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TWO SAMPLE RANK ESTIMATORS OF OPTIMAL NONPARAMETRIC SCORE-FUNCTIONS AND CORRESPONDING ADAPTIVE RANK STATISTICS

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In the general two-sample testing problem, $X_1, \cdots, X_m$ i.i.d. with continuous c.d.f. F, $Y_1, \cdots, Y_n$ i.i.d. with continuous c.d.f. G, and null hypothesis $H_0:F = G$ versus alternative $H_1:F \le G$, $F \ne G$, we construct uniformly consistent and tractable rank estimators of the underlying optimal nonparametric score-function for a large subclass of (fixed) alternatives. Moreover, we prove asymptotic normality of the corresponding adaptive rank statistics under any fixed alternative (F, G) from the same subclass, and compare the results with the corresponding results for the (local) asymptotically optimum linear rank statistic for H_0 versus (F, G). In addition we prove some results on the estimation of a density and its derivative in the i.i.d. case if the support is [0, 1], which are needed for a comparison argument in the case of rank estimators, but which may be of interest in other situations, too.

1. Introduction. In a recent paper Behnen and Neuhaus (1983) propose the adaption of two sample rank tests to general "stochastically larger" alternatives by estimating an adequate nonparametric score-function on the basis of ranks:

Let $X_1, \cdots, X_m, Y_1, \cdots, Y_n$ be independent real valued random variables and suppose that the distribution of $X_i[Y_j]$ is given by a continuous (cumulative) distribution function $F[G]$, $i = 1, \cdots, m$, $j = 1, \cdots, n$. Let $N = m + n$ be the size of the pooled sample and consider the testing problem

$$H_0:F = G \quad \text{versus} \quad H_1:F \le G, \quad F \ne G.$$

If we assume (for a moment) a simple alternative $(F, G) \in H_1$ then it is clear from Hájek (1974) that the (upper) test based on the rank statistic

$$\sum_{i=1}^{m} \log\left\{f_N\left(\frac{R_{1i} - 1/2}{N}\right) \Big/ g_N\left(\frac{R_{1i} - 1/2}{N}\right)\right\} \tag{1.1}$$

has the best exact Bahadur slope for testing H_0 vs. (F, G), where $f_N:[0, 1] \to [0, N/m]$, $g_N:[0, 1] \to [0, N/n]$ are Lebesgue-densities on [0, 1] (μ-densities) defined by

$$f_N = \frac{d(F \circ H_N^{-1})}{d\mu}, \quad g_N = \frac{d(G \circ H_N^{-1})}{d\mu},$$

$$H_N = \frac{m}{N} F + \frac{n}{N} G, \quad b_N := f_N - g_N, \quad -\frac{N}{n} \le b_N \le \frac{N}{m}, \tag{1.2}$$

$$f_N = 1 + \frac{n}{N} b_N, \quad g_N = 1 - \frac{m}{N} b_N, \quad \frac{m}{N} f_N + \frac{n}{N} g_N = 1,$$

where μ always denotes the Lebesgue measure on [0, 1], and where $R_{1i}[R_{2j}]$ always denote the ranks of $X_i[Y_j]$ in the pooled sample $X_1, \cdots, X_m, Y_1, \cdots, Y_n$.

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On the other hand Behnen and Neuhaus (1983) show that in local situations the statistic (1.1) may be substituted by

$$\sum_{i=1}^{m} b_N\left(\frac{R_{1i} - 1/2}{N}\right), \tag{1.3}$$

which defines (in local situations) an asymptotically optimum test for H_0 vs. (F, G), cf. Behnen (1972). Therefore, in the usual case of unknown $(F, G) \in H_1$ it seems natural to adapt the statistics (1.1) and (1.3) to the underlying (F, G) by estimating b_N from the data. As is discussed in Behnen and Neuhaus (1983) on the basis of invariance arguments such estimators $\hat{b}_N$ of b_N should be based on the ranks only. Moreover, that paper discusses in detail only the case of an easy estimator of b_N which only estimates some tendency of the function b_N and which surprisingly leads to the Galton rank test.

In this paper we construct an estimator $\hat{b}_N$ of b_N, which *depends only on the ranks* and which has the properties (in probability)

$$\sup_{0\le t\le 1} |\hat{b}_N(t) - b_N(t)| \to 0, \quad \int_0^1 |\hat{b}_N'(t) - b_N'(t)|\, dt \to 0. \tag{1.4}$$

Moreover, we prove asymptotic normality of the corresponding adaptive rank statistic

$$\sum_{i=1}^{m} \hat{b}_N\left(\frac{R_{1i} - 1/2}{N}\right) \tag{1.5}$$

under very general alternatives (F, G) from H_1.

REMARK. Throughout this paper we use the notation

$$Q_i = \frac{R_{1i} - 1/2}{N}, \quad \text{if } i = 1, \cdots, m, \qquad Q_{m+j} = \frac{R_{2j} - 1/2}{N}, \quad \text{if } j = 1, \cdots, n. \tag{1.6}$$

We use Q_i, $i = 1, \cdots, N$, instead of the usual $R_i/(N + 1)$, $i = 1, \cdots, N$, in order to get complete symmetry [cf. remark following formula (2.7)]. In a separate paper, cf. Behnen and Hušková (1983), the applicability of such adaptive procedures is demonstrated by presenting a simple algorithm for the adaption of scores, by proving the asymptotic normality of the adaptive statistic (1.5) under the null hypothesis H_0, and by revealing the adaptive behavior of (1.5) for sample sizes as low as $m = 10$, $n = 10$ in a Monte Carlo power simulation.

Moreover, similar adaption procedures have been constructed for other models in Behnen and Neuhaus (1982).

2. Estimators of b_N based on ranks. Throughout this section we assume the two-sample situation of Section 1 with fixed underlying continuous distribution functions F and G, and

$$\lambda_N := m/N \to \lambda, \quad 0 < \lambda < 1, \quad N = m + n \to \infty. \tag{2.1}$$

Moreover, we use the following notations and properties [cf. (1.2)]

$$f = \frac{d(F \circ H^{-1})}{d\mu}, \quad g = \frac{d(G \circ H^{-1})}{d\mu}, \quad H = \lambda F + (1 - \lambda)G$$

$$\frac{dF}{dH} = f \circ H, \quad \frac{dG}{dH} = g \circ H, \quad b = f - g, \quad \lambda f + (1 - \lambda)g = 1, \tag{2.2}$$

$$f = 1 + (1 - \lambda)b, \quad g = 1 - \lambda b, \quad 0 \le f \le \lambda^{-1}, \quad 0 \le g \le (1 - \lambda)^{-1}.$$

Finally, let $K:\mathbb{R} \to \mathbb{R}$ be a kernel such that

(2.3) K is a probability density on $\mathbb{R}$ symmetric about zero with absolutely continuous derivative K' and essentially bounded second derivative K'', and K is zero outside the interval $(-1, 1)$,

and let $\{a_N\}$ be a sequence of real numbers such that

$$0 < a_N < \tfrac{1}{2}, \quad a_N \to 0, \quad Na_N^6 \to \infty, \quad \text{as} \quad N \to \infty. \tag{2.4}$$

Now we define a (modified) kernel estimator $\hat{b}_N:[0, 1] \to \mathbb{R}$ on the basis of ranks according to

$$\hat{b}_N = \hat{f}_N - \hat{g}_N, \tag{2.5}$$

where [cf. (1.6)]

$$\begin{aligned} \hat{f}_N(t) &= \frac{1}{m} \textstyle\sum_{i=1}^m K_N(t, Q_i) = \int d\hat{F}_m K_N(t, \tilde{H}_N), \\ \hat{g}_N(t) &= \frac{1}{n} \textstyle\sum_{j=m+1}^N K_N(t, Q_j) = \int d\hat{G}_n K_N(t, \tilde{H}_N), \quad \tilde{H}_N = \hat{H}_N - \frac{1}{2N}, \end{aligned} \tag{2.6}$$

$$K_N(t, s) = \frac{1}{a_N}\left\{K\left(\frac{t+s}{a_N}\right) + K\left(\frac{t-s}{a_N}\right) + K\left(\frac{t-2+s}{a_N}\right)\right\}, \tag{2.7}$$

and $\hat{F}_m$, $\hat{G}_n$, $\hat{H}_N = \lambda_N \hat{F}_m + (1 - \lambda_N)\hat{G}_n$ denote the empirical distribution functions of the first sample, the second sample, and the pooled sample, respectively.

REMARK. $\hat{f}_N$ and $\hat{g}_N$ are usual kernel estimators with kernel K applied to the modified rank data

$$-Q_m, -Q_{m-1}, \cdots, -Q_1, Q_1, \cdots, Q_m, 2 - Q_m, \cdots, 2 - Q_1$$

and

$$-Q_N, -Q_{N-1}, \cdots, -Q_{m+1}, Q_{m+1}, \cdots, Q_N, 2 - Q_N, \cdots, 2 - Q_{m+1},$$

respectively. This artificial enlargement of the original rank data by their reflections at the points zero and one takes care of the estimation problems near the boundary of the compact support [0, 1] of f_N and g_N. As is shown in the following Lemma, the symmetry of K guarantees that $\hat{f}_N$ and $\hat{g}_N$ are probability densities on the interval [0, 1].

LEMMA 2.1. *Assume* (2.3) *and* $0 < a_N \le 1$. *Then, for* $\hat{f}_N$, $\hat{g}_N$, *and* $\hat{b}_N$ *according to* (2.5) *to* (2.7), *we have*

$$\int \hat{f}_N \, d\mu = \int \hat{g}_N \, d\mu = 1, \quad \int \hat{b}_N \, d\mu = 0. \tag{2.8}$$

PROOF.

$$\begin{aligned} \int \hat{f}_N \, d\mu &= \int d\hat{F}_m \left\{\int_{\tilde{H}_N/a_N}^{(1+\tilde{H}_N)/a_N} K(x)\,dx + \int_{-\tilde{H}_N/a_N}^{(1-\tilde{H}_N)/a_N} K(x)\,dx + \int_{(-2+\tilde{H}_N)/a_N}^{(-1+\tilde{H}_N)/a_N} K(x)\,dx\right\} \\ &= \int d\hat{F}_m \int_{(-2+\tilde{H}_N)/a_N}^{(1+\tilde{H}_N)/a_N} K(x)\,dx \qquad (\text{because of } K(x) = K(-x)) \\ &= \int d\hat{F}_m \int_{-1}^{+1} K(x)\,dx = 1, \end{aligned}$$

because of $0 < a_N \le 1$, $0 \le \hat{H}_N \le 1$, and (2.3). Similarly we have

$$\int \hat{g}_N \, d\mu = 1, \quad \text{and therefore} \quad \int \hat{b}_N \, d\mu = 0. \quad \square$$

THEOREM 2.2. *Assume* (2.1), (2.3), *and* (2.4). *For the given* (F, G) *let* b_N *and* b *be defined according to* (1.2) *and* (2.2), *respectively. Assume* b [*and therefore by Lemma* 2.5 *also* b_N] *to be absolutely continuous on the compact interval* [0, 1], *such that*

$$\int |b_N' - b'| \, d\mu \to 0 \quad as \quad N \to \infty. \tag{2.9}$$

Then, for $\hat{b}_N$ *defined in* (2.5) *to* (2.7), *we have in* (F, G)*-probability*

$$\|\hat{b}_N - b\| \to 0, \quad \int |\hat{b}_N' - b'| \, d\mu \to 0 \quad as \quad N \to \infty, \tag{2.10}$$

where $\|\cdot\|$ *always denotes the sup-norm.*

PROOF. Obviously, $\hat{b}_N$ is absolutely continuous on [0, 1] for each N and given rank data. For the rest of the proof we define auxiliary "estimators" $\tilde{f}_N$, $\tilde{g}_N$, and $\tilde{b}_N = \tilde{f}_N - \tilde{g}_N$ by substituting Q_i and Q_{m+j} in (2.6) by $H_N(X_i)$ and $H_N(Y_j)$, respectively.

Now, because of symmetry in the arguments, the proof of (2.10) is complete, if we prove in (F, G)-probability

$$\|\hat{f}_N^{(i)} - \tilde{f}_N^{(i)}\| \to_{N\to\infty} 0, \quad i = 0, 1. \tag{2.11}$$

$$\|\tilde{f}_N - f\| \to_{N\to\infty} 0, \quad \int \|\tilde{f}_N' - f'\| \, d\mu \to_{N\to\infty} 0. \tag{2.12}$$

PROOF OF (2.11). For $i = 0, 1$ and $0 \le t \le 1$ we have by definition and the assumptions of the theorem:

$$\begin{aligned}
|\hat{f}_N^{(i)}(t) - \tilde{f}_N^{(i)}(t)| \le{} & \frac{1}{a_N^{i+1}} \left| \int d\hat{F}_m(x) \int_{(t+H_N(x))/a_N}^{(t+\hat{H}_N(x))/a_N} dy K^{(i+1)}(y) \right| \\
& + \frac{1}{a_N^{i+1}} \left| \int d\hat{F}_m(x) \int_{(t-H_N(x))/a_N}^{(t-\hat{H}_N(x))/a_N} dy K^{(i+1)}(y) \right| \\
& + \frac{1}{a_N^{i+1}} \left| \int d\hat{F}_m(x) \int_{(t-2+H_N(x))/a_N}^{(t-2+\hat{H}_N(x))/a_N} dy K^{(i+1)}(y) \right| \\
\le{} & 3 \operatorname{ess\,sup}_{y \in \mathbb{R}} |K^{(i+1)}(y)| \, a_N^{-2-i} N^{-1/2} N^{1/2} \|\hat{H}_N - H_N\| = o_P(1),
\end{aligned}$$

because of

$$N^{1/2} \|\hat{H}_N - H_N\| \le \frac{N^{1/2}}{2N} + O(1) m^{1/2} \|\hat{F}_m - F\| + O(1) n^{1/2} \|\hat{G}_n - G\| = O_P(1).$$

PROOF OF (2.12). $H_N(X_1), \cdots, H_N(X_m)$ are i.i.d. random variables with μ-density f_N. Because of (1.2), (2.2), and (2.9) on one hand we have

$$\begin{gathered}
f \text{ and } f_N \text{ are absolutely continuous on } [0, 1], \\
\int |f_N' - f'| \, d\mu \to 0 \quad \text{as} \quad N \to \infty,
\end{gathered} \tag{2.13}$$

on the other hand [cf. Lemma 2.5]

$$\| f_N - f \| \to 0 \quad \text{as} \quad N \to \infty. \tag{2.14}$$

Therefore the subsequent Theorem 2.3 proves (2.12). □

THEOREM 2.3. *For each* $N \in \mathbb{N}$ *let* $Z_{N1}, \cdots, Z_{Nm}$ *be* i.i.d. *random variables with values in the compact interval* [0, 1] *and density* h_N *on* [0, 1] *such that*

$$h_N \quad \text{is absolutely continuous on} \quad [0, 1], \tag{2.15}$$

$$h_N(1) \to h(1), \quad \int | h_N' - h' | \, d\mu \to 0, \tag{2.16}$$
$$\text{as} \quad N \to \infty, \quad \text{for some absolutely continuous function } h \text{ on } [0, 1],$$

$$m = m(N) \to \infty, \quad 0 < a_N < 1/2, \quad a_N \to 0, \quad m a_N^4 \to \infty, \qquad \text{as} \quad N \to \infty. \tag{2.17}$$

For K_N *according to* (2.7) *define*

$$\hat{h}_N(t) = (1/m) \textstyle\sum_{i=1}^m K_N(t, Z_{Ni}), \quad 0 \le t \le 1. \tag{2.18}$$

Then, under the assumption (2.3), *we have in probability*

$$\| \hat{h}_N - h \| \to 0, \quad \int | \hat{h}_N' - h' | \, d\mu \to 0 \quad \text{as} \quad N \to \infty. \tag{2.19}$$

PROOF. Obviously, $\hat{h}_N$ is absolutely continuous on [0, 1] for each N and given $Z_{N1}, \cdots, Z_{Nm}$. Therefore it suffices to prove (as $N \to \infty$)

$$\| \hat{h}_N^{(i)} - E\hat{h}_N^{(i)} \| \to 0 \quad \text{in probability}, \quad i = 0, 1, \tag{2.20}$$

$$\| E\hat{h}_N - h_N \| \to 0, \tag{2.21}$$

$$\| h_N - h \| \to 0, \tag{2.22}$$

$$\int | E\hat{h}_N' - h' | \, d\mu \to 0. \tag{2.23}$$

PROOF OF (2.20). Let $\hat{H}_{Nm}$ denote the empirical distribution function of $Z_{N1}, \cdots, Z_{Nm}$ and H_N the distribution function of h_N. With the notation

$$K_N^{(0)}(t, s) = K_N(t, s), \quad K_N^{(1)}(t, s) = (\partial/\partial t) K_N(t, s)$$

we get for $0 \le t \le 1$ and $i = 0, 1$

$$\begin{aligned}
&| \hat{h}_N^{(i)}(t) - E\hat{h}_N^{(i)}(t) | \\
&= \left| \int_0^1 d\hat{H}_{Nm}(s) K_N^{(i)}(t, s) - \int_0^1 dH_N(s) K_N^{(i)}(t, s) \right| \\
&= \left| \int_0^1 (d\hat{H}_{Nm}(s) - dH_N(s)) \frac{1}{a_N^{i+1}} \int_{-1}^1 dy K^{(i+1)}(y) (1_{\{y a_N \le t+s\}} + 1_{\{y a_N \le t-s\}} + 1_{\{y a_N \le t+s-2\}}) \right| \\
&\le \frac{1}{a_N^{i+1}} \int_{-1}^1 dy \, | K^{(i+1)}(y) | \cdot 3 \, \| \hat{H}_{Nm} - H_N \| \\
&= O(1) m^{-1/2} a_N^{-1-i} m^{1/2} \| \hat{H}_{Nm} - H_N \| = o_p(1), \quad \text{if} \quad i = 0, 1.
\end{aligned}$$

PROOF OF (2.21). From (2.7) and (2.3) we get

$$\int_0^1 dxK_N(t,x) = \int_{(t-2)/a_N}^{(t+1)/a_N} dyK(y) = \int_{-1}^1 dyK(y) = 1 \quad \forall 0 \le t \le 1 \tag{2.24}$$

and therefore $\forall 0 \le t \le 1$

$$\begin{aligned}
&| E\hat{h}_N(t) - h_N(t) | \\
&= \left| \int_0^1 dxh_N(x)K_N(t,x) - h_N(t) \int_0^1 dxK_N(t,x) \right| \\
&= \left| \int_{t-a_N}^{t+a_n} dxK_N(t,x)(h_N(x) - h_N(t)) \right| \le \sup_{|x-t| \le a_N} | h_N(x) - h_N(t) | \\
&\le 2 \| h_N - h \| + \sup_{|x-t| \le a_N} | h(x) - h(t) | = o(1), \quad \text{because of (2.22).}
\end{aligned}$$

PROOF OF (2.22). Obviously, we get from (2.15) and (2.16) for $0 \le t \le 1$

$$| h_N(t) - h(t) | \le \int_0^1 dx \, | h'(x) - h'_N(x) | + | h_N(1) - h(1) | = o(1).$$

PROOF OF (2.23). In a first step for $0 \le t \le 1$ we get the representation

$$\begin{aligned}
E\hat{h}'_N(t) &= \int_0^1 dxh_N(x) \frac{\partial}{\partial t} K_N(t,x) = \int_0^1 dx\left(h_N(0) + \int_0^x dyh'_N(y)\right) \frac{\partial}{\partial t} K_N(t,x) \\
&= h_N(0) \frac{1}{a_N} \left\{ K\left(\frac{t+1}{a_N}\right) - K\left(\frac{t-2}{a_N}\right) \right\} \\
&\quad + \int_0^1 dyh'_N(y) \frac{1}{a_N} \left\{ K\left(\frac{t+1}{a_N}\right) - K\left(\frac{t+y}{a_N}\right) + K\left(\frac{t-y}{a_N}\right) - K\left(\frac{t+y-2}{a_N}\right) \right\} \\
&= \int_0^1 dyh'_N(y) \frac{1}{a_N} K\left(\frac{t-y}{a_N}\right) - \frac{1}{a_N} \int_0^1 dyh'_N(y) 1_{\{t+y<a_N\}} K\left(\frac{t+y}{a_N}\right) \\
&\quad - \frac{1}{a_N} \int_0^1 dyh'_N(y) 1_{\{t+y>2-a_N\}} K\left(\frac{t+y-2}{a_N}\right).
\end{aligned}$$

In a second step we note

$$\begin{aligned}
0 &\le \int_0^1 dt \frac{1}{a_N} \int_0^1 dy \, | h'_N(y) | \, 1_{\{t+y<a_N\}} K\left(\frac{t+y}{a_N}\right) \\
&\le \frac{1}{a_N} \int_0^{a_N} dy \, | h'_N(y) | \int_0^{a_N} dt \, \| K \| = o(1)
\end{aligned}$$

$$\begin{aligned}
0 &\le \frac{1}{a_N} \int_0^1 dt \int_0^1 dy \, | h'_N(y) | \, 1_{\{t+y>2-a_N\}} K\left(\frac{t+y-2}{a_N}\right) \\
&\le \frac{1}{a_N} \int_{1-a_N}^1 dy \, | h'_N(y) | \int_{1-a_N}^1 dt \, \| K \| = o(1).
\end{aligned}$$

Therefore we get from (2.16) and Lemma 2.4

$$\int |E\hat{h}'_N - h'|\, d\mu = \int_0^1 dt \left| \int_0^1 dy h'_N(y) \frac{1}{a_N} K\left(\frac{t-y}{a_N}\right) - h'(t) \right| + o(1)$$

$$= \int_0^1 dt \left| \int_0^1 dy h'(y) \frac{1}{a_N} K\left(\frac{t-y}{a_N}\right) - h'(t) \right| + o(1) = o(1). \quad \square$$

LEMMA 2.4. *Let $g:[0,1] \to \mathbb{R}$ be a measurable function such that $\int |g|\, d\mu < \infty$. Let K be given by* (2.3) *and* $(a_n, n \in \mathbb{N})$ *a sequence in $\mathbb{R}$ such that $0 < a_N \to 0$ as $n \to \infty$. Then*

$$\int_0^1 dt \left| \int_0^1 dy g(y) \frac{1}{a_n} K\left(\frac{t-y}{a_n}\right) - g(t) \right| \to 0 \quad as \quad n \to \infty.$$

PROOF. Let g^+ and g^- denote the positive part and the negative part of g, i.e., $g = g^+ - g^-$, $g^+ \geq 0$, $g^- \geq 0$. For $0 \leq t \leq 1$ put

$$g_n^{\pm}(t) := \int_0^1 dy g^{\pm}(y) \frac{1}{a_n} K\left(\frac{t-y}{a_n}\right),$$

i.e.,

$$g_n(t) := \int_0^1 dy g(y) \frac{1}{a_n} K\left(\frac{t-y}{a_n}\right) = g_n^+(t) - g_n^-(t), \quad g_n^+ \geq 0, \quad g_n^- \geq 0.$$

Now, on one hand we get from Dunford and Schwartz (1958), Vol. I, Theorem III.12.11, for $n \to \infty$

$$0 \leq g_n^+ \to g^+[\mu], \quad 0 \leq g_n^- \to g^-[\mu], \tag{2.25}$$

on the other hand because of (2.3) and $0 < a_n \to 0$

$$\int_0^1 dt g_n^+(t) \to \int_0^1 dt g^+(t), \quad \int_0^1 dt g_n^-(t) \to \int_0^1 dt g^-(t). \tag{2.26}$$

The properties (2.25) and (2.26) imply

$$0 \leq \int_0^1 dt\, |g_n(t) - g(t)| \leq \int_0^1 dt\, |g_n^+(t) - g^+(t)| + \int_0^1 dt\, |g_n^-(t) - g^-(t)| \to 0. \quad \square$$

LEMMA 2.5. *Assume* (2.1) *and let the μ-densities f_N and f and the distribution functions H_N and H be defined as in* (1.2) *and* (2.2), *repsectively. Then we have:*

a)
$$\|H_N - H\| = |\lambda_N - \lambda| \cdot \|F - G\| \leq |\lambda_N - \lambda| \to 0 \quad as \quad N \to \infty,$$

$$\left\| \frac{dH}{dH_N} - 1 \right\| \leq |\lambda_N - \lambda| \max\{\lambda_N^{-1}, (1-\lambda_n)^{-1}\} \to 0 \quad as \quad N \to \infty,$$

$$\|H \circ H_N^{-1} - Id_{[0,1]}\| = \|H - H_N\| \to 0 \quad as \quad N \to \infty.$$

b) *If f is continuous on* [0, 1], *then f_N is continuous on* [0, 1] *for each N, and $\|f_N - f\| \to 0$ as $N \to \infty$.*

c) *If f is absolutely continuous on* [0, 1], *then f_N is absolutely continuous on* [0, 1] *for each N.*

PROOF. a) The assertions are immediate consequences of the definitions of H_N and H, the properties (1.2) and (2.2), and the identities

$$F \circ H_N^{-1} \circ H_N = F, \quad G \circ H_N^{-1} \circ H_N = G.$$

b) Define $K_N := H \circ H_N^{-1}$, then K_N is absolutely continuous on [0, 1] with derivative

$$k_N = \lambda f_N + (1 - \lambda) g_N = 1 + (\lambda - \lambda_N) b_N > 0.$$

On the other hand we get from (1.2) and (2.2)

$$f_N \circ H_N = \frac{dF}{dH_N} = \frac{dF}{dH} \Big/ \frac{dH_N}{dH} = \frac{f \circ H}{\lambda_N f \circ H + (1 - \lambda_N) g \circ H}$$

and therefore

$$f_N = \frac{f \circ K_N}{\lambda_N f \circ K_N + (1 - \lambda_N) g \circ K_N} = \frac{f \circ K_N}{1 + (\lambda_N - \lambda) b \circ K_N}.$$

Since $1 + (\lambda_N - \lambda) b \geq \min\{\lambda_N/\lambda, (1 - \lambda_N)/(1 - \lambda)\} > 0$, we have continuity of f_N on [0, 1] for each N, and

$$\begin{aligned} \| f_N - f \| &\leq \| f_N - f \circ K_N \| + \| f \circ K_N - f \| \\ &\leq \| f \| \cdot \left\| \frac{(\lambda_N - \lambda) b}{1 + (\lambda_N - \lambda) b} \right\| + \| f \circ K_N - f \| \to_{N \to \infty} 0, \end{aligned}$$

because of $\| K_N - Id_{[0,1]} \| \to_{N \to \infty} 0$ (part a) and the uniform continuity of f on [0, 1].

c) From the representation of f_N and k_N in the proof of part b) we get (under the present assumptions)

$$f_N' = \frac{[1 - (\lambda_N - \lambda) b \circ K_N][f' \circ K_N][1 + (\lambda - \lambda_N) b_N] - [f \circ K_N][(\lambda_N - \lambda) b' \circ K_N][1 + (\lambda - \lambda_N) b_N]}{[1 + (\lambda_N - \lambda) b \circ K_N]^2}.$$

Moreover, $K_N(0) = K_N^{-1}(0) = 0$, $K_N(1) = K_N^{-1}(1) = 1$, and $K_N^{-1} = H_N \circ H^{-1}$ is absolutely continuous on [0, 1] with derivative [cf. proof of part b)] $1 + (\lambda_N - \lambda) b$. Thus

$$\int | f_N' | \, d\mu = (1 + o(1)) \int | f' \circ K_N | \, d\mu + | \lambda_N - \lambda | \cdot O(1) = \int | f' | \, d\mu + o(1). \quad \square$$

COROLLARY 2.6. *If, under the assumptions of Lemma* 2.5, f *is absolutely continuous on* [0, 1] *and* f' *is continuous in the open interval* (0, 1), *then*

$$\int | f_N' - f' | \, d\mu \to 0, \quad \int | g_N' - g' | \, d\mu \to 0 \quad \text{as} \quad N \to \infty. \tag{2.27}$$

PROOF. On one hand we obtain from the proof of part c of Lemma 2.5

$$\begin{aligned} \int | f_N' - f' | \, d\mu &\leq \int | f' - f' \circ K_N | \, d\mu + \int | f' \circ K_N - f_N' | \, d\mu \\ &= \int | f' - f' \circ K_N | \, d\mu + o(1), \end{aligned}$$

on the other hand $\| K_N - Id_{[0,1]} \| = o(1)$, the μ-integrability of f', and the uniform continuity of f' on each compact interval $[\varepsilon, 1 - \varepsilon]$, $0 < \varepsilon < ½$, imply [first $N \to \infty$, then $0 < \varepsilon \to 0$]

$$\int | f' - f' \circ K_N | \, d\mu \leq 2 \int_0^{\varepsilon} | f' | \, d\mu + 2 \int_{1-\varepsilon}^{1} | f' | \, d\mu + o(1) \to_{\varepsilon \to 0} 0. \quad \square$$

3. Asymptotic normality of rank statistics with estimated scores. According to Section 1 and Section 2 we want to consider the adaptive rank statistic (1.5), where the rank estimator $\hat{b}_N$ of b_N is given by (2.3) to (2.7). In a first step we consider more general statistics of the form [cf. (1.1), (1.3), (1.5) and (1.6)]

$$S_N(\psi_N) = \sum_{i=1}^{m} \psi_N(Q_i). \tag{3.1}$$

THEOREM 3.1. *Assume* (2.1), *let* ψ *be an absolutely continuous score function on* [0, 1], *let* $\hat{\psi}_N$ *be an estimator of* ψ *(based on* $X_1, \cdots, X_m, Y_1, \cdots, Y_n$*) such that* $\hat{\psi}_N$ *is absolutely continuous on* [0, 1] *for each N and given data, and*

$$\int |\hat{\psi}_N' - \psi'|\, d\mu = o_P(1) \quad as \quad N \to \infty. \tag{3.2}$$

Then, under (F, G) *and* $N \to \infty$, *we obtain*

$$S_N(\hat{\psi}_N) - m \int dF\hat{\psi}_N \circ H_N = S_N(\psi) - m \int dF\psi \circ H_N + o_P(N^{1/2}). \tag{3.3}$$

If, in addition, f and g are continuous on [0, 1], *then we obtain*

$$\mathscr{L}\left\{\sqrt{\frac{N}{mn}}\left(S_N(\psi) - m \int dF\psi \circ H_N\right)\right\} \to \mathscr{N}(0, \sigma^2(\psi, f, g)), \tag{3.4}$$

where

$$\sigma^2(\psi, f, g) = (1 - \lambda)\mathrm{Var}_f\left[\int_0^Z \psi' g\, d\mu\right] + \lambda\, \mathrm{Var}_g\left[\int_0^Z \psi' f\, d\mu\right], \tag{3.5}$$

and H_N *and* f, g *are given by* (1.2) *and* (2.2), *respectively.*

PROOF. For the proof of (3.3) we utilize the following representation [cf. (2.6)],

$$\begin{aligned} S_N(\hat{\psi}_N) &- m \int dF\hat{\psi}_N \circ H_N - S_N(\psi) + m \int dF\psi \circ H_N \\ &= m \int d\hat{F}_m(\hat{\psi}_N - \psi) \circ \tilde{H}_N - m \int dF(\hat{\psi}_N - \psi) \circ H_N \\ &= m \int (d\hat{F}_m - dF)(\hat{\psi}_N - \psi) \circ \tilde{H}_N + m \int dF\{(\hat{\psi}_N - \Psi) \circ \tilde{H}_N - (\hat{\psi}_N - \psi) \circ H_N\}. \end{aligned}$$

Now, in (F, G)-probability, we obtain from (3.2)

$$\begin{aligned} m^{1/2}&\left|\int (d\hat{F}_m - dF)(\hat{\psi}_N - \psi) \circ \tilde{H}_N\right| \\ &= m^{1/2}\left|\int_0^1 dx(\hat{\psi}_N'(x) - \psi'(x)) \int (d\hat{F}_m - dF)1_{[\tilde{H}_N^{-1}(x+(1/2N)),\infty)}\right| \\ &\le m^{1/2}\,\|\hat{F}_m - F\| \int d\mu\, |\hat{\psi}_N' - \psi'| = O_P(1)o_P(1) = o_P(1), \end{aligned}$$

and, because of

$$\frac{m}{N}\int dF|\cdots| \le \int\left(\frac{m}{N}dF + \frac{n}{N}dG\right)|\cdots| = \int dH_N|\cdots|,$$

$$\begin{aligned} m^{1/2}&\left|\int dF\{(\hat{\psi}_N - \Psi)\circ\hat{H}_N - (\hat{\psi}_N - \Psi)\circ H_N\}\right| \\ &= m^{1/2}\left|\int dF\int_0^1 dx(\hat{\psi}'_N(x) - \psi'(x))\{1_{[0,\hat{H}_N]}(x) - 1_{[0,H_N]}(x)\}\right| \\ &= m^{1/2}\left|\int_0^1 dx(\hat{\psi}'_N(x) - \psi'(x))\int dF\{1_{[\hat{H}_N^{-1}(x+1/2N),\infty)} - 1_{[H_N^{-1}(x),\infty)}\}\right| \\ &\le \frac{N}{m}m^{1/2}\int_0^1 dx\,|\hat{\psi}'_N(x) - \psi'(x)|\int dH_N 1_{[A_N(x),B_N(x))}, \end{aligned}$$

where $A_N(x) = \hat{H}_N^{-1}(x + \frac{1}{2}N) \wedge H_N^{-1}(x)$, $B_N(x) = \hat{H}_N^{-1}(x + \frac{1}{2}N) \vee H_N^{-1}(x)$. Therefore the proof of (3.3) is concluded by (2.1), (3.2), and

$$\begin{aligned} N^{1/2}&\|H_N\circ B_N - H_N\circ A_N\| \\ &= N^{1/2}\left\|H_N\circ\hat{H}_N^{-1}\circ\left(Id + \frac{1}{2N}\right) - H_N\circ H_N^{-1}\right\| \\ &\le N^{1/2}\|H_N - \hat{H}_N\| + N^{1/2}\left\|\hat{H}_N\circ\hat{H}_N^{-1}\circ\left(Id + \frac{1}{2N}\right) - Id\right\| \\ &= O_P(1) + N^{1/2}\frac{1}{2N} = O_P(1). \end{aligned}$$

Since ψ is absolutely continuous on $[0, 1]$, we get the first step of the proof of (3.4) from Theorem 6.1 of Govindarajulu et al. (1967), namely for $N \to \infty$,

$$\left\|\mathscr{L}_{(F,G)}\left\{\sqrt{N}\left(\int\psi\left(\frac{N}{N+1}\hat{H}_N\right)d\hat{F}_m - \int\psi(H_N)\,dF\right)\right\} - \mathscr{N}(0,\sigma_N^2)\right\|_{BL} \to 0, \tag{3.6}$$

where

$$\sigma_N^2 = \frac{(1-\lambda_N)^2}{\lambda_N}\operatorname{Var}_{f_N}\left[\int_0^Z\psi' g_N\,d\mu\right] + (1-\lambda_N)\operatorname{Var}_{g_N}\left[\int_0^Z\psi' f_N\,d\mu\right].$$

Because of (2.1) and the continuity of f and g on one hand Lemma 2.5.b implies

$$\frac{m}{n}\sigma_N^2 \to_{N\to\infty} (1-\lambda)\operatorname{Var}_f\left[\int_0^Z\psi' g\,d\mu\right] + \lambda\operatorname{Var}_g\left[\int_0^Z\psi' f\,d\mu\right] = \sigma^2(\psi, f, g),$$

on the other hand we have [cf. proof of (3.3)]

$$\begin{aligned} &\left|\sqrt{\frac{mN}{n}}\left\{\int\psi\left(\frac{N}{N+1}\hat{H}_N\right)d\hat{F}_m - \int\psi(H_N)\,dF\right\}\right. \\ &\qquad\qquad\left. - \sqrt{\frac{N}{mn}}\left\{S_N(\psi) - m\int\psi(H_N)\,dF\right\}\right| \\ &= \left|\sqrt{\frac{mN}{n}}\int d\hat{F}_m\left\{\psi\left(\frac{N}{N+1}\hat{H}_N\right) - \psi\left(\hat{H}_N - \frac{1}{2N}\right)\right\}\right| \end{aligned}$$

$$= \left| \sqrt{\frac{mN}{n}} \int d\hat{F}_m \int_0^1 dx\psi'(x)\{1_{[0,(N/(N+1))\hat{H}_N]}(x) - 1_{[0,\hat{H}_N-(1/2N)]}(x)\} \right|$$

$$\leq \sqrt{\frac{mN}{n}} \frac{N}{n} \int_0^1 dx\, |\psi'(x)| \int d\hat{H}_N\, | 1_{[\hat{H}_N^{-1}((N+1)/N)x),\infty)} - 1_{[\hat{H}_N^{-1}(x+(1/2N)),\infty)} |$$

$$\leq \sqrt{\frac{mN}{n}} \frac{N}{m} \int_0^1 dx\, |\psi'(x)| \frac{2}{N} = o(1) \quad \text{as} \quad N \to \infty.$$

Therefore (3.6) implies (3.4). □

Now we are in the position to prove the asymptotic normality of the adaptive rank statistic $S_N(\hat{b}_N)$.

THEOREM 3.2. *Assume* (2.1), (2.3), *and* (2.4). *For given* (F, G) *let* b_N *and* b *be defined according to* (1.2) *and* (2.2), *respectively. Assume* b *[and therefore by Lemma* 2.5 *also* b_N*] to be absolutely continuous on* [0, 1], *and assume* (2.9). *Then, for* $\hat{b}_N$ *defined in* (2.5) *to* (2.7), *we have under* (F, G) *and* $N \to \infty$

$$\mathscr{L}\left\{ \sqrt{\frac{N}{mn}} \left(S_N(\hat{b}_N) - \frac{mn}{N} \int \tilde{b}_N b_N \, d\mu \right) \right\} \to \mathscr{N}(0, 4\sigma^2(b, f, g)), \tag{3.7}$$

where

$$\tilde{b}_N(t) = \int_0^1 dx b_N(x) K_N(t, x), \quad 0 \leq t \leq 1, \tag{3.8}$$

and

$$\sigma^2(b, f, g) = (1 - \lambda)\mathrm{Var}_f\left[\int_0^Z b'g \, d\mu\right] + \lambda \, \mathrm{Var}_g\left[\int_0^Z b'f \, d\mu\right]. \tag{3.9}$$

Moreover,

$$\sigma^2(b, f, g) > 0, \quad \text{iff} \quad F \neq G. \tag{3.10}$$

PROOF. Because of Theorem 2.2 we get on one hand from Theorem 3.1

$$S_N(\hat{b}_N) = S_N(b) - m \int b \circ H_N \, dF + m \int \hat{b}_N \circ H_N \, dF + o_P(N^{1/2}), \tag{3.11}$$

on the other hand $dF/dH_N = f_N \circ H_N = 1 + (n/N)b_N \circ H_N$, Lemma 2.1, and the definition of $\hat{b}_N$ imply

$$m \int \hat{b}_N \circ H_N \, dF$$

$$= m \int \hat{b}_N\left(1 + \frac{n}{N} b_N\right) d\mu = \frac{mn}{N} \int \hat{b}_N b_N \, d\mu$$

$$= \frac{mn}{N} \left\{ \frac{1}{m} \textstyle\sum_{i=1}^m \displaystyle\int_0^1 dx b_N(x) K_N(x, Q_i) - \frac{1}{n} \textstyle\sum_{i=m+1}^N \displaystyle\int_0^1 dx b_N(x) K_N(x, Q_i) \right\} \tag{3.12}$$

$$= \frac{mn}{N} \left\{ \frac{1}{m} \textstyle\sum_{i=1}^m \tilde{b}_N(Q_i) - \frac{1}{n} \textstyle\sum_{i=m+1}^N \tilde{b}_N(Q_i) \right\} = S_N(\tilde{b}_N) - \frac{m}{N} \textstyle\sum_{i=1}^N \tilde{b}_N\left(\frac{i - ½}{N}\right).$$

Because of (2.13) and (2.14) we get completely analogous to the proof of (2.23)

$$\int |\tilde{b}_N' - b'| \, d\mu \to 0 \quad \text{as} \quad N \to \infty. \tag{3.13}$$

Therefore Theorem 3.1 and (2.24) imply

$$\begin{aligned} S_N(\tilde{b}_N) &= S_N(b) - m \int b \circ H_N \, dF + m \int \tilde{b}_N \circ H_N \, dF + o_P(N^{1/2}) \\ &= S_N(b) - m \int b \circ H_N \, dF + \frac{mn}{N} \int \tilde{b}_N b_N \, d\mu + o_P(N^{1/2}). \end{aligned} \tag{3.14}$$

By combination of (3.11) to (3.14) we obtain

$$\begin{aligned} &S_N(\hat{b}_N) - \frac{mn}{N} \int \tilde{b}_N b_N \, d\mu \\ &\quad = 2\left\{S_N(b) - m \int b \circ H_N \, dF\right\} - \frac{m}{N} \textstyle\sum_{i=1}^N \tilde{b}_N\left(\frac{i - 1/2}{N}\right) + o_P(N^{1/2}). \end{aligned} \tag{3.15}$$

Finally, because of $\int_0^1 \tilde{b}_N(x) \, dx = 0$, we get

$$\begin{aligned} \left| \textstyle\sum_{i=1}^N \tilde{b}_N\left(\frac{i - 1/2}{N}\right) \right| &= N \left| \frac{1}{N} \textstyle\sum_{i=1}^N \tilde{b}_N\left(\frac{i - 1/2}{N}\right) - \int_0^1 \tilde{b}_N(x) \, dx \right| \\ &= N \left| \textstyle\sum_{i=1}^N \int_{(i-1)/N}^{i/N} \left(\tilde{b}_N\left(\frac{i - 1/2}{N}\right) - \tilde{b}_N(x)\right) dx \right| \\ &\le N \textstyle\sum_{i=1}^N \int_{(i-1)/N}^{1/N} dx \int_0^1 dy \, |\tilde{b}_N'(y)| \cdot |1_{[x,1]}(y) - 1_{[(i-1/2)/N,1]}(y)| \\ &= N \int_0^1 dy \, |\tilde{b}_N'(y)| \textstyle\sum_{i=1}^N \int_{(i-1)/N}^{i/N} dx \left| 1_{[0,y]}(x) - 1_{[0,y]}\left(\frac{i - 1/2}{N}\right) \right| \\ &\le \int |\tilde{b}_N'| \, d\mu = \int |b'| \, d\mu + o(1) \quad [cf. \ (3.13)]. \end{aligned}$$

Therefore (3.15) and Theorem 3.1 prove (3.7).

For the proof of (3.10) we notice that $F = G$, i.e., $f = g$, $b = 0$, obviously implies $\sigma^2(b, f, g) = 0$.

Now assume $F \neq G$. Because of (2.2) the absolute continuity of b on [0, 1] implies the absolute continuity of f, g, f^2, and g^2 on [0, 1]. Therefore $\forall 0 \le z \le 1$

$$\int_0^z 2ff' \, d\mu = f^2(z) - f^2(0), \quad \int_0^z 2gg' \, d\mu = g^2(z) - g^2(0),$$

and

$$\mathrm{Var}_f\left[\int_0^Z 2gg' \, d\mu\right] = \mathrm{Var}_f[g^2(Z)] < \infty, \quad \mathrm{Var}_g\left[\int_0^Z 2ff' \, d\mu\right] = \mathrm{Var}_g[f^2(Z)] < \infty.$$

Moreover, $f' = (1 - \lambda)b'$, $g' = -\lambda b'$, and thus

$$b'f = \frac{1}{2(1 - \lambda)} 2ff', \quad b'g = -\frac{1}{2\lambda} 2gg',$$

and

$$\text{(3.16)}\qquad \begin{aligned} \operatorname{Var}_f\left[\int_0^Z b'g\,d\mu\right] &= \left(\frac{1}{2\lambda}\right)^2 \operatorname{Var}_f[g^2(Z)] < \infty, \\ \operatorname{Var}\left[\int_0^Z b'f\,d\mu\right] &= \left(\frac{1}{2(1-\lambda)}\right)^2 \operatorname{Var}_g[f^2(Z)] < \infty. \end{aligned}$$

In order to prove $\sigma^2(b, f, g) > 0$, let us assume $\sigma^2(b, f, g) = 0$. Because of (3.9), (3.16), and the continuity of the μ-densities f and g, this implies

$$0 = \left[g^2 - \int_0^1 dyf(y)g^2(y)\right]^2 f = \left[f^2 - \int_0^1 dyg(y)f^2(y)\right]^2 g,$$

$$gf = \left[\int_0^1 dyf(y)g^2(y)\right]^{1/2} f, \quad fg = \left[\int_0^1 dyg(y)f^2(y)\right]^{1/2} g.$$

Especially

$$\int fg\,d\mu = \left[\int g^2 f\,d\mu\right]^{1/2} = \left[\int f^2 g\,d\mu\right]^{1/2}, \quad fg = \left[\int fg\,d\mu\right] f = \left[\int fg\,d\mu\right] g.$$

Since $f = g$ is a contradiction to $F \neq G$, this implies

$$fg = 0.$$

But from $\lambda f + (1-\lambda)g = 1$ and $fg = 0$ we get

$$\lambda f^2 = f, \quad (1-\lambda)g^2 = g,$$

$$f1_{\{f>0\}} = (1/\lambda)1_{\{f>0\}}, \quad g1_{\{g>0\}} = (\lambda/(1-\lambda))1_{\{g>0\}},$$

$$\mu\{f > 0, g = 0\} = \mu\{f > 0\} = \lambda \int d\mu f 1_{\{f>0\}} = \lambda > 0,$$

$$\mu\{g > 0, f = 0\} = \mu\{g > 0\} = (1-\lambda) \int d\mu g 1_{\{g>0\}} = 1 - \lambda > 0,$$

$$b = f - g = (f-g)1_{\{f>0,g=0\}} + (f-g)1_{\{f=0,g>0\}} = (1/\lambda)1_{\{f>0,g=0\}} - (1/(1-\lambda))1_{\{f=0,g>0\}},$$

which contradicts the continuity of b, because

$$\mu\{f > 0, g = 0\} = \lambda > 0, \quad \mu\{f = 0, g > 0\} = 1 - \lambda > 0. \quad \square$$

COROLLARY 3.3. *Given $(F, G) \in H_1$ such that $fg = 0$, then $b = f - g$ is not continuous.*

The proof is immediate from the proof of (3.10).

COROLLARY 3.4. *The assumptions of Theorem 3.2 imply, under (F, G) and $N \to \infty$,*

$$\text{(3.17)}\qquad \mathscr{L}\left\{\sqrt{\frac{N}{mn}}\left(S_N(b_N) - \frac{mn}{N}\int b_N^2\,d\mu\right)\right\} \to \mathscr{N}(0, \sigma^2(b, f, g)).$$

PROOF. Immediate from Theorem 3.1. $\square$

So the smoothness assumption on $b = f - g$ automatically assures that F and G are not disjoint measures, and this implies that the limiting laws (3.7) and (3.17) are nondegenerate for underlying $(F, G) \in H_1$.

If we consider the statistic $S_N(b_N)$, which is (locally) optimum for H_0 versus the simple alternative $(F, G) \in H_1$, under the null hypothesis H_0 then Theorem 3.1 implies the nondegenerate limiting law $(N \to \infty)$

$$\mathscr{L}_{H_0}\left\{\sqrt{\frac{N}{mn}}\, S_N(b_N)\right\} \to \mathscr{N}\left(0, \int b^2\, d\mu\right). \tag{3.18}$$

The situation is different if we consider the adaptive statistic $S_N(\hat{b}_N)$ under the null hypotheses H_0: Now Theorem 2.2 implies for $N \to \infty$

$$\|\hat{b}_N\| \to 0, \quad \int |\hat{b}_N'|\, d\mu \to 0 \quad \text{in } H_0\text{-probability.} \tag{3.19}$$

Therefore Theorem 3.2 implies for $N \to \infty$

$$\sqrt{\frac{N}{mn}}\, S_N(\hat{b}_N) \to 0 \quad \text{in } H_0\text{-probability.} \tag{3.20}$$

Behnen and Hušková (1983) prove nondegenerate asymptotic normality of $S_N(\hat{b}_N)$ under H_0 for suitable centering and standardization.

We conclude this paper by a comparison of the centering and the scale of $S_N(\hat{b}_N)$ and $S_N(b_N)$, respectively, which lead to standard normal law $\mathscr{N}(0, 1)$ as limiting distribution under the given simple alternative $(F, G) \in H_1$ corresponding to b_N and b. On one hand this may be utilized in order to obtain approximations of the type II error probabilities at $(F, G) \in H_1$ of the level α tests for H_0 based on $S_N(\hat{b}_N)$ and $S_N(b_N)$, respectively; on the other hand we may use these values in order to define a measure of performance.

Let $\alpha \in (0, 1)$ be given, and let $k_{N\alpha}$ and $l_{N\alpha}$ denote the critical values of the upper level α tests for H_0 based on the statistics $\sqrt{N/mn}\, S_N(\hat{b}_N)$ and $\sqrt{N/mn}\, S_N(b_N)$, respectively. Then (3.20) and (3.18) imply

$$k_{N\alpha} \to 0, \quad l_{N\alpha} \to u_\alpha\left(\int b^2\, d\mu\right)^{1/2}, \tag{3.21}$$

where u_α is the upper α-fractile of $\mathscr{N}(0, 1)$. And from (3.7) and (3.17) we obtain the following "approximations" of the type II error probabilities at the given $(F, G) \in H_1$

$$P_{F,G}\left\{\sqrt{\frac{N}{mn}}\, S_N(\hat{b}_N) \le k_{N\alpha}\right\} = \Phi\left(\frac{k_{N\alpha} - \sqrt{\frac{mn}{N}} \int \hat{b}_N b_N\, d\mu}{2\sigma(b, f, g)}\right) + o(1), \tag{3.22}$$

$$P_{F,G}\left\{\sqrt{\frac{N}{mn}}\, S_N(b_N) \le l_{N\alpha}\right\} = \Phi\left(\frac{l_{N\alpha} - \sqrt{\frac{mn}{N}} \int b_N^2\, d\mu}{\sigma(b, f, g)}\right) + o(1), \tag{3.23}$$

where Φ denotes the distribution function of $\mathscr{N}(0, 1)$. Obviously, these "approximations" are not very good, because both sides of (3.22) and (3.23) are of the order $o(1)$. But nevertheless they seem to give some orientation for small and medium sample sizes.

In order to have a well defined asymptotic measure of performance we may consider the standardized shifts divides by $\sqrt{N}$ of $\sqrt{N/mn}\, S_N(\hat{b}_N)$ and $\sqrt{N/mn}\, S_N(b)$, respectively, which lead to $\mathscr{N}(0, 1)$ under the given alternative $(F, G) \in H_1$, i.e.,

$$\frac{\sqrt{\frac{mn}{N^2}} \int \hat{b}_N b_N\, d\mu}{2\sigma(b, f, g)} \to_{N\to\infty} \frac{\sqrt{\lambda(1-\lambda)} \int b^2\, d\mu}{2\sigma(b, f, g)}, \tag{3.24}$$

and

$$\frac{\sqrt{\dfrac{mn}{N^2}}\displaystyle\int b_N\,d\mu}{\sigma(b,f,g)} \to_{N\to\infty} \frac{\sqrt{\lambda(1-\lambda)}\displaystyle\int b^2\,d\mu}{\sigma(b,f,g)}. \tag{3.25}$$

Obviously, the standardized asymptotic shift in the $\sqrt{N}$-scale of the adaptive $S_N(\hat{b}_N)$-test is half of the corresponding asymptotic shift of the (locally) optimum $S_N(b_N)$-test for any alternative $(F, G) \in H_1$ such that b is absolutely continuous on [0, 1] and (2.9) is fulfilled.

Finally, let us remark that (3.24) and (3.25) are closely connected with the definition of approximate Hodges-Lehmann asymptotic relative efficiency because of

$$-\frac{2}{N}\log\Phi\left(\frac{k_{N\alpha}-\sqrt{\dfrac{mn}{N}}\displaystyle\int \hat{b}_N b_n\,d\mu}{2\sigma(b,f,g)}\right) \to_{N\to\infty} \frac{\lambda(1-\lambda)\left(\displaystyle\int b^2\,d\mu\right)^2}{4\sigma^2(b,f,g)},$$

$$-\frac{2}{N}\log\Phi\left(\frac{l_{N\alpha}-\sqrt{\dfrac{mn}{N}}\displaystyle\int b_N^2\,d\mu}{\sigma(b,f,g)}\right) \to_{N\to\infty} \frac{\lambda(1-\lambda)\left(\displaystyle\int b^2\,d\mu\right)^2}{\sigma^2(b,f,g)}.$$

Nothing is known about the exact Hodges-Lehmann asymptotic relative efficiency of the adaptive $S_N(\hat{b}_N)$-test.

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