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SOBOLEV TESTS FOR SYMMETRY OF DIRECTIONAL DATA

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For testing a probability distribution on a compact Riemannian manifold for symmetry under the action of a given group of isometries, two classes of invariant tests are proposed and some properties noted. These tests are based on Sobolev norms and generalize Giné's Sobolev tests of uniformity. For general compact manifolds randomization tests analogous to Wellner's tests for the two-sample case are suggested. For the circle, distribution-free tests of symmetry based on uniform scores are provided.

1. Introduction. A natural hypothesis in directional statistics is that a given distribution is symmetrical, for example that a circular distribution has antipodal symmetry. In general, for a probability distribution on a compact Riemannian manifold, this hypothesis is that the distribution is invariant under a specified group of isometries. The purpose of this paper is to introduce two general classes of invariant tests for such symmetry. Although tests for symmetry about an axis of a circular distribution have been considered by Schach (1969a) and by Mardia (1972, page 195), our tests have little connection with these. The tests introduced here are based on the machinery introduced by Giné (1975) to test uniformity and used by Wellner (1979) for the two-sample case.

In Section 2 we review the material on Riemannian manifolds and Sobolev norms which we shall need. Proposition 2.1 provides the basis for decomposing each of Giné's tests of uniformity into the sum of a test of symmetry and a test of uniformity on a quotient manifold. In Section 3 we introduce randomization tests of symmetry and consider their consistency properties and asymptotic distributions. Finally, Section 4 provides a class of distribution-free tests of symmetry on the circle. The asymptotic null distributions and a consistency result are given and some examples are considered.

2. Preliminaries. We summarize in this section those properties of Riemannian manifolds, isometry groups, and Sobolev norms which we shall need. Details can be found in Giné (1975) and the references given there.

Let $\mathbb{X}$ be a compact Riemannian manifold. Denote by $C(\mathbb{X})$ the set of continuous functions on $\mathbb{X}$, by $\mathscr{M}(\mathbb{X})$ the set of bounded Borel measures on $\mathbb{X}$, and by $\mathscr{P}(\mathbb{X})$ the Borel probability measures on $\mathbb{X}$. The Riemannian metric determines the uniform measure μ in $\mathscr{P}(\mathbb{X})$. Let G be a subgroup of the isometry group of $\mathbb{X}$. Then G acts on $C(\mathbb{X})$ and on $\mathscr{M}(\mathbb{X})$ as follows. If g in G sends x to $g.x$ then, for $f \in C(\mathbb{X})$, g sends f to $g^*f = f \circ g$ and for $\nu \in \mathscr{M}(X)$, g sends ν to $g_*\nu = \nu \circ g^{-1}$. The hypothesis of symmetry which we wish to test is that ν is invariant under this action, i.e. that $g_*\nu = \nu$ for all $g \in G$. If ν is invariant under G, then it is also invariant under the closure of G. Therefore we shall assume that G is closed in the isometry group. It follows that G is a compact Lie group. (See Theorem 3.4 on page 239 of Kobayashi and Nomizu, 1963.) Thus we can use normalized Haar measure λ on G to average the actions on $C(\mathbb{X})$ and $\mathscr{M}(\mathbb{X})$. For $f \in C(\mathbb{X})$ we define $\bar{f} \in C(\mathbb{X})$ by

$$\bar{f}(x) = \int_G f(g.x)\, d\lambda(g).$$

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Similarly, for $\nu \in \mathscr{M}(\mathbb{X})$ we define $\bar{\nu} \in \mathscr{M}(\mathbb{X})$ by

$$\int_{\mathbb{X}} f \, d\bar{\nu} = \int_{\mathbb{X}} \bar{f} \, d\nu.$$

Now $C(\mathbb{X})$ and $\mathscr{P}(\mathbb{X})$ have the important direct-sum decompositions

$$C(\mathbb{X}) = C(\mathbb{X})_+ \oplus C(\mathbb{X})_- \quad \text{and} \quad \mathscr{P}(\mathbb{X}) \subset \mathscr{P}(\mathbb{X})_+ \oplus \mathscr{M}(\mathbb{X})_-$$

where $\mathscr{P}(\mathbb{X})_+ = \{\nu \in \mathscr{P}(\mathbb{X}) : \bar{\nu} = \nu\}, \quad \mathscr{M}(\mathbb{X})_- = \{\nu \in \mathscr{M}(\mathbb{X}) : \bar{\nu} = 0\}$

and similarly for $C(\mathbb{X})_\pm$. In particular, the hypothesis of symmetry can be rewritten as $H_0 : \nu \in \mathscr{P}(\mathbb{X})_+$.

Denote by $N(G)$ the normaliser of G in the isometry group of $\mathbb{X}$, i.e. the set of isometries γ satisfying $\gamma^{-1}G\gamma = G$. Then $N(G)$ sends each of $\mathscr{P}(\mathbb{X})_+$, $C(\mathbb{X})_\pm$ into itself. In particular the null hypothesis is invariant under $N(G)$, so it is reasonable to seek tests which are also invariant under $N(G)$. Our tests have this property.

The Laplacian Δ of $\mathbb{X}$ acts on $L^2(\mathbb{X}, \mu)$, the space of square-integrable functions on $\mathbb{X}$. If E_k denotes the kth eigenspace of Δ with eigenvalue σ_k, for $k = 0, 1, \cdots$, then $E_k \subset C(\mathbb{X})$ and $L^2(\mathbb{X}, \mu) = \oplus_{k=0}^{\infty} E_k$. Let the functions $\{f_i\}$ be an orthonormal basis of $L^2(\mathbb{X}, \mu)$ consisting of eigenfunctions of Δ. Then the function $\mathbf{t}_k$: $\mathbb{X} \to E_k$ defined by $\mathbf{t}_k(x) = \sum_{f_i \in E_k} f_i(x) f_i$ is well-defined (as it does not depend on the basis $\{f_i\}$).

In defining the semi-norms introduced by Giné (1975), the following definition will be useful.

DEFINITION. A sequence $\{\alpha_k\}_{k=1}^{\infty}$ of real numbers satisfies Condition C if

$$\sup_k |\alpha_k \sigma_k^{s/2}| < \infty \quad \text{for some} \quad s > (\dim \mathbb{X})/2.$$

If $A = \{\alpha_k\}_{k=1}^{\infty}$ satisfies condition C, the corresponding Sobolev semi-norm $\|\cdot\| = \|\cdot\|_A$ on $\mathscr{M}(\mathbb{X})$ is defined by

$$\|\nu\|^2 = \sum_{k=1}^{\infty} \alpha_k^2 \sum_{f_i \in E_k} \left(\int f_i \, d\nu \right)^2. \tag{2.1}$$

Thus $\|\nu\|^2$ is a weighted sum of squares of Fourier coefficients of ν. A neat expression for $\|\nu\|^2$ can be obtained from the function $\mathbf{t} : \mathbb{X} \to L^2(\mathbb{X}, \mu)$ defined by $\mathbf{t}(x) = \sum_{k=1}^{\infty} \alpha_k \mathbf{t}_k(x)$. If $\{\alpha_k\}_{k=1}^{\infty}$ satisfies condition C then

$$\|\nu\|^2 = \left\| \int \mathbf{t} \, d\nu \right\|_2^2 \tag{2.2}$$

where $\|\cdot\|_2$ is the L^2 norm on $L^2(\mathbb{X}, \mu)$. There are other useful expressions for $\|\nu\|^2$ generalizing equations (2.7) and (4.7) of Giné's (1975) paper.

If G is a group of isometries of $\mathbb{X}$, then the decomposition $C(\mathbb{X}) = C(\mathbb{X})_+ \oplus C(\mathbb{X})_-$ gives a decomposition of each E_k into $E_{k+} \oplus E_{k-}$ and similarly of $L^2(\mathbb{X}, \mu)$ into $L^2(\mathbb{X}, \mu)_+ \oplus L^2(\mathbb{X}, \mu)_-$. Let $\mathbf{t}_k = \mathbf{t}_{k+} + \mathbf{t}_{k-}$ form the decomposition of $\mathbf{t} : \mathbb{X} \to L^2(\mathbb{X}, \mu)$ above and define $\|\cdot\|_+$, $\|\cdot\|_-$ by $\|\nu\|_\pm^2 = \|\int \mathbf{t}_\pm \, d\nu\|_2^2$. The following proposition is immediate.

PROPOSITION 2.1. *For any sequence* $\{\alpha_k\}_{k=1}^{\infty}$ *satisfying condition* C *and for all* $\nu \in \mathscr{M}(\mathbb{X})$,

$$\text{(i)} \quad \|\nu\|^2 = \|\nu\|_-^2 + \|\nu\|_+^2$$

$$\text{(ii)} \quad \|\gamma \cdot \nu\|_\pm^2 = \|\nu\|_\pm^2 \quad \gamma \in N(G).$$

Let ε_n denote the empirical distribution of a sample of size n. From Proposition 2.1 (i) and some invariance arguments we obtain

$$n\|\varepsilon_n - \mu\|^2 = n\|\varepsilon_n\|_-^2 + n\|\bar{\varepsilon}_n - \mu\|_+^2. \tag{2.3}$$

The term $n\|\bar{\varepsilon}_n - \mu\|_+^2$ is the test statistic of one of Giné's (1975) Sobolev tests for uniformity on the quotient space $\mathbb{X}/G$; while we shall use $n\|\varepsilon_n\|_-^2$ as a test for symmetry under G. We thus have the fundamental partitioning of each Sobolev test of uniformity into a test of symmetry and a test of uniformity on a quotient space.

3. Randomization tests of symmetry. Giné (1975) used the test statistics $n\|\varepsilon_n - \mu\|^2$ to measure the distance between ε_n, the empirical distribution of a sample of size n, and μ, the uniform distribution. He thus obtained a class of invariant tests for uniformity on compact Riemannian manifolds. These tests include Rayleigh's (1919), Watson's (1961) U^2, and Ajne's (1968) A_n. Wellner (1979) considered the corresponding two-sample problem and used the statistics $n_1n_2(n_1+n_2)^{-1}\|\varepsilon_{n_1}^{(1)} - \varepsilon_{n_2}^{(2)}\|^2$ to measure the distance between $\varepsilon_{n_1}^{(1)}$ and $\varepsilon_{n_2}^{(2)}$, the empirical distributions of samples of sizes n_1 and n_2 from the two populations. Similarly, we test our hypothesis $H_0: \nu = \bar{\nu}$ by measuring the distance between the sample analogues ε_n and $\bar{\varepsilon}_n$ of ν and $\bar{\nu}$ by $n\|\varepsilon_n - \bar{\varepsilon}_n\|^2$. Note that if G is equal to the isometry group of $\mathbb{X}$, then our hypothesis is that of uniformity and our statistics are Giné's. For other subgroups G, however, our hypothesis is composite and the asymptotic null distribution of $n\|\varepsilon_n - \bar{\varepsilon}_n\|^2$ depends on ν. Under the null hypothesis, $\bar{\varepsilon}_n$ is a sufficient statistic, so we follow Wellner's randomization approach.

More precisely, let $\{\alpha_k\}_{k=1}^\infty$ be a sequence of real numbers satisfying condition C and let $\|\cdot\|$ be the corresponding Sobolev semi-norm on $\mathscr{M}(\mathbb{X})$. Then, given a sample $(x_1, \cdots, x_n)$ with empirical measure ε_n, define

$$T_n = n\|\varepsilon_n - \bar{\varepsilon}_n\|^2 = n\|\varepsilon_n\|_-^2. \tag{3.1}$$

The observed value of T_n is compared with its null distribution conditional on $\bar{\varepsilon}_n$, and the null hypothesis of symmetry is rejected for large values of T_n.

A difficulty with these tests as with all randomization tests is that of determining the null distribution of T_n conditional on $\bar{\varepsilon}_n$. If G is finite, of order l say, this can in principle be done by enumeration but involves $O(l^n)$ operations. Accordingly, we suggest following Wellner (1979) in simulating the distribution by sampling from the distribution under H_0 of $(X_1, \cdots, X_n)$ conditional on $\bar{\varepsilon}_n$, where $X_1, \cdots, X_n$ are i.i.d. random variables on $\mathbb{X}$. As this distribution is the image of the uniform distribution on G^n, such simulation is straightforward.

By construction these randomization tests are similar tests. Also, these tests are invariant under $N(G)$ by Proposition 2.1 (ii).

A simple criterion for consistency is given in the following theorem. The proof is analogous to the corresponding proofs in Giné (1975) and Wellner (1979) but uses the central limit theorem for triangular arrays (Gnedenko and Kolmogorov, 1954, page 128) rather than the usual or permutational versions. Details are given in Jupp and Spurr (1982).

THEOREM 3.1. *The sequence of tests based on $n\|\varepsilon_n\|_-^2$ conditional on $\bar{\varepsilon}_n$ is consistent against an alternative ν if and only if $\|\nu\|_-^2 > 0$. In particular, a sequence of tests is consistent against all alternatives if and only if $\alpha_k \neq 0$ for all k with $E_{k-} \neq \{0\}$.*

The asymptotic distribution of T_n under local and under fixed alternatives are given in the next two theorems. Again, the results are analogous to those of Giné (1975) and Wellner (1979). As in Giné (1975), $Z^{(\nu)}(f)$ denotes the Gaussian process indexed by $f \in L^2(\mathbb{X}, \nu)$ with mean zero and covariance structure given by

$$\operatorname{Cov}(Z^{(\nu)}(f), Z^{(\nu)}(g)) = \int_{\mathbb{X}} \left(f - \int_{\mathbb{X}} f\,d\nu\right)\left(g - \int_{\mathbb{X}} g\,d\nu\right) d\nu.$$

Also, $\to_{w^*}$ and $\to_d$ denote respectively convergence in the weak (star) topology of $\mathscr{P}(\mathbb{X})$ and convergence in distribution.

THEOREM 3.2 (*Local alternatives*). *Let* $\{\nu_n\}_{n=1}^{\infty}$ *be a sequence in* $\mathscr{P}(\mathbb{X})$ *satisfying* $\nu_n \rightarrow_{w^*} \nu$ *for some* $\nu \in \mathscr{P}(\mathbb{X})_+$ *such that*

$$\lim_{n\to\infty} n^{1/2} \int_{\mathscr{X}} f_i \, d(\nu_n - \nu) = d_i \quad \text{for} \quad f_i \in E_{k-} \quad \text{with} \quad \sum_{k=1}^{\infty} \alpha_k^2 \sum_{f_i \in E_{k-}} d_i^2 < \infty.$$

Then

$$T_n \rightarrow_d \sum_{k=1}^{\infty} \alpha_k^2 \sum_{f_i \in E_{k-}} \{Z^{(\nu)}(f_i) + d_i\}^2.$$

THEOREM 3.3 (*Fixed alternatives*). *For random samples from* $\nu \in \mathscr{P}(\mathbb{X})$ *with* $\| \nu \|_-^2 > 0$,

$$n^{-1/2}(T_n - n \| \nu \|_-^2) \rightarrow_d \mathscr{N}(0, \operatorname{Var}_\nu(u))$$

where $u(x) = 2 \sum_{k=1}^{\infty} \alpha_k^2 \sum_{f_i \in E_{k-}} \left(\int f_i \, d\nu \right) f_i(x)$ i.e. $u = 2\grave{E}_\nu[\mathbf{t}]$.

4. Uniform scores tests. A distinguishing feature of the circle, S^1, among compact Riemannian manifolds is that, once an origin and orientation have been chosen, each probability distribution ν on S^1 determines a unique probability integral transform $H_\nu: S^1 \rightarrow S^1$ defined by $H_\nu(\theta) = 2\pi \int_{0+}^{\theta} d\nu$ where S^1 is considered as the circle of unit radius. In the two-sample case, H_ν transforms the combined sample to uniform scores. By applying tests of uniformity to the uniform scores of one sample, Wheeler and Watson (1964), Mardia (1967) and Beran (1969) derived two-sample tests for the circle which are distribution-free under the null hypothesis of a continuous common distribution. Closely-related tests were considered by Schach (1969b). Similarly, we obtain invariant distribution-free tests for symmetry on the circle by using the symmetrized empirical distribution to define uniform scores of the observed sample and by applying a Sobolev test of uniformity to these. Thus, given the empirical distribution ε_n, the probability integral transform of its symmetrization $\bar{\varepsilon}_n$ transforms ε_n into the uniform-scores distribution η_n and our test statistic is

$$T_n^* = n \| \eta_n - \bar{\eta}_n \|^2 = n \| \eta_n \|_-^2. \tag{4.1}$$

Symmetry is rejected for large values of T_n^*.

The isometry group of the circle is $O(2)$, the orthogonal group of $\mathbb{R}^2$, consisting of rotations and reflections. The only closed subgroups G are:

(i) $G = O(2)$ or $G = SO(2)$, the rotation group. In either case, $N(G) = O(2)$, G-invariance is the same as uniformity, and our tests are those of Giné (1975).

(ii) $G = \mathbb{Z}_l$, a cyclic group of rotations, generated by $\theta \rightarrow \theta + 2\pi/l$ for some positive integer l. Then $N(G) = O(2)$.

(iii) G is generated by a cyclic group of rotations, $\mathbb{Z}_l$, and by a reflection. Then $N(G)$ is generated by this reflection and by $\mathbb{Z}_{2l}$.

We now give explicit versions of our tests in the case $G = \mathbb{Z}_l$. The null hypothesis in this case is that the distribution function $F(\theta)$ of ν satisfies $F(\theta + 2\pi l^{-1}) = F(\theta) + l^{-1}$. If $\theta_1, \cdots, \theta_n$ is a sample from ν, let $\{\phi_1, \cdots, \phi_{nl}\}$ be

$$\{\theta_i + j2\pi l^{-1}: 1 \le i \le n, 1 \le j \le l\} \quad \text{with} \quad 0 \le \phi_1 \le \cdots \le \phi_{nl} \le 2\pi.$$

The distribution assigning mass $(ln)^{-1}$ to each ϕ_k is $\bar{\varepsilon}_n$. Choose an origin 0 and an orientation for the circle and let $F_{\bar{\varepsilon}_n}$ be the corresponding distribution function of $\bar{\varepsilon}_n$. In the absence of ties between the ϕ_k's, the uniform scores β_i, $1 \le i \le n$, are defined by $\beta_i = 2\pi F_{\bar{\varepsilon}_n}(\theta_{(i)})$, where $\theta_{(1)}, \cdots, \theta_{(n)}$ are the order-statistics of $\theta_1, \cdots \theta_n$. For computational purposes it is useful to put T_n^* in the form used by Beran (1969)

$$T_n^* = (1/n) \sum_{i=1}^{n} \sum_{j=1}^{n} h_-(\beta_i - \beta_j) \tag{4.2}$$

where $h_-(\theta) = 2 \sum_{k \not\equiv 0 (\mathrm{mod}\, l)} \alpha_k^2 \cos k\theta$. Equivalence of the two forms follows from Proposition

5.2 of Giné (1975). Note that, as $\| \cdot \|_{-}^{2}$ is invariant under $O(2)$, T_n^* is well-defined independently of the origin and orientation of S^1. Note also that T_n^* is invariant under $O(2)$.

The case where G is a finite group which includes a reflection is similar except that the origin may be taken as any point on an axis of symmetry and that the analogue of the computational formula (4.2) contains extra terms.

An important property of the uniform scores tests is that they are distribution-free for sampling from continuous distributions. This is because under the null hypothesis of symmetry and in the absence of ties, the distribution of η_n conditional on $\bar{\eta}_n$ is the image of a uniform distribution on G^n. The null distribution of T_n^* may be determined by enumeration or by sampling. This null distribution is not in general the same as that of the corresponding test for uniformity or two-sample tests. However the following theorem combined with Theorem 4.1 of Giné (1975) and Theorem 1 of Beran (1969) shows that the asymptotic distributions are the same.

THEOREM 4.1. *Under random sampling from a continuous circular distribution which is invariant under G,*
(i) *if $G = \mathbb{Z}_l$ is a group of rotations,*

$$T_n^* \to_d \textstyle\sum_{k \not\equiv 0 (\mathrm{mod}\, l)} \alpha_k^2 H_k,$$

(ii) *if G is generated by a reflection and by a group $\mathbb{Z}_l$ of rotations,*

$$T_n^* \to_d \textstyle\sum_{k=1}^{\infty} \alpha_k^2 H_k.$$

Here $\{H_k\}_{k=1}^{\infty}$ is a sequence of independent chi-squared random variables with two degrees of freedom if $k \not\equiv 0 (\mathrm{mod}\, l)$ and one degree of freedom otherwise.

There is also a consistency result analogous to Theorem 2 of Beran (1969).

THEOREM 4.2 (*consistency*). *Let ν be a continuous distribution on the circle and let $H: S^1 \to S^1$ be a probability integral transform of $\bar{\nu}$. Then, if $\| \nu \circ H^{-1} \|_{-}^{2} > 0$, the sequence of tests based on T_n^* is consistent against ν. If $\{\alpha_k k^s\}$ is bounded for some $s > 3/2$, then the sequence of tests based on T_n^* is consistent against ν if and only if $\| \nu \circ H^{-1} \|_{-}^{2} > 0$.*

We conclude this section with two examples.

EXAMPLE 1. A quick test for $\mathbb{Z}_l$-symmetry is obtained by taking $\alpha_1 = 1$ and $\alpha_k = 0$, for $k \geq 2$. Then $T_n^* = 2n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{n} \cos(\beta_i - \beta_j) = 2n\bar{R}^2$ where $\bar{R}$ is the mean resultant length of the uniform scores. This is the analogue of the Rayleigh test for uniformity (Mardia, 1972, page 133) and of the uniform scores two-sample test of Wheeler and Watson (1964) and Mardia (1967). The asymptotic null distribution of T_n^* is χ_2^2.

EXAMPLE 2. If $\alpha_k = k^{-1}$ for $k \geq 1$, then the corresponding test is consistent against all non-symmetric alternatives. For testing $\mathbb{Z}_l$-symmetry, $T_n^* = 4\pi^2\{U_n^2 - (12nl^2)^{-1}\}$ where U_n^2 denotes Watson's (1961) U^2-statistic for testing uniformity applied to the uniform scores $\beta_1, \cdots, \beta_n$.

In the case of antipodal symmetry ($l = 2$), we have $T_n^* = \pi^2 A_n = \pi^2 \chi_n^2/4$ where A_n is Watson's (1967) A_n-statistic of Ajne's (1968) test and χ_n^2 is Rao's (1972) averaged χ^2 statistic on the circle for 2 intervals, each statistic being applied to the uniform scores.

For any integer $l \geq 2$, it can be shown using the method of Watson (1961, pages 111–112) that

$$\lim_{n\to\infty} P(T_n^* > x) = (2l/\pi) \textstyle\sum_{m=1}^{\infty} (-1)^{m-1} m^{-1} \sin(m\pi/l) \exp(-mx^2/2).$$

Some critical values of T_n^* for $l = 2$ are given in Jupp and Spurr (1982).

5. An astronomical example. In the study of the orbits of long-period comets two questions of interest arise:

(i) are the perihelion directions uniformly distributed over the celestial sphere?
(ii) are the orbital planes of comets with a given perihelion direction distributed with circular symmetry about that axis?

The orientation of a comet's orbit can be represented by an element of the rotation group $SO(3)$. The unit vector $\mathbf{x}_1$ in the direction of the perihelion and the unit vector $\mathbf{x}_2$ normal to the plane of the orbit (with direction specified by the sense of rotation) determine a unit vector $\mathbf{x}_3$ such that $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ is in $SO(3)$. Jupp and Mardia (1979) considered just $(\mathbf{x}_1, \mathbf{x}_2)$. The symmetry in the second question above is that of $\mathbf{R}$ in $G = SO(3)$ acting on $\mathbb{X} = SO(3)$ by premultiplication by block diag$[1, \mathbf{R}]$. Taking $\mathbf{t}$ to be the inclusion of $SO(3)$ in the space of 3×3 matrices yields the Rayleigh-type test of uniformity on $SO(3)$ considered by Khatri and Mardia (1977) and by Prentice (1978). Let $\bar{\mathbf{X}}$, $\bar{\mathbf{x}}_1$, and $\bar{\mathbf{X}}_1$ denote the sample means of $\mathbf{X}$, $\mathbf{x}_1$, and $(\mathbf{x}_2, \mathbf{x}_3)$. Then the terms in the decomposition (2.3) of the corresponding test statistic are

$$n \| \varepsilon_n - \mu \|^2 = n \operatorname{tr}(\bar{\mathbf{X}}'\bar{\mathbf{X}}), \quad T_n = n \| \varepsilon_n \|_-^2 = n \operatorname{tr}(\bar{\mathbf{X}}_1'\bar{\mathbf{X}}_1), \quad \text{and}$$

$$n \| \bar{\varepsilon}_n - \mu \|_+^2 = n\bar{\mathbf{X}}_1' \, \bar{\mathbf{x}}_1$$

with respective asymptotic null distributions

$$3^{-1}\chi_9^2, \; 3^{-1}\chi_6^2, \text{ and } 3^{-1}\chi_3^2.$$

For the data set of 240 comets considered by Jupp and Mardia (1979) we obtain $3n\bar{\mathbf{x}}_1'\bar{\mathbf{x}}_1 = 213.0$ and $3n \operatorname{tr}(\bar{\mathbf{X}}_1'\bar{\mathbf{X}}_1) = 11.8$. Thus uniformity of perihelion directions is rejected strongly as in Jupp and Mardia (1979) and as in Mardia's (1975) analysis of a similar data set. On the other hand, as $P(\chi_6^2 > 11.8) > 0.05$, we may accept symmetry of the orbital planes.

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