$

where $0 < \theta < 1$ and $\partial = (\partial/\partial x_1, \dots, \partial/\partial x_p)'$. Noting that $\mathbf{U} \sim N_q(\mathbf{0}, I_q)$, we have

$$E_{\mathbf{U}}\left[(\mathbf{U}'B'\partial)^j\Phi(\mathbf{x})\right] = \begin{cases}\dfrac{j!}{2^{j/2}(j/2)!}(\partial'BB'\partial)^{j/2}\Phi(\mathbf{x}), & \text{if } j \text{ is even},\\ 0, & \text{if } j \text{ is odd}.\end{cases}$$

Inserting these results into (3.1) we hence obtain

$$F(\mathbf{x}) = \Phi_k(\mathbf{x}) + E_B[\Delta_k(\mathbf{x}, B)], \tag{3.2}$$

where

$$\Phi_k(\mathbf{x}) = \Phi(\mathbf{x}) + \sum_{j=1}^{k-1}\frac{1}{j!2^j}E_B\left[(\partial'BB'\partial)^j\Phi(\mathbf{x})\right] \tag{3.3}$$

and

$$\Delta_k(\mathbf{x}, B) = E_{\mathbf{U}}\left[\frac{1}{(2k)!}(\mathbf{U}'B'\partial)^{2k}\Phi(\mathbf{x} + \theta B\mathbf{U})\right]. \tag{3.4}$$

Now we define

$$M_{p,k} = \sup_{|\alpha|=2k} \sup_{\mathbf{x}} \left| \frac{\partial^{2k}}{\partial x_1^{\alpha_1} \cdots \partial x_p^{\alpha_p}} \Phi(\mathbf{x}) \right|, \tag{3.5}$$

where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_p)'$ and the α_i's are nonnegative integers. Then we have

$$|\Delta_k(\mathbf{x}, B)| \le \frac{M_{p,k}}{(2k)!} p^k E_{\mathbf{U}}\left[(\mathbf{U}'B'B\mathbf{U})^k\right]. \tag{3.6}$$

We then obtain from (3.2) and (3.6) the following theorem.

THEOREM 3.1. *Under Assumption* 1

$$\sup_{\mathbf{x}} |F(\mathbf{x}) - \Phi_k(\mathbf{x})| \le \frac{M_{p,k}}{(2k)!} p^k E_{\mathbf{U},B}\left[(\mathbf{U}'B'B\mathbf{U})^k\right], \tag{3.7}$$

where $\Phi_k(\mathbf{x})$ *and* $M_{p,k}$ *are defined by* (3.3) *and* (3.5), *respectively.*

COROLLARY 3.1 (The cases $k = 1$ and 2).

$$\sup_{\mathbf{x}} |F(\mathbf{x}) - \Phi(\mathbf{x})| \le \frac{p}{2\sqrt{2\pi e}} E\left[\operatorname{tr}(\Sigma - I_p)\right] \tag{3.8}$$

and

$$\sup_{\mathbf{x}} |F(\mathbf{x}) - \Phi_2(\mathbf{x})| \le \frac{1.39}{4!\sqrt{2\pi}} E\left[\left\{\operatorname{tr}(\Sigma - I_p)\right\}^2 + 2\operatorname{tr}(\Sigma - I_p)^2\right]. \tag{3.9}$$

PROOF. The formulas (3.8) and (3.9) can be obtained from (3.7) by putting $k = 1$ and 2, respectively, and noting that

$$M_{p,1} = \max_{i,j} \sup_{\mathbf{x}} \left| \frac{\partial^2}{\partial x_i \partial x_j} \Phi(\mathbf{x}) \right| = \sup_x |\Phi^{(2)}(x)| = \frac{1}{\sqrt{2\pi e}},$$

$$\begin{aligned} M_{p,2} &= \max_{i,j,k,l} \sup_{\mathbf{x}} \left| \frac{\partial^4}{\partial x_i \partial x_j \partial x_k \partial x_l} \Phi(\mathbf{x}) \right| \\ &= \sup_x |\Phi^{(4)}(x)| = \frac{1.38\cdots}{\sqrt{2\pi}}, \end{aligned}$$

$$E_{\mathbf{U}}\left[(\mathbf{U}'B'B\mathbf{U})^k\right] = (\operatorname{tr} BB')^2 + 2\operatorname{tr}(BB')^2. \qquad \square$$

When $p = 1$, the results (3.7), (3.8) and (3.9) agree with the known results due to Fujikoshi (1985). For example, letting $\Sigma = \sigma^2$, (3.7) is expressed as

$$\begin{aligned} \sup_{\mathbf{x}} |F(\mathbf{x}) - \Phi(\mathbf{x})| &\le \frac{1}{2^k k!} \sup |H_{2k-1}(x)\varphi(x)| E(\sigma^2 - 1)^k \\ &\le \frac{1}{2k\pi} E(\sigma^2 - 1)^k \quad \text{[Fujikoshi (1987)]}, \end{aligned} \tag{3.10}$$

where $\varphi(x)$ is the probability density function of the standard normal variable and

$$\Phi_k(x) = \Phi(x) - \sum_{j=1}^{k-1} \frac{1}{2^j j!} H_{2j-1}(x)\varphi(x)E(\sigma^2 - 1)^j, \tag{3.11}$$

and where $H_j(x)$ is the Hermite polynomial defined by $(d^j/dx^j)\ \varphi(x) = (-1)^j H_j(x)\varphi(x)$.

4. Uniform upper bound for $|P(\mathbf{X} \in A) - P(\mathbf{Z} \in A)|$. A stronger version of (3.10) is given by

$$\sup_{A \in C} \left| P(x \in A) - \int_A d\phi_k(x) \right| \le \frac{2}{\pi} E(\sigma^2 - 1)^k, \tag{4.1}$$

which is obtained as a special case of Shimizu (1987), where C is the set of all Borel measurable subsets in R^1. However, it seems difficult to apply the method used in the case of $p = 1$ to the case of $p > 1$. We only consider the quantity

$$\sup_{A \in C} |P(\mathbf{X} \in A) - P(\mathbf{Z} \in A)|, \tag{4.2}$$

where C is the set of all measurable subsets in R^p, and we give its upper bound by adopting a different method. Since the conditional distribution of $\mathbf{X}$ given Σ is $N_p(\mathbf{0}, \Sigma)$, we have

$$\begin{aligned} P(\mathbf{X} \in A) &= \int_A E_\Sigma[\varphi_\Sigma(\mathbf{x})]\, d\mathbf{x} \\ &= E_\Sigma\left[\int_A \varphi_\Sigma(\mathbf{x})\, d\mathbf{x}\right], \end{aligned}$$

where $\varphi_\Sigma(\mathbf{x}) = |\Sigma|^{-1/2}\varphi(\Sigma^{-1/2}\mathbf{x})$ and $\varphi(\mathbf{x}) = (\sqrt{2\pi})^{-p}\exp(-\frac{1}{2}\mathbf{x}'\mathbf{x})$. From this expression we obtain

$$\begin{aligned} |P(\mathbf{X} \in A) - P(\mathbf{Z} \in A)| &= \left| E_\Sigma\left[\int_A \{\varphi_\Sigma(\mathbf{x}) - \varphi(\mathbf{x})\}\, d\mathbf{x}\right]\right| \\ &\le E_\Sigma\left[\int_{R^p} |\varphi_\Sigma(\mathbf{x}) - \varphi(\mathbf{x})|\, d\mathbf{x}\right]. \end{aligned} \tag{4.3}$$

We derive an upper bound by evaluating the last expression of (4.3).

THEOREM 4.1. *Let C be the set of all Borel measurable subsets in R^p. Then under Assumption* 1

$$\sup_{A \in C} |P(\mathbf{X} \in A) - P(\mathbf{Z} \in A)| \le E\left[(|\Sigma|^{1/2} - 1) + \tfrac{1}{2}|\Sigma|^{1/2}(\operatorname{tr}\Sigma - p)\right]. \tag{4.4}$$

PROOF. We can write

$$\varphi_\Sigma(\mathbf{x}) - \varphi(\mathbf{x}) = \left[1 - |\Sigma| \exp\left\{-\tfrac{1}{2}\mathbf{x}'\left(I_p - \Sigma^{-1}\right)\mathbf{x}\right\}\right]. \tag{4.5}$$

By Taylor's theorem, we have

$$\exp\left\{-\tfrac{1}{2}\mathbf{x}'\left(I_p - \Sigma^{-1}\right)\mathbf{x}\right\} = 1 - \tfrac{1}{2}\mathbf{x}'\left(I_p - \Sigma^{-1}\right)\mathbf{x}\exp\left\{-\tfrac{1}{2}\theta\mathbf{x}'\left(I_p - \Sigma^{-1}\right)\mathbf{x}\right\},$$

where $0 < \theta < 1$. Substituting this equality into (4.5) and noting that $I_p - \Sigma^{-1} \geq 0$, we obtain

$$|\varphi_\Sigma(\mathbf{x}) - \varphi(\mathbf{x})| \leq \left[\left(|\Sigma|^{1/2} - 1\right) + \tfrac{1}{2}|\Sigma|^{1/2}\mathbf{x}'\left(I_p - \Sigma^{-1}\right)\mathbf{x}\right]\varphi_\Sigma(\mathbf{x})$$

and hence

$$\int_{R^p}|\varphi_\Sigma(\mathbf{x}) - \varphi(\mathbf{x})|\,d\mathbf{x} \leq |\Sigma|^{1/2} - 1 + \tfrac{1}{2}|\Sigma|^{1/2}(\operatorname{tr}\Sigma - p).$$

The desired result is obtained by using (4.3). □

It may be noted that the result (4.4) in the case of $p = 1$ is not the same as the result (4.1) with $k = 1$.

5. Applications. Consider the distribution of the standardized statistic V of the maximum likelihood estimate $\hat{\Xi}$ in the model (2.1). We have seen in Section 2 that under Assumptions 1 and 2 the distribution is the same as that of the random vector $\mathbf{X}$ in (1.1). It is hence enough to consider reductions on what we have obtained in Sections 3 and 4, assuming that B has the structure (1.4).

LEMMA 5.1. *Suppose that B has the structure* (1.4). *Then*:

(i) *If* $n - r - 1 > 0$,

$$E(BB') = \frac{r}{n - r - 1}I_{p'}. \tag{5.1}$$

(ii) *If* $n - r - 3 > 0$,

$$E\left[(\operatorname{tr}BB')^2 + 2\operatorname{tr}(BB')\right] = bgr\beta(n, b, g, r), \tag{5.2}$$

where

$$\begin{aligned}\beta(n, b, g, r) = {}& \frac{1}{(n - r)(n - r - 1)(n - r - 3)}\\ &\times\left[\{(bg + 2)r + 2(b + g + 1)\}n\right.\\ &\left.\quad - (bg + 2)(r + 2)r + 2(b - 1)(g - 1)\right].\end{aligned} \tag{5.3}$$

PROOF. (i) follows from $BB' = I_b \otimes LW^{-1}L'$ and $E(W^{-1}) = (n-r-1)^{-1}I_r$ [see, e.g., Muirhead (1982)]. The left-hand side of (5.2) is equal to

$$E_{L,W}\left[b^2(\operatorname{tr} L'LW^{-1})^2 + b\operatorname{tr}(L'LW^{-1})^2\right]$$
$$= bgE\left[(bg+2)(\operatorname{tr} W^{-1})^2 + 2(b+g+1)\operatorname{tr} W^{-2}\right].$$

The desired result is a consequence of

$$E_W\begin{bmatrix}(\operatorname{tr}\Omega W^{-1})^2\\ \operatorname{tr}(\Omega W^{-1})^2\end{bmatrix} = \frac{1}{(n-r)(n-r-1)(n-r-3)} \times\begin{bmatrix}n-r-2 & 2\\ 1 & n-r-1\end{bmatrix}\begin{bmatrix}(\operatorname{tr}\Omega)^2\\ \operatorname{tr}\Omega^2\end{bmatrix} \tag{5.4}$$

which is obtained from (10) in Constantine (1966). □

Using Lemma 5.1, we can insert the results (i) and (ii) into Corollary 3.1 as follows:

(i) If $n-r-1>0$,

$$\sup_{\mathbf{x}}|F(\mathbf{x}) - \Phi(\mathbf{x})| \le \frac{p^2 r}{2\sqrt{2\pi e}\,(n-r-1)}. \tag{5.5}$$

(ii) If $n-r-3>0$,

$$\sup_{\mathbf{x}}|F(\mathbf{x}) - \Phi_2(\mathbf{x})| \le \frac{1.39}{4!\sqrt{2\pi}}p^3 r\beta(n,b,g,r), \tag{5.6}$$

where

$$\Phi_2(\mathbf{x}) = \Phi(\mathbf{x}) + \frac{r}{n-r-1}\sum_{i=1}^{p}\left[\frac{\Phi^{(2)}(x_i)}{\Phi(x_i)}\right]\Phi(\mathbf{x}). \tag{5.7}$$

Next we consider a further reduction for (4.4) under Assumption 2.

LEMMA 5.2. *Supose that $\Sigma = I_p + BB'$ and B has the structure* (1.4). *Then, if $n-b-r-1>0$,*

$$E\left[|\Sigma|^{1/2}\right] = \alpha(n,b,g), \tag{5.8}$$

$$E\left[|\Sigma|^{1/2}(\operatorname{tr}\Sigma - p)\right] = \frac{pr\alpha(n,b,g)}{n-b-r-1}, \tag{5.9}$$

where

$$\alpha(n,b,g) = \prod_{j=1}^{r}\frac{\Gamma\left[\frac{1}{2}(n+g-j+1)\right]\Gamma\left[\frac{1}{2}(n-b-j+1)\right]}{\Gamma\left[\frac{1}{2}(n-j+1)\right]\Gamma\left[\frac{1}{2}(n-b+g-j+1)\right]}$$
$$= \prod_{j=1}^{g}\frac{\Gamma\left[\frac{1}{2}(n+g-j+1)\right]\Gamma\left[\frac{1}{2}(n+g-b-r-j+1)\right]}{\Gamma\left[\frac{1}{2}(n+g-r-j+1)\right]\Gamma\left[\frac{1}{2}(n+g-b-j+1)\right]}. \tag{5.10}$$

PROOF. We prove (5.8) and (5.9) for the case $g \geq r$. Letting $H = (W + L'L)^{-1/2}W(W + L'L)^{-1/2}$, we have

$$|\Sigma| = |H|^{-b}, \qquad \operatorname{tr}\Sigma - p = b(\operatorname{tr}H^{-1} - r).$$

Noting that H is distributed as the multivariate beta distribution $B_r(\frac{1}{2}n, \frac{1}{2}g)$, we have

$$E\left[|\Sigma|^{1/2}\right] = E\left[|H|^{-b/2}\right] = \alpha(n, b, g).$$

Similarly

$$\begin{aligned} E\left[|\Sigma|^{1/2}(\operatorname{tr}\Sigma - p)\right] &= bE\left[|H|^{-b/2}(\operatorname{tr}H^{-1} - r)\right] \\ &= b\alpha(n, b, g)E\left[\operatorname{tr}\tilde{H}^{-1} - r\right] \\ &= b\alpha(n, b, g)r\left[\frac{n - b + g - r - 1}{n - b - r - 1} - 1\right], \end{aligned}$$

where $\tilde{H}$ is distributed as $B_r(\frac{1}{2}(n - b), \frac{1}{2}g)$. For the last reduction, see, for example, Constantine (1966). Similarly we can get the same result for the case $r \geq g$. □

Using Lemma 5.2, we can write the result (4.4) as

$$\text{(5.11)} \qquad \sup_{A \in C}|P(X \in A) - P(\mathbf{Z} \in A)| \leq \alpha(n, b, g) - 1 + \frac{bgr\alpha(n, b, g)}{2(n - b - r - 1)}$$

if $n - b - r - 1 > 0$. It is easy to see that the right-hand side of (5.11) can be expanded as

$$\text{(5.12)} \qquad \frac{pr}{n - b - r - 1} + O(n^{-2})$$

and hence the asymptotic bound in the case of $b = g = 1$ is $r/(n - r - 2)$. On the other hand, the right-hand side of (4.1) with $k = 1$ is equal to $(2\pi^{-1})r/(n - r - 2)$. It will be interesting to derive another error bound whose special case of $p = 1$ agrees with (4.1).

Acknowledgments. The authors wish to thank the referee and an Associate Editor for their comments and suggestions. This research was partly supported by the ISM Cooperative Research Program (86-ISM-CRP-7) and carried out while the first-named author was a visiting professor at the Institute of Statistical Mathematics (1986–1987).

REFERENCES

CONSTANTINE, A. G. (1966). The distribution of Hotelling's generalized T_0^2. *Ann. Math. Statist.* **37** 215–225.

FUJIKOSHI, Y. (1985). An error bound for asymptotic expansion of the distribution function of an estimate in a multivariate linear model. *Ann. Statist.* **13** 827–831.

FUJIKOSHI, Y. (1987). Error bounds for asymptotic expansions of the distribution of the MLE in a GMANOVA model. *Ann. Inst. Statist. Math.* **39** 153–161.

GLESER, L. J. and OLKIN, I. (1970). Linear models in multivariate analysis. In *Essays in Probability and Statistics* (R. C. Bose, ed.) 267–292. Univ. North Carolina Press, Chapel Hill, N.C.

MUIRHEAD, R. J. (1982). *Aspects of Multivariate Statistical Theory.* Wiley, New York.

POTTHOFF, R. F. and ROY, S. N. (1964). A generalized multivariate analysis of variance model useful especially for growth curve problems. *Biometrika* **51** 313–326.

SHIMIZU, R. (1987). Error bounds for asymptotic expansion of the scale mixtures of the normal distribution. *Ann. Inst. Statist. Math.* **39** 611–622.

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
HIROSHIMA UNIVERSITY
HIROSHIMA 730
JAPAN

INSTITUTE OF STATISTICAL MATHEMATICS
4-6-7 MINAMI-AZABU
MINATO-KU TOKYO 106
JAPAN