The Annals of Statistics
1983, Vol. 11, No. 3, 746–752

A CHARACTERIZATION OF CERTAIN STATISTICS IN EXPONENTIAL MODELS WHOSE DISTRIBUTIONS DEPEND ON A SUB-VECTOR OF PARAMETERS ONLY

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Let W be a $(k + r)$-dimensional random (column) vector with distribution F_ξ^W belonging to a $(k + r)$-parameter exponential family, $\Pi = \{F_\xi^W : \xi \in \Omega \subset R^{k+r}\}$. Let μ be a σ-finite measure which dominates Π such that for all $\xi \in \Omega$, F_ξ^W has a density, with respect to μ, of the form $f^W(w:\xi) = h(w) \exp\{\xi' w + c(\xi)\}$. Consider the partitions $W' = (U', T')$ and $\xi' = (\theta', \nu')$, where $U, \theta \in R^k$ and $T, \nu \in R^r$. It is proven that the conditional covariance matrix of U given $T = t$ does not depend on t for almost all values of t if and only if there exists a unique measurable vector-valued function $g(T) = (g_1(T), \cdots, g_k(T))'$, such that the random vector $Z = U - g(T)$ is stochastically independent of T under any member of Π. Furthermore, the distribution of Z is shown to constitute a k-parameter exponential family with θ as the vector of natural parameters. Further results are obtained and exemplified.

1. Introduction. Let W be a random vector taking values in a Borel set $\mathscr{X}$ of a $(k + r)$-dimensional Euclidean space with associated σ-field $\mathscr{A}$ and a $(k + r)$-parameter exponential family of distributions $\Pi = \{F_\xi^W : \xi \in \Omega \subset R^{k+r}\}$ defined on $(\mathscr{X}, \mathscr{A})$. Let μ be a σ-finite measure defined on $(\mathscr{X}, \mathscr{A})$ which dominates Π such that for all $\xi \in \Omega$, F_ξ^W has a density, with respect to (w.r.t.) μ, of the form

$$f^W(w:\xi) = h(w) \exp\{\xi' w + c(\xi)\} \tag{1.1}$$

(where ξ' denotes a transpose of a $(k + r)$ column vector ξ). It is assumed that the representation given by (1.1) is minimal and that Π is regular (i.e., the natural parameter space Ω is a non-empty open subset of R^{k+r}). The assumption that Π is regular is made for simplicity only as the results in the sequel hold on int Ω.

Consider a partition of W and ξ into $W' = (U', T')$, $\xi' = (\theta', \nu')$ where $U, \theta \in R^k$ and T, $\nu \in R^r$ ($k \geq 1, r \geq 1$). Let Θ and Υ stand for the projections of Ω onto $\theta = (\theta_1, \cdots, \theta_k)'$ and $\nu = (\nu_1, \cdots, \nu_r)'$ respectively, and denote by $\Pi^T = \{F_\xi^T : \xi \in \Omega\}$ the family of marginal distributions of T.

The present study is concerned with the problem of delineating cases where certain kinds of statistics are available whose distributions depend on a subvector of parameters only. The motivation for such a study has been stimulated by Lehmann's results on the construction of UMPU tests based on a single test statistic (c.f. Lehmann, 1959, Chapter 5, and also a discussion in Bar-Lev and Reiser, 1982). Consider the case where $k = 1$, r being arbitrary. The problem in question is of testing $\theta_1 = \theta_1^0$ vs $\theta_1 \neq \theta_1^0$, where θ_1^0 is specified. Theorem 1 of Lehmann (1959, Chapter 5) states that if there exists a statistic of the form $Z_1 = a_1(T)U_1 - g_1(T)$, where $a_1(T) > 0$ a.e. Π^T, such that Z_1 and T are independent under $\Pi_{\theta_1^0} = \{F_\xi^W : \xi' = (\theta_1^0, \nu')\}$, then a UMPU test for such a hypothesis can be given in terms of the statistic Z_1. It turns out that if Z_1 is independent of T not only when $\theta_1 = \theta_1^0$ but under any member of Π, then Z_1 can also be used for deriving UMPU tests for certain other composite hypotheses concerning θ_1 that are indicated in the above reference. At the end of Section 2 we show that in such a case, i.e., where Z_1 and T are

Received August 1982; revised March 1983.
AMS 1970 *subject classifications.* Primary 62E15; secondary 62E10.
Key words and phrases. Exponential family, ancillarity, UMPU tests.

independent under Π, then $a_1(T) \equiv$ constant a.e. Π^T and this constant can be chosen to be one without any loss of generality. Thus the statistic Z_1 is reduced to be of the form $U_1 - g_1(T)$. The general question then arises: For arbitrary k and r, what sort of subfamily of (1.1) admits a similar property to that for the case $k = 1$ and r is arbitrary. The results of Section 2 supply an answer to this question. These results are mainly derived by imposing some conditions on the structure of the covariance matrix, say $V[U \mid T = t]$, of the conditional distribution of U given $T = t$. Theorem 2.1 shows that $V[U \mid T = t]$ does not depend on t for almost all values of t, if and only if there exists a unique (up to an affine transformation) vector valued function $g(T) = (g_1(T), \cdots, g_k(T))'$ such that the random vector $Z = U - g(T)$ is stochastically independent of T under any member of Π. Moreover, the distribution of Z does not depend on ν and constitutes a k-parameter exponential family with θ as the vector of natural parameters. Further results are obtained in Theorem 2.2 by requiring that only some elements of $V[U \mid T = t]$ do not depend on t. In Theorem 2.3 we treat the case where $V[U \mid T = t]$ is functionally independent of t on some Borel set of values of t. We end Section 2 by proving our claim connected with Lehmann's results (see above). Some illustrative examples are presented in Section 3. It should be emphasized that even though we started our discussion with hypotheses testing considerations, the problem of delineating statistics whose distributions depend on a sub-vector of parameters is of a more general interest, and the derivation of UMPU tests represents only one of many possible applications of this effect.

2. The main results. Let $(\mathscr{J}, \mathscr{B}_T)$ be the range space of T. By well known results connected with the exponential family (see Lehmann, 1959, Barndorff-Nielsen, 1978, Andersen, 1973, and Johansen, 1979), it follows that the members of Π^T and of the family of conditional distributions of U given $T = t$ have the forms

$$dF^T(t:\theta, \nu) = \exp\{\nu' t + c(\theta, \nu) + \log b(\theta, t)\}\, d\lambda^*(t) \tag{2.1}$$

$$dF^{U|t}(u:\theta) = h(u, t) \exp\{\theta' u - \log b(\theta, t)\}\, d\bar{\lambda}_t(u), \tag{2.2}$$

where

$$b(\theta, t) = \int h(u, t) \exp\{\theta' u\}\, d\bar{\lambda}_t(u) \tag{2.3}$$

and the existence of the measures λ^* and $\bar{\lambda}_t$ is ensured by Lemma 8 of Lehmann (1959) (see also Andersen, 1973).

Since (2.2) constitutes a k-parameter exponential family with θ as the natural parameter vector, we obtain the following expressions for the mean vector, the covariance matrix and the characteristic function of the conditional distribution of U given $T = t$, which are valid for all $\theta \in \Theta$ and a.e. Π^T:

$$E[U \mid T = t] = \partial \log b(\theta, t)/\partial\theta, \quad V[U \mid T = t] = \partial^2 \log b(\theta, t)/\partial\theta' \partial\theta \tag{2.4}$$

$$E\{e^{is'U} \mid T = t\} = b(\theta + is, t)/b(\theta, t), \quad s = (s_1, \cdots, s_k)'. \tag{2.5}$$

The results in the sequel are mainly connected with imposing conditions on the form of $V[U \mid T = t]$. As the methods of proof for the theorems and corollaries following Lemma 2.1 and Theorem 2.1 are quite similar to those of the latter, their proofs are omitted for the sake of brevity.

LEMMA 2.1. *$V[U \mid T = t]$ depends on θ only* a.e. $\Pi^T \Leftrightarrow \log b(\theta, t)$ *is of the form*

$$\log b(\theta, t) = H(\theta) + \theta' g(t) + r(t), \tag{2.6}$$

where $H(\theta)$ is a function of θ only and $g(t) = (g_1(t), \cdots, g_k(t))'$ and $r(t)$ are functions of t only.

PROOF. The non-trivial part of the lemma is $\Rightarrow$. Assume that $V[U \mid T = t] =$

$\partial^2 H(\theta)/\partial\theta'\partial\theta$ for some function $H(\theta)$, where the (i, j) element of the latter matrix, say $H^{(ij)}(\theta)$, is equal to $\partial^2 H(\theta)/\partial\theta_i\partial\theta_j$, $i, j = 1, \cdots, k$. Let $H^{(i)}(\theta)$ denote $\partial H(\theta)/\partial\theta_i$, $i = 1, \cdots, k$. By (2.4), $H^{(ij)}(\theta) = \partial^2 \log b(\theta, t)/\partial\theta_i\partial\theta_j$, $i, j = 1, \cdots, k$ and thus $\partial \log b(\theta, t)/\partial\theta_i = \int H^{(ij)}(\theta)\, d\theta_j + s_j^i(\theta^{(j)}, t)$, where $s_j^i(\theta^{(j)}, t)$ is a function of t and θ only through $\theta^{(j)} = (\theta_1, \cdots, \theta_{j-1}, \theta_{j+1}, \cdots, \theta_k)'$. Since the $s_j^i(\theta^{(j)}, t)$'s are equal for all $j = 1, \cdots, k$ it follows that $s_j^i(\theta^{(j)}, t) = g_i(t)\ \forall\, j = 1, \cdots, k$, where $g_i(t)$ is a function of t only, i.e.,

$$\partial \log b(\theta, t)/\partial\theta_i = H^{(i)}(\theta) + g_i(t), \quad i = 1, \cdots, k. \tag{2.7}$$

By similar argument we obtain $\log b(\theta, t) = H(\theta) + \theta_i g_i(t) + m_i(\theta^{(i)}, t)$ for $i = 1, \cdots, k$. But by (2.7), $\partial m_i(\theta^{(i)}, t)/\partial\theta_j = g_j(t)$ for $j \neq i$ and $j = 1, \cdots, k$, $\Rightarrow m_i(\theta^{(i)}, t) = \theta_j g_j(t) + n_{ij}(\theta^{(i,j)}, t)$ where $n_{ij}(\theta^{(i,j)}, t)$ is a function of t and θ excluding the components θ_i and θ_j. Repetition of the same argument for all $j \neq i$ results in $m_i(\theta^{(i)}, t) = \sum_{j\neq i}\theta_j g_j(t) + r(t)$, and thus the desired result. □

THEOREM 2.1. *The following three conditions are equivalent*:

(i) $\log b(\theta, t)$ *is of the form given by* (2.6).

(ii) $Z = U - g(T)$ *and* T *are stochastically independent under any number of* Π *for some function* $g(T)$.

(iii) *The distribution of* Z *depends on* ξ *only through* θ.

Moreover, when these conditions hold, the distribution of Z *constitutes a* k*-parameter exponential family with natural parameter vector* θ *and characteristic function* $\exp\{H(\theta + is) - H(\theta)\}$.

PROOF. We first show that (ii) ⇔ (iii) and then (i) ⇔ (ii).

(iii) ⇒ (ii): For fixed θ, Π^T is complete and T is sufficient for Π, whereas the distribution of Z is independent of ν. Basu's Theorem (see Sverdrup, 1966, Theorem 10) is now applicable and thus Z and T are independent.

(ii) ⇒ (iii): Since the members of Π are equivalent, no two distributions of Π are singular (for definition, see Sverdrup, 1966, page 318). Also, for fixed θ, T is sufficient for Π, whereas, by assumption, Z is stochastically independent of T. We can now apply another theorem due to Basu (see Theorem 11 of Sverdrup, 1966) to obtain that the distribution of Z is independent of ν.

(i) ⇒ (ii): Using (2.6) in (2.5), we obtain

$$E\{\exp(is'U) \mid T = t\} = \exp\{H(\theta + is) - H(\theta) + is'g(t)\} \tag{2.8}$$

or

$$E\{\exp[is'(U - g(T))] \mid T = t\} = \exp\{H(\theta + is) - H(\theta)\}, \tag{2.9}$$

from which it follows that Z and T are independent and that

$$\varphi^Z(s) = E\{\exp(is'Z)\} = \exp\{H(\theta + is) - H(\theta)\}, \tag{2.10}$$

a function of θ only.

Since $F^{U|t}(u{:}\theta)$ is a k-parameter exponential family, $\log b(\theta, t)$ is analytic in θ, and thus, by (2.6), $H(\theta)$ is analytic also. This, together with the structure of $\varphi^Z(s)$ in (2.10), are shown by Patil (1963, 1965) to characterize a k-parameter exponential family with θ as its natural parameter vector.

(ii) ⇒ (i): We have

$$\begin{aligned} \log E\{\exp(is'Z) \mid T = t\} &= -is'g(t) + \log E\{\exp(is'U) \mid T = t\} \\ &= -is'g(t) + \log b(\theta + is, t) - \log b(\theta, t), \end{aligned} \tag{2.11}$$

which, by assumption, does not depend on t. Thus, the second order partials $\partial^2 \log b(\theta + is, t)/\partial s_i \partial s_j$, $i, j = 1, \cdots, k$, are also independent of t. This, as in the proof of Lemma 2.1, leads to

$$\log b(\theta + is, t) = H(\theta + is) + (\theta + is)'m(t) + r(t), m(t) = (m_1(t), \cdots, m_k(t))'.$$

Substituting this expression for $\log b(\theta + is, t)$ in (2.11), we obtain that $m(t) = g(t)$ a.e. Π^T and thus the desired result. □

REMARK. If one is interested in constructing UMPU tests for hypotheses concerning one of the components of θ, say θ_1, the tests can be based on the conditional distribution of Z_1 given $(Z_2, \cdots, Z_k)'$. This follows since, by the above theorem, the family of distributions of Z is k-parameter exponential, so that the results of Lehmann (1959, Chapter 4) are applicable.

The following two corollaries are obtained as straightforward applications of Theorem 2.1.

COROLLARY 2.1. *Assume that* $\log b(\theta, t)$ *is of the form given by* (2.6). *Let* $1 \le p \le k$ *and* $1 \le i_1 < \cdots < i_p \le k$ *be arbitrary, and* $c_{i_1}, \cdots, c_{i_p}$ *be constants. Then* $g_{i_j}(T) = c_{i_j}$, *for* $j = 1, \cdots, p$ a.e. Π^T *if and only if the random vectors* $U^* = (U_{i_1}, \cdots, U_{i_p})'$ *and* T *are stochastically independent. In particular, for the case* $p = k$, U *and* T *are independent such that their marginal distributions are* k *and* r*-parameter exponential families with* θ *and* ν *as their natural parameter vectors, respectively.* □

Corollary 2.1 supplies trivial applications of Theorem 2.1 for the case $p = k$, just by taking U and T, which are distributed as k and r-parameter exponential families respectively, to be stochastically independent.

COROLLARY 2.2. *Let* $\log b(\theta, t)$ *be of the form* (2.6). *For arbitrary* $1 \le p \le k - 1$ *and* $1 \le i_1 < \cdots < i_p \le k$ *let* $T^p = (T_{i_1}, \cdots, T_{i_p})'$ *and* T^{pp} *denote the vector of components of* T *not included in* T^p. *Then* $g(T)$ *is a function of* T *only through* T^p (a.e. Π^T) $\Leftrightarrow$ U *and* T^{pp} *are conditionally independent given* $T^p = t^p$. □

We proceed by treating the case where only a principal sub-matrix of $V[U \mid T = t]$, of order $p \times p (1 \le p \le k - 1)$, depends on θ only. For simplicity, we consider the case where $p = 1$ (the case where $p > 1$ can be constructed by a similar argument). Assume that for fixed j, the (j, j) term of $V[U \mid T = t]$ is a function of θ only (a.e. Π^T), i.e., $\partial^2 \log b(\theta, t)/\partial\theta_j^2 = H^{(jj)}(\theta)$. As in the proof of Lemma 2.1, this assumption can be shown to imply that

$$\log b(\theta, t) = H(\theta) + \theta_j g(\theta^{(j)}, t) + r(\theta^{(j)}, t). \tag{2.12}$$

Thus we have the following theorem analogous to Theorem 2.1.

THEOREM 2.2. *The following two conditions are equivalent*:
(i) $\log b(\theta, t)$ *is of the form given by* (2.12).
(ii) $V_j = U_j - g(\theta^{(j)}, T)$ *and* T *are independent for some* $g(\theta^{(j)}, t)$.

Moreover, when these conditions hold, the distribution of V_j *depends on* θ *only and for fixed* $\theta^{(j)}$ *constitutes a 1-parameter exponential family with* θ_j *as the natural parameter.* □

Note that if in the above theorem $g(\theta^{(j)}, t) = s(t)$ where $s(t)$ does not involve $\theta^{(j)}$, then T and $Z_j = U_j - s(T)$ are stochastically independent and the distribution of Z_j depends on ξ only through θ.

In the above developments, the conditions imposed on the form of $V[U \mid T = t]$ were required to hold for almost all values of t. Weakening the requirement so that it holds on only a Borel set $B \in \mathscr{B}_T$, for which $P_\xi(T \in B) > 0$ for some $\xi \in \Omega$ (and thus for all $\xi \in \Omega$), still enables us to obtain results similar to those obtained above. The following theorem summarizes this idea.

THEOREM 2.3. *Let B be a Borel set of values of t for which $P_\xi(T \in B) > 0$, $l(\theta, t)$ a function of θ and t and $g(t) = (g_1(t), \cdots, g_k(t))$. Then the following conditions are equivalent:*

(i) $\log b(\theta, t) = \{H(\theta) + \theta' g(t) + r(t)\} I_B(t) + \log l(\theta, t) I_{B^c}(t)$, *where $I_B(t)$ is the indicator function of the set B and B^c is its complement.*

(ii) $V[U \mid T = t] = (H^{(ij)}(\theta)) I_B(t) + (\partial^2 \log l(\theta, t)/\partial\theta_i \partial\theta_j) I_{B^c}(t)$, $i, j = 1, \cdots, k$, *where $(H^{(ij)}(\theta))$ and $(\partial^2 \log l(\theta, t)/\partial\theta_i\partial\theta_j)$ are matrices of second order partials of $H(\theta)$ and $\log l(\theta, t)$, respectively.*

(iii) $Z = U - g(T)$ *and T are conditionally independent given B.* □

We now consider the problem raised in Section 1, concerning Lehmann's results on the construction of UMPU tests based on a single test statistic. For the case treated by Lehmann (i.e., $k = 1$ and r arbitrary), we assume that T and $Z_1 = a_1(T)U_1 - g_1(T)$, ($a_1(T) > 0$ a.e. Π^T) are independent under any member of Π. Then, $a_1(T) =$ constant (a.e. Π^T), as is shown below. To simplify the derivation, without losing the essence, we further assume that Z_1 possesses a finite second moment. A more involved proof is available without this condition. Now, $V(Z_1 \mid T = t) = a_1^2(t) V[U_1 \mid T = t] = a_1^2(t)\, \partial^2 \log b(\theta_1, t)/\partial\theta_1^2$ which, by the independence of Z_1 and T, depends on θ_1 only. This, as in Lemma 2.1, leads to the relation $\partial \log b(\theta_1, t)/\partial\theta_1 = H^{(1)}(\theta_1)/a_1^2(t) + m_1(t)/a_1^2(t)$, where $m_1(t)$ depends on t only. Since $E[Z_1 \mid T = t] = H^{(1)}(\theta_1)/a_1(t) + m_1(t)/a_1(t) - g_1(t)$ is functionally independent of t, we obtain that $a_1(t) = c_1$ and $g_1(t) = m_1(t)/c_1 + c_2$ for arbitrary constants c_1 and c_2.

3. Examples. Before providing some illustrative examples, we note that in order to employ the above results for specific distributions, it is sufficient to derive only the marginal distribution of T, from which $\log b(\theta, t)$ can be obtained (see (2.1)). Then if $\log b(\theta, t)$ is one of the required forms indicated in Section 2, we can derive the structure of Z and its characteristic function.

The set of examples given below is divided according to three different forms (cases) of $\log b(\theta, t)$ given by (2.6), (2.12) and (i) of Theorem 2.3, respectively.

CASE I. $\log b(\theta, t)$ is of the form given by (2.6).

We first treat this case for $k = 1$, $r = 1$. Let $a(x) \exp\{\theta v_1(x) + \nu v_2(x) + c(\theta, \nu)\}$ be a density w.r.t. the Lebesgue measure on the real line and $X_1, \cdots, X_n$ be i.i.d. r.v.'s from this density. Set $U = \sum_{j=1}^n v_1(X_j)$, $T = \sum_{j=1}^n v_2(X_j)$. Bar-Lev and Reiser (1982) prove that $\log b(\theta, T)$ is of the form specified by (2.6) if ν can be represented as $-\theta\varphi'(\eta)$, where $\eta = E[v_2(X_1)]$ and $\varphi'(\eta) = d\varphi(\eta)/d\eta$ for some function $\varphi(\eta)$. Under this representation of ν, one obtains $g(T) = n\varphi(T/n)$ and $H(\theta) = nM(\theta) - M(n\theta)$. The normal, gamma and inverse Gaussian distributions were shown to admit such a representation. However, the following two examples show that this condition for the structure of ν is only sufficient but not necessary for $\log b(\theta, T)$ to be of the form (2.6).

EXAMPLE 1: Let X and Y be independent r.v.'s with densities (w.r.t. Lebesgue measure on $(0, \infty)$),

$$f^X(x{:}\alpha_1) = x^{\alpha_1 - 1} e^{-x}/\Gamma(\alpha_1), \quad f^Y(y{:}\alpha_2) = y^{\alpha_2 - 1} e^{-y}/\Gamma(\alpha_2),$$

respectively.

(i) Set $W' = (U = \log X, T = \log(Y/X))$; $\theta = \alpha_1 + \alpha_2$, $\nu = \alpha_2$ $(\infty > \theta > \nu > 0)$; the density of W' (w.r.t. Lebesgue measure on R^2) is of the form (1.1) with $c(\theta, \nu) = -\log\Gamma(\theta - \nu) - \log\Gamma(\nu)$.

(ii) $f^T(t:\theta, \nu) = \exp\{\nu t + c(\theta, \nu) - \theta \log(1 + e^t) + \log \Gamma(\theta)\}$, $-\infty < t < \infty$ $\log b(\theta, t) = -\theta \log(1 + e^t) + \log \Gamma(\theta)$; $g(t) = -\log(1 + e^t)$; $H(\theta) = \log \Gamma(\theta)$.
(iii) $Z = U - g(T) = \log X + \log(1 + e^T) = \log(X + Y)$ is independent of $T = \log(Y/X)$.
(iv) $\varphi^Z(s) = \Gamma(\theta + is)/\Gamma(\theta)$. □

EXAMPLE 2. Let X and Y be independent r.v.'s having exponential distributions with expectations $1/\alpha$ and $1/\beta$, respectively.
(i) $W' = (U = X, T = Y - X)$; $\theta = -(\alpha + \beta)$, $\nu = -\beta (0 > \nu > \theta)$; $c(\theta, \nu) = \log(\nu - \theta) + \log(-\nu)$.
(ii) $f^T(t:\theta, \nu) = \exp\{\nu t + c(\theta, \nu) - \theta t I_{(-\infty,0)}(t) - \log(-\theta)\}$; $\log b(\theta, t) = -\log(-\theta) - \theta t I_{(-\infty,0)}(t)$; $g(t) = -tI_{(-\infty,0)}(t)$, $H(\theta) = -\log(-\theta)$.
(iii) $Z = U - g(T) = X + TI_{(-\infty,0)}(T) = \min(X, Y)$ and $T = Y - X$ are independent.
(iv) $\varphi^Z(s) = \exp\{-\log[-(\theta + is)] - \log(-\theta)\} = (\alpha + \beta)/(\alpha + \beta - is)$, i.e., Z has an exponential distribution with expectation $1/(\alpha + \beta)$. □

EXAMPLE 3. $(k = 2, r = 1)$. Consider the linear regression model treated by Lehmann (1959, Chapter 5.7, pages 180–181). Let $Y_1, \cdots, Y_n$ be independent where Y_i is distributed as $N(\alpha + \beta x_i, \sigma^2)$.
(i) $W' = (U_1, U_2, T)$ where $U_1 = \sum_{i=1}^n Y_i^2$, $U_2 = \sum_{i=1}^n x_i Y_i$, $T = \sum_{i=1}^n Y_i$; $\theta_1 = -1/(2\sigma^2)$, $\theta_2 = \beta/\sigma^2$, $\nu = \alpha/\sigma^2$; $c(\theta_1, \theta_2, \nu) = n\nu^2/(4\theta_1) + \nu\theta_2 \sum_{i=1}^n x_i/(2\theta_1) + \theta_2^2 \sum_{i=1}^n x_i^2/(4\theta_1) + (n/2)\log(-2\theta_1)$.
(ii) $T \sim N(n\alpha + \beta \sum_{i=1}^n x_i, n\sigma^2)$,
$f^T(t:\theta_1, \theta_2, \nu) = \exp\{\nu t + c(\theta_1, \theta_2, \nu) + \theta_1 t^2/n + \theta_2 t \sum_{i=1}^n x_i/n - [\theta_2^2/(4\theta_1)] \sum_{i=1}^n (x_i - \bar{x}_n)^2 - [(n-1)/2] \log(-2\theta_1) - \log(2\pi n)^{1/2}\}$;
$g(t) = (g_1(t), g_2(t))'$, $g_1(t) = t^2/n$, $g_2(t) = t \sum_{i=1}^n x_i/n$;
$H(\theta_1, \theta_2) = -[\theta_2^2/(4\theta_1)] \sum_{i=1}^n (x_i - \bar{x}_n)^2 - [(n-1)/2] \log(-2\theta_1) - \log(2\pi n)^{1/2}$.
(iii) $Z' = (Z_1, Z_2)$, where $Z_1 = U_1 - g_1(T) = \sum_{i=1}^n (Y_i - \bar{Y}_n)^2$ and $Z_2 = U_2 - g_2(T) = \sum_{i=1}^n (x_i - \bar{x}_n)(Y_i - \bar{Y}_n)$, is independent of $T = \sum_{i=1}^n Y_i$.
(iv) $\varphi^Z(s) = \exp\{H(\theta_1 + is_1, \theta_2 + is_2) - H(\theta_1, \theta_2)\}$. □

For the discrete case it has been conjectured that, unless U and T are stochastically independent, there does not exist a statistic of the form $Z = U - g(T)$ which is ancillary for ν in the presence of θ (i.e., whose distribution depends on θ only). The following example of a discrete distribution, suggested to the author by Professor Lawrence D. Brown, demonstrates that this conjecture is not valid. (Professor Brown has suggested a general method for constructing further examples of this type. This will not be discussed here.)

EXAMPLE 4. $(k = 1, r = 1)$. Let $W' = (U, T)$ have a probability function

$$f^{U,T}(u, t:\theta, \nu) = \exp\{\theta u + \nu t + c(\theta, \nu)\}, \quad c(\theta, \nu) = -\log\{(1 + e^\theta)(1 + e^{\theta+\nu})\}$$

w.r.t. a counting measure on the set $\{(u, t):(u, t) = (0, 0), (1, 0), (1, 1), (2, 1)\}$.
(i) $f^T(t:\theta, \nu) = \exp\{\nu t + c(\theta, \nu) + \theta t + \log(1 + e^\theta)\} I_{\{0,1\}}(t)$;
$\log b(\theta, t) = \theta t + \log(1 + e^\theta)$; $g(t) = t$, $H(\theta) = \log(1 + e^\theta)$.
(ii) $Z = U - T$ and T are independent r.v.'s; $\varphi^Z(s) = [1 + \exp(\theta + is)]/[1 + \exp(\theta)]$. □

CASE II. $\log b(\theta, t)$ is of the form given by (2.12).

EXAMPLE 5. $(k = 3, r = 2)$. Let $\{(X_i, Y_i)'\}_{i=1}^n$ be a random sample from a bivariate normal distribution with unknown parameters $E(X_i) = \xi$, $E(Y_i) = \eta$, $V(X_i) = \sigma^2$, $V(Y_i) = \tau^2$, and $|\rho| \neq 1$, whose exponential representation is given by Lehmann (1959, Section 5.11).

(i) $W' = (U', T')$; $U' = (U_1, U_2, U_3)$; $T' = (T_1, T_2)$ where $U_1 = \sum_{i=1}^n X_i Y_i$,

$U_2 = \sum_{i=1}^n Y_i^2$, $U_3 = \sum_{i=1}^n Y_i$, $T_1 = \sum_{i=1}^n X_i$, $T_2 = \sum_{i=1}^n X_i^2$; $\theta' = (\theta_1, \theta_2, \theta_3)$,
$\nu' = (\nu_1, \nu_2)$ where $\theta_1 = \rho/[\sigma\tau(1-\rho^2)]$, $\theta_2 = -[2\tau^2(1-\rho^2)]^{-1}$,
$\theta_3 = \{\eta/\tau^2 - \xi\rho/(\sigma\tau)\}/(1-\rho^2)$, $\nu_1 = \{\xi/\sigma^2 - \eta\rho/(\sigma\tau)\}/(1-\rho^2)$,
$\nu_2 = -[2\sigma^2(1-\rho^2)]^{-1}$.

Expressions for $c(\theta, \nu)$ and $H(\theta)$ will not be given for sake of brevity.

(ii) $f^{T_1,T_2}(t_1, t_2{:}\theta, \nu) = \exp\{\nu_1 t_1 + \nu_2 t_2 + c(\theta, \nu) - [\theta_1\theta_3/2\theta_2]T_1 - [\theta_1^2/(4\theta_2)]T_2 + H(\theta) + a(t_1, t_2)\}I_{(t_1^2/n,\infty)}(t_2)I_{(-\infty,\infty)}(t_1)$,
where $a(t_1, t_2) = \log\{[nt_2 - t_1^2]^{(n-3)/2}/(nt_2)^{(n-2)/2}\}$.

(iii) Applying Theorem 2.2 for $j = 3$, we obtain $g(\theta^{(3)}, t_1, t_2) = -(\theta_1/2\theta_2)t_1$ and $r(\theta^{(3)}, t_1, t_2) = [-\theta_1^2/(4\theta_2)]t_2 + a(t_1, t_2)$. Thus, $V_3 = U_3 - g(\theta^{(3)}, t_1, t_2) = \sum_{i=1}^n Y_i + (\theta_1/2\theta_2)\sum_{i=1}^n X_i = \sum_{i=i}^n Y_i - (\rho\tau/\sigma)\sum_{i=1}^n X_i$ and T are independent. □

CASE III. $\log b(\theta, t)$ is of the form given by (i) of Theorem 2.3.

EXAMPLE 6. ($k = 1$, $r = 1$). Let Y be exponentially distributed with expectation $1/\alpha$ and let X, independent of Y, belong to a one-parameter exponential family with density, w.r.t. the Lebesgue measure on $(0, \infty)$ of the form $h(x)\exp\{\gamma x + d(\gamma)\}$. We assume that the natural parameter space of the latter density is a nonempty interval of R.

(i) $W' = (U = X, T = Y - X)$; $\theta = \gamma - \alpha$, $\nu = -\alpha$; $c(\theta, \nu) = \log(-\nu) + d(\theta - \nu)$.

(ii) $f^T(t{:}\theta, \nu) = \exp\{\nu t + c(\theta, \nu) + [\theta\cdot 0 - d(\theta)]I_{(0,\infty)}(t) + [\log\int_{-t}^{\infty} h(x)e^{\theta x}dx]I_{(-\infty,0)}(t)\}$.

(iii) By applying Theorem 2.3 we obtain that for $t > 0$, $g(t) \equiv 0$ and thus $Z = U - g(T) = X$ and $T = Y - X$ are conditionally independent given $T > 0$ ($Y > X$). □

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