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ON HOTELLING'S FORMULA FOR THE VOLUME OF TUBES AND NAIMAN'S INEQUALITY[1]

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Two new derivations of the Hotelling–Naiman results on the volume of tubes about curves in spheres are presented. The first involves simple differential inequalities. The second is probabilistic, using the concept of upcrossing borrowed from the theory of Gaussian processes. The upcrossings method is extended to an harmonic regression problem not covered by the Hotelling–Naiman formulation.

1. Introduction. Motivated by the question of testing for a nonlinear parameter in a regression model with independent, homoscedastic normal residuals, Hotelling (1939) was led to consider the geometric problem of computing the volume of a tube of given radius around a curve in S^{n-1}, the unit sphere in $\mathbb{R}^n$. The answer involves only the arc length of the curve and not its curvature, providing the radius of the tube is sufficiently small that there is no self-overlap in the tube. Starting from a somewhat different statistical setting Naiman (1986) arrived at the same geometric problem and showed that Hotelling's result (properly interpreted) is an upper bound for the volume of a tube of arbitrary radius.

As a technical ingredient of a longer calculation Estermann (1926), page 94, derived the analog of Naiman's inequality for tubes about curves in Euclidean space. His method is easily adapted to prove Naiman's inequality. Hotelling's argument does not yield the Estermann–Naiman inequalities, nor do the methods of Estermann and Naiman allow one to obtain the exact volume of tubes of small radii.

The purpose of this article is to give two new, unified derivations of the Hotelling–Naiman results. The first involves differential inequalities. The second is probabilistic, using the concept of upcrossing borrowed from the theory of Gaussian processes. In the context of Gaussian processes Knowles (1987) has observed that approximations obtained from Hotelling's result and bounds derived via upcrossings are related.

Hotelling's statistical motivation and geometric problem are reviewed briefly in Section 2, which also establishes our basic notation. Sections 3 and 4 contain our derivations.

For a more extensive discussion of applications and several numerical examples, see Johansen and Johnstone (1988) and Knowles and Siegmund (1988).

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2. The problem. Assume $y_i = \beta f_i(\theta) + \varepsilon_i$, $i = 1, 2, \ldots, n$, where the f_i are known functions depending on an unknown parameter θ and the ε_i are independent $N(0, \sigma^2)$ errors. In principle, one can also consider the more general model $y_i = \langle \beta, x_i \rangle + \beta_{p+1} f_i(\theta) + \varepsilon_i$, where β and the x_i are p-dimensional vectors, but for our purposes the simpler one suffices.

The primary example given by Hotelling is $f_i(\theta) = \cos(\mu t_i + \omega)$, where the t_i are known constants and $\theta = (\mu, \omega)$. A second example is the broken line regression $f_i(\theta) = (t_i - \theta)^+$. See Davies (1987) for an interesting discussion of both these examples.

The likelihood ratio statistic for testing H_0: $\beta = 0$ against H_1: $\beta \neq 0$ is easily seen to be equivalent to

$$\max_\theta \left\{ \left[\sum f_i(\theta) y_i \right]^2 / \left[\sum f_i^2(\theta) \sum y_i^2 \right] \right\}.$$

Letting $f(\theta) = (f_1(\theta), \ldots, f_n(\theta))$ and $y = (y_1, \ldots, y_n)$, we can write this as

$$\max_\theta \left\{ \langle f(\theta), y \rangle^2 / \left[\| f(\theta) \|^2 \, \| y \|^2 \right] \right\}.$$

Putting $\gamma(\theta) = f(\theta)/\| f(\theta) \|$ and $U = y/\| y \|$, we see that the rejection region of the likelihood ratio test,

$$\max_\theta \langle \gamma(\theta), U \rangle^2 > w^2,$$

is the union of the two tubes, one about $\gamma(\theta)$, the other about $-\gamma(\theta)$, of geodesic radius $\cos^{-1} w$. Here the tube about $\gamma(\theta)$ of geodesic radius φ is the set of all points $U \in S^{n-1}$, the unit sphere in $\mathbb{R}^n$, within geodesic distance φ of the curve $\gamma(\theta)$. Under H_0, U is distributed uniformly on S^{n-1}, and hence the significance level of the likelihood ratio test is the normalized surface area on S^{n-1} of the union of the two tubes.

If γ, $U \in S^{n-1}$, then $\langle \gamma, U \rangle = 1 - 2^{-1} \| \gamma - U \|^2$. Hence the tube about $\gamma(\theta)$ of geodesic radius $\varphi = \cos^{-1} w$ can also be defined as the set of all points in S^{n-1} within Euclidean distance $[2(1 - w)]^{1/2}$ of the curve. See Figure 1.

A simpler geometric problem is to compute the volume of a tube about a curve in Euclidean space. In this context Hotelling's result is very easy to state. If the curve is smooth, closed, and there is no self-overlap in the tube (precise

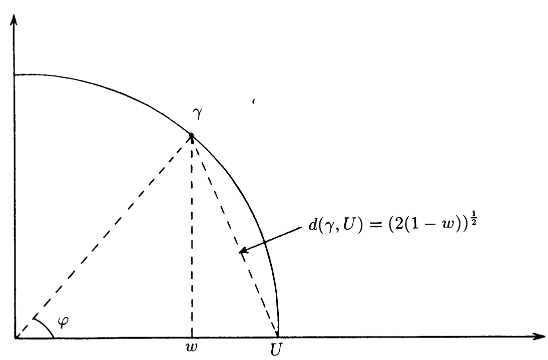

FIG. 1.

definitions are given below), the volume of the tube is the product of the arc length of the curve and the cross-sectional area of the tube. If the curve is not closed, the volume of two hemispherical caps must be added to account for parts of the tube associated with endpoints of the curve.

We shall use the following notation. Given a closed interval I of real numbers, $\alpha: I \to \mathbb{R}^n$ is a piecewise regular (continuous and piecewise continuously differentiable with nonvanishing derivative) curve of arc length $|\alpha|$. The Euclidean distance between two points is $d(x, y) = \|x - y\|$, between a point and a set is $d(x, B) = \inf\{\|x - y\|: y \in B\}$, and between two sets is $d(A, B) = \sup\{d(x, B): x \in A\}$. The tube (in $\mathbb{R}^n$) of radius R about α is $\alpha^R = \{x: d(x, \alpha(I)) < R\}$. For any (measurable) $A \subset S^{n-1}$ or $A \subset \mathbb{R}^n$, $V(A)$ = volume of A. Also let Ω_n denote the volume of the n-dimensional unit ball in $\mathbb{R}^n$ and ω_{n-1} the volume (surface area) of S^{n-1}, the unit sphere in $\mathbb{R}^n$($\Omega_n = \pi^{n/2}/\Gamma(n/2 + 1)$, $\omega_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$). Finally, let $\dot{\alpha}(t) = d\alpha(t)/dt$.

Hotelling's result in $\mathbb{R}^n$ says that if α is twice continuously differentiable and there is no self-overlap in the tube, then

$$V(\alpha^R) = |\alpha| \, \Omega_{n-1} R^{n-1} + \Omega_n R^n \tag{2.1}$$

if the curve is not closed and

$$V(\alpha^R) = |\alpha| \, \Omega_{n-1} R^{n-1} \tag{2.2}$$

if the curve is closed.

If α does not actually intersect itself, the condition of no self-overlap is essentially the condition that R be sufficiently small. Estermann (1926), page 94, and Naiman (1986) have proved the elegant result that if α is only assumed piecewise regular,

$$V(\alpha^R) \le |\alpha| \, \Omega_{n-1} R^{n-1} + \Omega_n R^n \tag{2.3}$$

for all $R \ge 0$.

Precise analogs of (2.1) and (2.3) for tubes about curves in S^{n-1} are given in Section 3.

REMARK. It is illuminating to consider the case where α is the unit circle in $\mathbb{R}^2$. One easily verifies (2.2) for $R \le 1$ and (2.3) for all $R \ge 0$. Also (2.3) is asymptotically sharp as $R \to \infty$. We do not have a simple geometric explanation why the second term on the right-hand side of (2.3), which is obviously necessary when α is not closed, works efficiently and in complete generality.

Anticipating applications to the examples presented above, we note that the broken line regression requires Naiman's formulation because the curve $\gamma(\theta)$ is only piecewise smooth. Hotelling's problem of testing for a periodic component in a regression model poses other difficulties because the parameter $\theta = (\mu, \omega)$ is two dimensional and hence the "curve" $\gamma(\theta)$ is a surface in S^{n-1}. Weyl (1939) in a companion paper to Hotelling's calculates the volume of a tube of small radius about an arbitrary closed differentiable manifold imbedded in $\mathbb{R}^n$ or in S^{n-1}. However, his results must be modified for manifolds with boundary before they can be applied to the problem at hand. See Knowles and Siegmund (1988) for an appropriate modification and numerical examples. In Section 4 we show that the special structure of Hotelling's problem allows one to use an upcrossing argument to give an upper bound for the significance level of the likelihood ratio test.

3. The Hotelling–Naiman theorem. We begin with a technical lemma which summarizes several well-known facts about the uniform distribution on S^{n-1}.

LEMMA 3.1. *Suppose $U = (U_1, \dots, U_n)$ is uniformly distributed on S^{n-1}.*

(i) *The distribution of U_1^2 is Beta$(1/2, (n-1)/2)$; the probability density function of U_1 is*

$$f_{n-1}(x) = \frac{\Gamma(n/2)}{\pi^{1/2}\Gamma[(n-1)/2]}(1-x^2)^{(n-3)/2}, \qquad |x| < 1.$$

(ii)

$$E(U_1^+) = \Gamma(n/2)/\{2\pi^{1/2}\Gamma[(n+1)/2]\}.$$

(iii) *For $k < n$, given $U_1, \dots, U_k$, the conditional distribution of $(U_{k+1}, \dots, U_n)$ is uniform on a sphere of dimension $n-k-1$ and radius $(1 - \sum_1^k U_i^2)^{1/2}$.*

(iv) *The random variable $U_1^2 + U_2^2$ has a Beta$(1, (n-2)/2)$ distribution and is independent of U_1^2/U_2^2.*

PROOF. All these results can be proved by means of the representation $U_i = Z_i/(Z_1^2 + \cdots + Z_n^2)^{1/2}$, where $Z_1, \dots, Z_n$ are independent, standard normal random variables, and Basu's theorem [cf. Lehmann (1986), page 191]. □

A simple picture underlies the differential inequalities approach to Naiman's inequality. A sphere centered at one end of the curve is sliced along the plane perpendicular to the curve at that endpoint. The hemisphere intersecting the curve is moved along the curve at unit speed. The volume swept out is greatest when the curve is a geodesic, which leads to the Hotelling–Naiman result.

It is easiest to begin with Estermann's bound: the version of Naiman's bound for tubes in Euclidean space.

THEOREM 3.1. *Let $\alpha: I \to \mathbb{R}^n$ be a piecewise regular curve of length $|\alpha|$, and for $R \geq 0$ let $\alpha^R = \{x \in \mathbb{R}^n: d(x, \alpha(I)) < R\}$. Then $V(\alpha^R)$ satisfies (2.3) for all $R \geq 0$.*

PROOF. Without loss of generality we can assume α is parameterized by arc length, so $I = [0, |\alpha|]$. Let the image of α on a subinterval $[a, b]$ of I be denoted by $\alpha_{[a,b]}$. Let $v(s) = V(\alpha_{[0,s]}^R)$. Clearly, $v(0) = \Omega_n R^n$, and to complete the proof, it suffices to show that v is absolutely continuous and $\dot{v}(s) \leq \Omega_{n-1} R^{n-1}$ for a.e. s.

We say a point x is R-close to $\alpha_{[a,b]}$ if $x \in \alpha^R_{[a,b]}$. Let $\delta > 0$. If a point is R-close to $\alpha_{[0,s+\delta]}$ but not R-close to $\alpha_{[0,s]}$, it must be R-close to $\alpha_{[s,s+\delta]}$ but not R-close to $\alpha_{[s,s]}$. Hence

$$v(s+\delta) - v(s) = V\big(\alpha^R_{[0,s+\delta]}\big) - V\big(\alpha^R_{[0,s]}\big) \le V\big(\alpha^R_{[s,s+\delta]}\big) - V\big(\alpha^R_{[s,s]}\big). \tag{3.1}$$

Recall that for two sets A and B, $d(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|$, and observe that $d(A, B) \le \eta$ implies $A^R \subset B^{R+\eta}$. Since α is right differentiable by assumption, the linear approximation to $\alpha(s+t)$ for $t \ge 0$, namely $\beta(t) = \alpha(s) + t\dot{\alpha}(s+)$ satisfies

$$d\big(\alpha_{[s,s+\delta]}, \beta_{[0,\delta]}\big) = \eta(\delta) = o(\delta) \tag{3.2}$$

uniformly in s provided we exclude left neighborhoods of length δ of discontinuity points of $\dot{\alpha}$. An explicit evaluation starting from (3.1) yields

$$\begin{aligned} 0 \le v(s+\delta) - v(s) &\le V\big(\beta^{R+\eta}_{[0,\delta]}\big) - V\big(\alpha^R_{[s,s]}\big) \\ &= \delta\Omega_{n-1}(R+\eta)^{n-1} + \Omega_n(R+\eta)^n - \Omega_n R^n \\ &= \delta\Omega_{n-1}R^{n-1} + O(\eta(\delta)). \end{aligned} \tag{3.3}$$

It follows that v is increasing and absolutely continuous, hence that $\dot{v}(s)$ exists a.e. and satisfies the required bound. □

REMARKS. (i) One can avoid the appeal to Lebesgue theory by showing directly that $v(s)$ is Lipschitz continuous with Lipschitz constant $M = \Omega_{n-1}R^{n-1}$. Fix $\varepsilon > 0$. Formula (3.3) and the analogous inequality for $\delta < 0$ imply that about each $s \in I$ there is a neighborhood N_s, open relative to I, of points s' satisfying $|v(s') - v(s)| \le (M + \varepsilon)|s' - s|$. The compactness of I provides a finite subcover $\{N_{s_i}\}$ from which it follows by chaining that $v(s)$ is Lipschitz $(M + \varepsilon)$ on I. Since $\varepsilon > 0$ is arbitrary, we recover (2.3) from the decomposition $v(s) = v(0) + [v(s) - v(0)]$.

(ii) Lalley and Robbins (1988) have also exploited the idea of volume swept out by a moving sphere in a differential games setting.

To see that equality holds in (2.3) when α is continuously differentiable, not closed and R is sufficiently small, we define the cross section $C[\alpha(s)]$ of the tube α^R at the point $\alpha(s)$ as the set of all $x \in \alpha^R$ such that $\langle x - \alpha(s), \dot{\alpha}(s)\rangle \le 0$, $= 0$ or ≥ 0 according as $s = 0$, $s \in (0, |\alpha|)$ or $s = |\alpha|$. Clearly,

$$\alpha^R = \bigcup_{s \in [0, |\alpha|]} C[\alpha(s)],$$

and we say that no self-overlap occurs in the tube if this union is disjoint. For a *closed* curve, we require only that the union over $s \in (0, |\alpha|)$ be disjoint. The critical radius is $R_c = \inf\{R \ge 0\colon \text{self-overlap occurs}\}$. If the curve does not actually intersect itself, $R_c > 0$, but its exact value may be difficult to determine analytically. Johansen and Johnstone (1988) give an easily computed bivariate function whose minimum is R_c.

THEOREM 3.2. *If* $\alpha: I \to \mathbb{R}^n$ *is regular and closed (resp. not closed), then the equality* (2.2) [*resp.* (2.1)] *holds for all* $R \le R_c$.

PROOF. Assume first that α is not closed. Again assume α is parameterized by arc length and that $R < R_c$. The key is to show that equality holds in (3.1): Indeed if a point is R-close to $\alpha_{[s,s+\delta]}$ but not to $\alpha_{[s,s]}$, then it lies in $\bigcup_{t \in (s,|\alpha|]} C[\alpha(t)]$ but not $\alpha^R_{[s,s]}$ and hence not in $\alpha^R_{[0,s]}$.

If we combine equality in (3.1) with the inequality $V(\alpha^R_{[s,s+\delta]}) \ge V(\beta^{R-\eta}_{[0,\delta]})$ and calculate as in (3.3), we obtain in addition to (3.3) the inequality

$$v(s+\delta) - v(s) \ge \delta \Omega_{n-1} R^{n-1} + O(\eta(\delta)).$$

The two inequalities show that $\dot{v}(s) = \Omega_{n-1}R^{n-1}$ for all $s \in [0, |\alpha|]$. Combined with the initial value $v(0) = \Omega_n R^n$, this implies (2.1).

Finally, suppose that α is closed. The idea is to split α into two nonclosed curves and apply what we have just proved. Choose a pair (s_0, t_0) maximizing the distance function $(s,t) \to \|\alpha(s) - \alpha(t)\|^2$. By reparameterization, we may assume that $s_0 = 0$. Since $\alpha(0) - \alpha(t_0)$ is orthogonal to both $\dot{\alpha}(0)$ and $\dot{\alpha}(t_0)$ it follows that $\|\alpha(0) - \alpha(t_0)\| \ge 2R$. In turn, this implies that neither of the tubes $\alpha^R_{[0,t_0]}$ and $\alpha^R_{[t_0,|\alpha|]}$ suffers any self-overlap. Equality (2.1) applies to each of these nonclosed curves. Since the two tubes intersect in precisely two disjoint balls, equality (2.2) is established by subtracting the double-counted portions. □

REMARK. It is interesting to note that the curvature of α plays no role in the preceding argument. This contrasts with Hotelling's argument where the curvature appears and then is found after an integration to have a coefficient of zero.

To discuss tubes in spheres, let S^{n-1} be the unit sphere in $\mathbb{R}^n$ and let $\gamma: I \to S^{n-1}$ be a piecewise regular curve parameterized by arc length.

Although it is appropriate to use geodesic distance to define tube radii, in order to adapt as directly as possible the preceding arguments we shall think of S^{n-1} imbedded in $\mathbb{R}^n$ and use Euclidean distance. The relation between geodesic distance φ, $w = \cos\varphi$, and Euclidean distance $R = [2(1-w)]^{1/2}$ is illustrated in Figure 1. The tube of radius R about γ in S^{n-1} is

$$\gamma^R = \left\{ y \in S^{n-1} \colon \max_s \langle y, \gamma(s) \rangle > \cos\varphi \right\} = \left\{ y \in S^{n-1} \colon d(y, \gamma(I)) < R \right\}.$$

Now the "linear continuation" of $\gamma(s)$ is continuation along a geodesic, defined by $\beta(t) = \gamma(s)\cos t + \dot{\gamma}(s+)\sin t$. Since Euclidean distances on S^{n-1} are inherited from $\mathbb{R}^n$, (3.1) and (3.2) remain valid with this new definition of β.

To complete the calculation analogous to (3.3), we must evaluate $v(0)$ and $V(\beta^{R+\eta}_{[0,\delta]})$. Let $U = (U_1, \dots, U_n)$ be uniformly distributed on S^{n-1} and assume without loss of generality that $\gamma(0) = (1, 0, \ldots, 0)$. Then

$$\begin{aligned} v(0) &= V\{(u_1, \ldots, u_n) \in S^{n-1}: u_1 > w\} \\ &= \omega_{n-1} P\{U_1 > w\} \\ &= \omega_{n-2} \int_w^1 (1 - x^2)^{(n-3)/2}\, dx \end{aligned}$$

by Lemma 3.1. To compute $V(\beta_{[0,\delta]}^{R(w)})$ observe that since $\langle \gamma(s), \dot{\gamma}(s+) \rangle = 0$ we can without loss of generality assume that $\gamma(s) = (1, 0, \ldots, 0)$ and $\dot{\gamma}(s+) = (0, 1, 0, \ldots, 0)$. Then $\beta_{[0,\delta]}$ is a portion of the equator subtending an angle δ at the origin. The portion of the tube which does not involve the two caps, $B = \{u \in S^{n-1}: \sum_{i \geq 3} u_i^2 < 1 - w^2, 0 \leq u_2/u_1 \leq \tan\delta, u_1 \geq 0\}$, has volume

$$V(B) = 4^{-1}\omega_{n-1} P\{U_1^2 + U_2^2 > w^2, U_2^2/U_1^2 \leq \tan^2\delta\},$$

which can be evaluated by Lemma 3.1. Hence

$$V(\beta_{[0,\delta]}^{R(w)}) = (2\pi)^{-1}\delta\omega_{n-1}(1 - w^2)^{(n-2)/2} + \omega_{n-2}\int_w^1 (1 - x^2)^{(n-3)/2}\, dx.$$

If $\tilde{R} = R + \eta$, the corresponding $\tilde{w} = 1 - \tilde{R}^2/2 = w - R\eta + \eta^2/2 = w + O(\eta)$. A calculation like (3.3) yields

$$0 \leq v(s + \delta) - v(s) \leq (2\pi)^{-1}\delta\omega_{n-1}(1 - w^2)^{(n-2)/2} + O(\eta(\delta)),$$

from which follows $\dot{v}(s) \leq (2\pi)^{-1}\omega_{n-1}(1 - w^2)^{(n-2)/2}$ a.e. s since $\eta(\delta) = o(\delta)$. From this inequality we obtain Naiman's bound for a tube in S^{n-1}.

An argument analogous to that of Theorem 3.2 shows that there is equality in the Hotelling–Naiman bound for tubes in S^{n-1} whenever the curve is smooth and the tube radius is less than R_c, the radius of first overlap. Details of the definition of R_c together with a computational method for evaluating R_c and some statistical examples are given in Johansen and Johnstone (1988).

The Naiman–Hotelling result we have just derived is summarized in Theorem 3.3, which is then proved by a different method.

THEOREM 3.3. *Let γ: $[0, t_0] \to S^{n-1}$ be a regular curve. Let U be uniformly distributed on S^{n-1} and put $Z(t) = \langle \gamma(t), U \rangle$. Then for any $0 < w < 1$,*

$$P\Big\{\max_{0 \leq t \leq t_0} Z(t) > w\Big\} \leq \frac{\Gamma(n/2)}{\pi^{1/2}\Gamma[(n-1)/2]}\int_w^1 (1 - x^2)^{(n-3)/2}\, dx + (2\pi)^{-1}|\gamma|(1 - w^2)^{(n-2)/2}. \tag{3.4}$$

If $\gamma(0) \neq \gamma(t_0)$ and no self-overlap occurs in the tube $\gamma_{[0,t_0]}^R$, where $R = [2(1 - w)]^{1/2}$, there is equality in (3.4). *If $\gamma(0) = \gamma(t_0)$ and no self-overlap occurs, the probability on the left-hand side of* (3.4) *equals the second term on the right-hand side. The inequality* (3.4) *continues to hold if γ is only assumed to be continuous and piecewise regular.*

PROOF. The argument uses the notion of an upcrossing of the level w, which plays an important role in the theory of Gaussian processes. See Leadbetter, Lindgren and Rootzén (1983), Chapter 7, for the definition and basic properties. Let $N_w(t_0)$ denote the number of upcrossings of the level w by $Z(t)$, $0 \le t \le t_0$. Then

$$
\begin{aligned}
P\Big\{\max_{0\le t\le t_0} Z(t) > w\Big\} &= P\{Z(0) > w\} + P\{Z(0) < w, N_w(t_0) \ge 1\} \\
&\le P\{Z(0) > w\} + E\{N_w(t_0)\}.
\end{aligned} \tag{3.5}
$$

The proof of (3.4) is completed by the evaluation of the right-hand side of (3.5) given in Lemmas 3.1 and 3.2, which follows.

If $\gamma(0) \ne \gamma(t_0)$ and there is no self-overlap in S^{n-1} in the tube of geodesic radius $\cos^{-1}(w)$ about γ, then no sample path $Z(t)$, $0 \le t \le t_0$, can downcross and subsequently upcross the level w. It follows that $P\{Z(0) > w, N_w(t_0) \ge 1\} + P\{N_w(t_0) \ge 2\} = 0$, so equality holds in (3.5), hence also in (3.4). If $\gamma(0) = \gamma(t_0)$ and no self-overlap occurs, $\{\max_{0\le t\le t_0} Z(t) > w\} = \{N_w(t) \ge 1\}$ and $P\{N_w(t) > 1\} = 0$, so the stated result is an immediate consequence of Lemma 3.2.

Since a continuous, piecewise regular curve is the uniform limit of a sequence of regular curves, the arc lengths of which also converge to that of the given curve, one sees from Fatou's lemma that (3.4) continues to hold under the weaker condition of continuity and piecewise regularity. □

LEMMA 3.2. *If γ is a regular curve in S^{n-1}, the expected number of upcrossings $N_w(t_0)$ of the level w by the process $Z(t)$, $0 \le t \le t_0$, is given by*

$$EN_w(t_0) = (2\pi)^{-1}|\gamma|(1 - w^2)^{(n-2)/2}.$$

PROOF. Assume without loss of generality that γ is parameterized by arc length $s = s(t)$, and let $s_0 = s(t_0)$. Then $\dot\gamma(s) = d\gamma/ds \in S^{n-1}$. A standard argument [cf. Leadbetter, Lindgren and Rootzén (1983), Chapter 7] shows that if as $h \to 0$ the joint density of $Z(s)$ and $[Z(s+h) - Z(s)]/h$ satisfies certain regularity conditions discussed below, then

$$EN_w(s_0) = f_{n-1}(w) \lim_{m\to\infty} 2^{-m} \sum_{k=1}^{2^m s_0} E\big[\dot Z^+(k/2^m) | Z(k/2^m) = w\big],$$

where f_{n-1} is the density function of $Z(s)$ given in Lemma 3.1. Since $\langle \gamma(s), \dot\gamma(s)\rangle = 0$, for the purpose of evaluating the conditional distribution of $\dot Z(s) = \langle \dot\gamma(s), U\rangle$ given $Z(s) = \langle \gamma(s), U\rangle$, we can let $U = (U_1, \dots, U_n)$ and by a rotation of the coordinate axes assume that $\gamma(s) = (1, 0, \dots, 0)$ and $\dot\gamma(s) = (0, 1, 0, \dots, 0)$. This means that

$$EN_w(s_0) = s_0 f_{n-1}(w) E\big[U_2^+ | U_1 = w\big]. \tag{3.6}$$

The conditional expectation in (3.6) is evaluated with the help of Lemma 3.1(ii) and (iii).

To justify the preceding calculation, it suffices that as $h \to 0$ the joint density function of $Z(s)$ and $[Z(s+h) - Z(s)]/h$, say $p_{s,h}(x, y)$, converge uniformly in s and x, at least for x in a neighborhood of w, to the joint density of $Z(s)$ and $\dot{Z}(s)$ [cf. Leadbetter, Lindgren and Rootzén (1983), Theorem 7.2.4].

By a rotation of the coordinate axes so that $\gamma(s) = (1, 0, \dots, 0)$ and $\dot{\gamma}(s) = (0, 1, 0, \dots, 0)$, we see that the joint density function of $Z(s)$ and $[Z(s+h) - Z(s)]/h = \dot{Z}(s) + \{[Z(s+h) - Z(s)]/h - \dot{Z}(s)\}$ has the form

$$P\{Z(s) \in dx, [Z(s+h) - Z(s)]/h \in dy\} = P\left\{U_1 \in dx, U_2 + \sum_{i \geq 1} \varepsilon_i U_i \in dy\right\}, \tag{3.7}$$

where the $\varepsilon_i \to 0$ uniformly in s as $h \to 0$, because γ is regular.

Given $U_1 = x$, $(U_2, \dots, U_n)$ are uniformly distributed on an $(n-2)$-dimensional sphere of radius $(1 - x^2)^{1/2}$ (cf. Lemma 3.1), and consequently the right-hand side of (3.7) equals

$$f_{n-1}(x)\, dx P\left\{\tilde{U}_2(1 + \varepsilon_2) + \sum_{i \geq 3} \varepsilon_i \tilde{U}_i \in (dy - \varepsilon_1 x)/(1 - x^2)^{1/2}\right\},$$

where $(\tilde{U}_2, \dots, \tilde{U}_n)$ is uniformly distributed on S^{n-2}. This last probability can be written as an integral with respect to the joint density of $\tilde{U}_2$ and $\sum_{i \geq 3} \varepsilon_i \tilde{U}_i$, and by a similar conditioning argument it can be shown to converge uniformly in s and $|x|$ bounded away from 1 to

$$P\left\{\tilde{U}_2 \in dy/(1 - x^2)^{1/2}\right\} = P\{\langle \dot{\gamma}(s), U \rangle \in dy | \langle \gamma(s), U \rangle = x\}.$$

The details are omitted. □

REMARK. It is possible (although not particularly natural) to derive Theorems 3.1 and 3.2 by a (down)crossing argument. Since the tube cannot be defined by an inner product, the appropriate process is $Z(s) = \|U - \alpha(s)\|$, where U is uniformly distributed in a box large enough to contain α^R.

4. Testing for an harmonic. As indicated in Section 2, Hotelling's problem of testing for an harmonic of undetermined frequency and phase does not fall within the scope of the results of Section 3 because it involves two nonlinear parameters. However, by writing

$$\beta \cos(\mu t_i + \omega) = \beta_1 \cos \mu t_i + \beta_2 \sin \mu t_i,$$

where $\beta_1 = \beta \cos \omega$, $\beta_2 = -\beta \sin \omega$, and observing that $\beta = 0$ if and only if $\beta_1 = \beta_2 = 0$, we reduce the number of nonlinear parameters to one. The upshot is a likelihood ratio test with a rejection region of the form

$$\sup_{\theta} \left[\langle \gamma_1(\theta), U \rangle^2 + \langle \gamma_2(\theta), U \rangle^2\right]^{1/2} > w,$$

where $\gamma_i(\theta) \in S^{n-1}$, $\langle \gamma_1(\theta), \gamma_2(\theta) \rangle = 0$ for all θ, and under H_0: $\beta_1 = \beta_2 = 0$, U is uniformly distributed on S^{n-1}. Although in this form the rejection region does not have a simple geometric interpretation, its one-dimensional structure permits one to obtain an inequality based on an upcrossings argument. The following theorem is similar to a result of Davies (1987), who assumes that σ^2 is known and therefore can modify known results about upcrossings of χ^2 processes.

THEOREM 4.1. *For $i = 1, 2$ let γ_i: $[0, t_0] \to S^{n-1}$ be regular curves. Assume $\langle \gamma_1(t), \gamma_2(t)\rangle = 0$ for all t. Let U be uniformly distributed on S^{n-1} and put $Z(t) = \{\langle \gamma_1(t), U\rangle^2 + \langle \gamma_2(t), U\rangle^2\}^{1/2}$. Then for $0 < w < 1$*

$$P\Big\{\max_{0\le t\le t_0} Z(t) > w\Big\} \le (1 - w^2)^{(n-2)/2}$$

$$+\frac{\Gamma(n/2)w(1-w^2)^{(n-3)/2}}{2\pi^{3/2}\Gamma[(n-1)/2]} \tag{4.1}$$

$$\times\int_0^{t_0}\int_0^{2\pi}\big[\|\dot\gamma_1\cos\omega + \dot\gamma_2\sin\omega\|^2 - \langle\dot\gamma_1, \gamma_2\rangle^2\big]^{1/2}\,d\omega\,dt.$$

REMARKS. (i) Theorem 4.1 has been formulated with a view toward application to Hotelling's problem of testing for an harmonic of undetermined frequency and phase. There has been no attempt at generality.

(ii) A special case of Knowles and Siegmund's (1988) formula for the volume of a tube about a surface imbedded in S^{n-1} shows that equality holds in (4.1) for all w sufficiently close to 1. However, their method does not yield the inequality (4.1) for all w.

(iii) See Knowles and Siegmund (1988) for a numerical example related to Theorem 4.1.

PROOF OF THEOREM 4.1. The inequality (3.5) is again applicable. The probability density function of $Z(t)$ is (cf. Lemma 3.1)

$$f_{n-1}^{(2)}(x) = (n-2)x(1-x^2)^{(n-4)/2}, \tag{4.2}$$

and hence $P\{Z(t) \ge w\} = (1 - w^2)^{(n-2)/2}$. The standard recipe for calculating $EN_w(t_0)$ yields

$$EN_w(t_0) = f_{n-1}^{(2)}(w)\int_0^{t_0} E\big[\dot Z^+(t)|Z(t) = w\big]\,dt. \tag{4.3}$$

Since

$$\dot Z = \frac{\langle\gamma_1, U\rangle\langle\dot\gamma_1, U\rangle + \langle\gamma_2, U\rangle\langle\dot\gamma_2, U\rangle}{\big[\langle\gamma_1, U\rangle^2 + \langle\gamma_2, U\rangle^2\big]^{1/2}},$$

after rotation of axes so that $\gamma_1 = (1, 0, \dots, 0)$ and $\gamma_2 = (0, 1, 0, \dots, 0)$, the conditional expectation in (4.3) equals

$$w^{-1}E\big(E\{[U_1\langle\dot\gamma_1, U\rangle + U_2\langle\dot\gamma_2, U\rangle]^+|U_1, U_2\}|U_1^2 + U_2^2 = w^2\big). \tag{4.4}$$

Defining α_1 and α_2 by $\dot\gamma_1 = \langle\dot\gamma_1, \gamma_2\rangle\gamma_2 + \alpha_1$ and $\dot\gamma_2 = \langle\dot\gamma_2, \gamma_1\rangle\gamma_1 + \alpha_2$, we see that

α_1 and α_2 are orthogonal to both γ_1 and γ_2, and satisfy

$$\|\alpha_1\|^2 = \|\dot{\gamma}_1\|^2 - \langle \dot{\gamma}_1, \gamma_2 \rangle^2, \qquad \|\alpha_2\|^2 = \|\dot{\gamma}_2\|^2 - \langle \gamma_1, \dot{\gamma}_2 \rangle^2, \qquad (4.5)$$
$$\langle \alpha_1, \alpha_2 \rangle = \langle \dot{\gamma}_1, \dot{\gamma}_2 \rangle.$$

Since $\langle \dot{\gamma}_1, \gamma_2 \rangle = -\langle \gamma_1, \dot{\gamma}_2 \rangle$, we easily obtain $U_1\langle \dot{\gamma}_1, U \rangle + U_2\langle \dot{\gamma}_2, U \rangle = U_1\langle \alpha_1, U \rangle + U_2\langle \alpha_2, U \rangle$, and hence the inner conditional expectation in (4.4) equals

$$E\{[U_1\langle \alpha_1, U \rangle + U_2\langle \alpha_2, U \rangle]^+ | U_1, U_2\}.$$

By decomposing α_2 into a component along α_1 and a component orthogonal to α_1 and then rotating the coordinate axes while leaving the first two coordinate directions fixed, we see that this conditional expectation equals

$$\|\alpha_1 U_1 + \alpha_2 U_2\| \, E(U_3^+ | U_1, U_2). \qquad (4.6)$$

Using (4.5) and Lemma 3.1 in (4.6) and substituting the result into (4.4), one can easily complete the proof of the theorem. □

Note added in proof. Berman [*Comm. Statist. Stochastic Models* 4 1–43 (1988)] has studied a general class of processes, $Z(t)$, which contains the process of Theorem 3.3 as a special case, and has used an upcrossing argument to give an upper bound for $P\{\max Z(t) > w\}$. He has not observed that for a subclass of his processes, for all sufficiently large w the inequality is in fact an equality. The authors thank S. Cambanis for pointing out this reference.

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