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ESTIMATION VIA LINEARLY COMBINING TWO GIVEN STATISTICS

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We consider the problems of (i) covariance adjustment of an unbiased estimator, (ii) combining two unbiased estimators, and (iii) improving upon an unbiased estimator. All these problems consist in determining a minimum dispersion linear unbiased combination of given two statistics, one of which is an unbiased estimator of a vector parameter $\boldsymbol{\theta} \in \mathscr{H}$, and the expectation of the other is a zero vector in the problem of covariance adjustment, is equal to $\boldsymbol{\theta}$ in the problem of combining, and is equal to a subvector of $\boldsymbol{\theta}$ in the problem of improving. The solutions obtained are substantial generalizations of known results, in the sense that they are valid for an arbitrary joint dispersion matrix of the given statistics as well as for the parameter space $\mathscr{H}$ being an arbitrary subspace of $\mathscr{R}^k$.

1. Introduction. When estimating a k-vector parameter $\boldsymbol{\theta} \in \mathscr{H}$, there are cases in which two linear statistics are available and the problem then arising is how to combine most profitably the whole information at disposal. Three such cases are discussed in the present paper. In each of them, the problem is specified as that of determining a best estimator of $\boldsymbol{\theta}$ within the class of all unbiased estimators of $\boldsymbol{\theta}$ obtainable via linearly combining the available two statistics, the term "best" being used in the usual sense of the nonnegative definite partial ordering between the dispersion matrices of estimators. Furthermore, it is commonly assumed that one of the statistics is an unbiased estimator of $\boldsymbol{\theta}$ and that the joint dispersion matrix of the two statistics is known apart only from a positive scalar multiplier. The cases differ, however, in the expectation of the second of the available statistics. If this expectation is a zero vector, the problem is labelled as *covariance adjustment of an unbiased estimator*; if the second statistic is an unbiased estimator of $\boldsymbol{\theta}$, the problem is labelled as *combining two unbiased estimators*; and if the second statistic is an unbiased estimator of a subvector of $\boldsymbol{\theta}$, the problem is labelled as *improving upon an unbiased estimator*.

The problem of covariance adjustment of an unbiased estimator and the problem of combining two unbiased estimators have hitherto been solved in the literature (Rao, 1967; Lewis and Odell, 1971, Sections 3.9 and 8.3; Baksalary and Kala, 1979) under certain additional, considerably restrictive, conditions on the joint dispersion matrix of the given two statistics. Solutions obtained in the present paper, however, are valid for an arbitrary dispersion matrix as well as for the parameter space $\mathscr{H}$ being an arbitrary, proper or improper, subspace of $\mathscr{R}^k$. The same level of generality is maintained when solving the problem of improving upon an unbiased estimator, which, according to the authors' knowledge, has not yet been discussed in the literature.

2. Preliminaries. Throughout this paper, $\mathscr{M}_{m,n}$ will denote the set of all $m \times n$ real matrices. We write $\mathbf{A} \in \mathscr{M}_n^s$ if $\mathbf{A} \in \mathscr{M}_{n,n}$ and is symmetric, $\mathbf{A} \in \mathscr{M}_n^{\geq}$ if $\mathbf{A} \in \mathscr{M}_n^s$ and is nonnegative definite, and $\mathbf{A} \in \mathscr{M}_n^{>}$ if $\mathbf{A} \in \mathscr{M}_n^{\geq}$ and is nonsingular. Given $\mathbf{A} \in \mathscr{M}_{m,n}$, the symbols $\mathscr{C}(\mathbf{A})$, $\mathbf{A}'$, and $\mathbf{A}^{-1}$ will denote the column space, transpose, and inverse, respectively, of $\mathbf{A}$, whereas $\mathbf{A}^-$ will stand for a generalized inverse of $\mathbf{A}$, that is, for any solution to the matrix equation $\mathbf{A}\mathbf{A}^-\mathbf{A} = \mathbf{A}$. $\mathrm{E}(\cdot)$ and $\mathrm{D}(\cdot)$ will denote the expectation and dispersion matrix of a random vector argument.

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The main results of this paper will be proved by making use of the following three lemmas.

LEMMA 1. *Let* $\{\mathbf{Y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{V}\}$ *denote a general Gauss-Markoff model and let* $\mathbf{C}\boldsymbol{\beta}$ *be a set of parametric functions estimable therein. Then* $\mathbf{AY}$ *is a minimum dispersion linear unbiased estimator (MDLUE) of* $\mathbf{C}\boldsymbol{\beta}$ *if and only if*

$$\mathbf{AG} = \mathbf{C}(\mathbf{X}'\mathbf{G}^{-}\mathbf{X})^{-}\mathbf{X}', \qquad \textit{where } \mathbf{G} = \mathbf{V} + \mathbf{XZX}', \tag{2.1}$$

with $\mathbf{Z}$ *being any nonnegative definite matrix for which* $\mathscr{C}(\mathbf{X}) \subseteq \mathscr{C}(\mathbf{G})$. *Moreover, the dispersion matrix of the MDLUE is*

$$\mathrm{D}(\mathbf{AY}) = \mathbf{C}(\mathbf{X}'\mathbf{G}^{-}\mathbf{X})^{-}\mathbf{C}' - \mathbf{CZC}'.$$

LEMMA 2. *Let* $\mathbf{A} \in \mathscr{M}_{n,m}$ *and* $\mathbf{B} \in \mathscr{M}_n^s$ *be such that*

$$\mathscr{C}(\mathbf{A}) \subseteq \mathscr{C}(\mathbf{B}). \tag{2.2}$$

Then, for any nonsingular $\mathbf{F} \in \mathscr{M}_{n,n}$, *and for any generalized inverses* $\mathbf{B}^{-}$ *and* $(\mathbf{FBF}')^{-}$,

$$\mathbf{A}'\mathbf{B}^{-}\mathbf{A} = \mathbf{A}'\mathbf{F}'(\mathbf{FBF}')^{-}\mathbf{FA}. \tag{2.3}$$

LEMMA 3. *Let* $\mathbf{A} \in \mathscr{M}_{m+n}^s$ *be partitioned as*

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}.$$

Then $\mathbf{A} \in \mathscr{M}_{m+n}^{\geq}$ *if and only if* $\mathbf{A}_{22} \in \mathscr{M}_n^{\geq}$, $\mathscr{C}(\mathbf{A}_{21}) \subseteq \mathscr{C}(\mathbf{A}_{22})$, *and*

$$\mathbf{A}_{*} \equiv \mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-}\mathbf{A}_{21} \in \mathscr{M}_m^{\geq}.$$

If this is the case, one of the generalized inverses of $\mathbf{A}$ *is*

$$\mathbf{A}^{-} = \begin{pmatrix} \mathbf{A}_{*}^{-} & -\mathbf{A}_{*}^{-}\mathbf{A}_{12}\mathbf{A}_{22}^{-} \\ -\mathbf{A}_{22}^{-}\mathbf{A}_{21}\mathbf{A}_{*}^{-} & \mathbf{A}_{22}^{-} - \mathbf{A}_{22}^{-}\mathbf{A}_{21}\mathbf{A}_{*}^{-}\mathbf{A}_{12}\mathbf{A}_{22}^{-} \end{pmatrix}.$$

Lemma 1 follows by similar arguments as those used by Rao (1978) in the proof of his Theorem 1. Lemma 2 is a consequence of the fact that $(\mathbf{F}')^{-1}\mathbf{B}^{-}\mathbf{F}^{-1}$ is a generalized inverse of $\mathbf{FBF}'$ and that the condition (2.2) assures the invariance of the two sides of (2.3) with respect to the choices of the generalized inverses involved; cf. Rao and Mitra (1971, page 43) and Hall and Meyer (1975, page 433). Lemma 3 follows by combining Theorem 1 in Albert (1969) with the theorem in Marsaglia and Styan (1974).

3. Covariance adjustment of an unbiased estimator. The main result of this section is the following.

THEOREM 1. *Let* $\mathbf{T}_1 \in \mathscr{R}^k$ *and* $\mathbf{T}_2 \in \mathscr{R}^\ell$ *be given statistics such that* $\mathrm{E}(\mathbf{T}_1) = \boldsymbol{\theta} \in \mathscr{H}$, $\mathrm{E}(\mathbf{T}_2) = \mathbf{0}$, *and*

$$\mathrm{D}\begin{pmatrix} \mathbf{T}_1 \\ \mathbf{T}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{V}_{11} & \mathbf{V}_{12} \\ \mathbf{V}_{21} & \mathbf{V}_{22} \end{pmatrix} \equiv \mathbf{V} \in \mathscr{M}_{k+\ell}^{\geq} \tag{3.1}$$

where $\boldsymbol{\theta}$ *is an unknown parameter, the parameter space* $\mathscr{H}$ *is a given subspace of* $\mathscr{R}^k$, *and* $\mathbf{V}$ *is known apart from a positive scalar multiplier. Then a best estimator of* $\boldsymbol{\theta}$ *in the class*

$$\mathscr{T}_{\mathscr{H}} = \{\mathbf{T} = \mathbf{K}_1\mathbf{T}_1 + \mathbf{K}_2\mathbf{T}_2 : \mathbf{K}_1 \in \mathrm{M}_{k,k}, \mathbf{K}_2 \in \mathscr{M}_{k,\ell}, \mathrm{E}(\mathbf{T}) = \boldsymbol{\theta} \ \forall\ \boldsymbol{\theta} \in \mathscr{H}\}$$

is expressible in the form

$$\mathbf{T}_{\mathscr{H}}^{*} = \mathbf{H}(\mathbf{H}'\mathbf{Q}^{-}\mathbf{H})^{-}\mathbf{H}'\mathbf{Q}^{-}(\mathbf{T}_1 - \mathbf{V}_{12}\mathbf{V}_{22}^{-}\mathbf{T}_2),$$

where $\mathbf{H}$ *is any matrix such that* $\mathscr{C}(\mathbf{H}) = \mathscr{H}$, *and*

$$\mathbf{Q} = \mathbf{HZH}' + \mathbf{V}_{11} - \mathbf{V}_{12}\mathbf{V}_{22}^{-}\mathbf{V}_{21},$$

with $\mathbf{Z}$ *being any nonnegative definite matrix for which* $\mathscr{C}(\mathbf{H}) \subseteq \mathscr{C}(\mathbf{Q})$. *The dispersion matrix of* $\mathbf{T}^*_{\mathscr{H}}$ *is*

$$\mathrm{D}(\mathbf{T}^*_{\mathscr{H}}) = \mathbf{H}(\mathbf{H}'\mathbf{Q}^{-}\mathbf{H})^{-}\mathbf{H}' - \mathbf{HZH}'.$$

PROOF. It is clear that the assumptions of the theorem lead to a Gauss-Markoff model of the form

$$\left\{\begin{pmatrix}\mathbf{T}_1\\ \mathbf{T}_2\end{pmatrix}, \begin{pmatrix}\mathbf{H}\\ \mathbf{O}\end{pmatrix}\tau, \begin{pmatrix}\mathbf{V}_{11} & \mathbf{V}_{12}\\ \mathbf{V}_{21} & \mathbf{V}_{22}\end{pmatrix}\right\}, \tag{3.2}$$

where τ is a new parameter, related to the original one by the equality $\boldsymbol{\theta} = \mathbf{H}\tau$. Consequently, the result of the theorem follows by verifying, with the aid of Lemma 3, that the condition (2.1) of Lemma 1 is fulfilled for the model (3.2) in the case of $\mathbf{C} = \mathbf{H}$ and

$$\mathbf{A} = \mathbf{H}(\mathbf{H}'\mathbf{Q}^{-}\mathbf{H})^{-}\mathbf{H}'\mathbf{Q}^{-}(\mathbf{I} \,\vdots\, -\mathbf{V}_{12}\mathbf{V}_{22}^{-}). \quad \square$$

To reveal the extent to which Theorem 1 generalizes results known in the literature, let us consider two particular cases. First observe that if $\mathbf{V}$ in (3.1) is positive definite, then so is the Schur complement

$$\mathbf{V}_0 = \mathbf{V}_{11} - \mathbf{V}_{12}\mathbf{V}_{22}^{-1}\mathbf{V}_{21}, \tag{3.3}$$

and thus the zero matrix may be used as $\mathbf{Z}$ in Theorem 1. This leads to the following.

COROLLARY 1.1. *If in Theorem* 1 $\mathbf{V} \in \mathscr{M}^{>}_{k+\ell}$, *then the best estimator of* $\boldsymbol{\theta} \in \mathscr{H}$ *in the class* $\mathscr{T}_{\mathscr{H}}$ *is*

$$\mathbf{T}^0_{\mathscr{H}} = \mathbf{H}(\mathbf{H}'\mathbf{V}_0^{-1}\mathbf{H})^{-}\mathbf{H}'\mathbf{V}_0^{-1}(\mathbf{T}_1 - \mathbf{V}_{12}\mathbf{V}_{22}^{-1}\mathbf{T}_2),$$

where $\mathbf{V}_0$ *is as given in* (3.3). *The dispersion matrix of* $\mathbf{T}^0_{\mathscr{H}}$ *is*

$$\mathrm{D}(\mathbf{T}^0_{\mathscr{H}}) = \mathbf{H}(\mathbf{H}'\mathbf{V}_0^{-1}\mathbf{H})^{-}\mathbf{H}'.$$

Next observe that if the parameter space equals $\mathscr{R}^k$ then, irrespective of the rank of $\mathbf{V}$, the identity matrix may be used as $\mathbf{H}$ in Theorem 1. According to the definition, $\mathbf{Z}$ may then be chosen as any element of $\mathscr{M}^{\geq}_k$ such that $\mathbf{Z} + \mathbf{V}_{11} - \mathbf{V}_{12}\mathbf{V}_{22}^{-}\mathbf{V}_{21} \in \mathscr{M}^{>}_k$. This results in

COROLLARY 1.2. *If in Theorem* 1 *the parameter space* $\mathscr{H}$ *equals* $\mathscr{R}^k$, *then a best estimator of* $\boldsymbol{\theta}$ *in the class* $\mathscr{T}_{\mathscr{R}^k}$ *is expressible in the form*

$$\mathbf{T}^* = \mathbf{T}_1 - \mathbf{V}_{12}\mathbf{V}_{22}^{-}\mathbf{T}_2.$$

The dispersion matrix of $\mathbf{T}^*$ *is*

$$\mathrm{D}(\mathbf{T}^*) = \mathbf{V}_{11} - \mathbf{V}_{12}\mathbf{V}_{22}^{-}\mathbf{V}_{21}.$$

The result given in Corollary 1.1 has actually been obtained in Baksalary and Kala (1979), but not stated there in an explicit form. On the other hand, Corollaries 1.1 and 1.2 both cover a result established by Rao (1967) for the case wherein simultaneously $\mathscr{H} = \mathscr{R}^k$ and $\mathbf{V} \in \mathscr{M}^{>}_{k+\ell}$. Rao's formulae for the best estimator of $\boldsymbol{\theta}$ and its dispersion matrix follow from Corollary 1.1 by setting $\mathbf{H} = \mathbf{I}$ while from Corollary 1.2 by replacing $\mathbf{V}_{22}^{-}$ by $\mathbf{V}_{22}^{-1}$.

4. Combining two unbiased estimators. Similarly as in the previous section, we begin with providing a general solution to the problem.

THEOREM 2. *Let* $\mathbf{U}_1, \mathbf{U}_2 \in \mathscr{R}^k$ *be given statistics such that* $\mathrm{E}(\mathbf{U}_1) = \mathrm{E}(\mathbf{U}_2) = \boldsymbol{\theta} \in \mathscr{H}$, *and*

$$\mathrm{D}\begin{pmatrix}\mathbf{U}_1\\ \mathbf{U}_2\end{pmatrix} = \begin{pmatrix}\boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12}\\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22}\end{pmatrix} \equiv \boldsymbol{\Sigma} \in \mathscr{M}_{2k}^{\geq},$$

where $\boldsymbol{\theta}$ *is an unknown parameter, the parameter space* $\mathscr{H}$ *is a given subspace of* $\mathscr{R}^k$, *and* $\boldsymbol{\Sigma}$ *is known apart from a positive scalar multiplier. Moreover, let* $\mathbf{H}$ *be any matrix such that* $\mathscr{C}(\mathbf{H}) = \mathscr{H}$, *and let*

$$\mathbf{R} = \mathbf{HZH}' + \boldsymbol{\Sigma}_{11} - (\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12})\boldsymbol{\Sigma}_*^-(\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{21}),$$

where $\mathbf{Z}$ *is any nonnegative definite matrix for which* $\mathscr{C}(\mathbf{H}) \subseteq \mathscr{C}(\mathbf{R})$, *and*

$$\boldsymbol{\Sigma}_* = \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12} - \boldsymbol{\Sigma}_{21} + \boldsymbol{\Sigma}_{22}. \tag{4.1}$$

Then a best estimator of $\boldsymbol{\theta}$ *in the class*

$$\mathscr{U}_{\mathscr{H}} = \{\mathbf{U} = \mathbf{L}_1\mathbf{U}_1 + \mathbf{L}_2\mathbf{U}_2 : \mathbf{L}_1, \mathbf{L}_2 \in \mathscr{M}_{k,k},\ \mathrm{E}(\mathbf{U}) = \boldsymbol{\theta}\ \forall\ \boldsymbol{\theta} \in \mathscr{H}\}$$

is expressible in the form

$$\mathbf{U}_{\mathscr{H}}^* = \mathbf{H}(\mathbf{H}'\mathbf{R}^-\mathbf{H})^-\mathbf{H}'\mathbf{R}^-\mathbf{U}^*,$$

with

$$\mathbf{U}^* = [\mathbf{I} - (\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12})\boldsymbol{\Sigma}_*^-]\mathbf{U}_1 + (\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12})\boldsymbol{\Sigma}_*^-\mathbf{U}_2. \tag{4.2}$$

The dispersion matrix of $\mathbf{U}_{\mathscr{H}}^*$ *is*

$$\mathrm{D}(\mathbf{U}_{\mathscr{H}}^*) = \mathbf{H}(\mathbf{H}'\mathbf{R}^-\mathbf{H})^-\mathbf{H}' - \mathbf{HZH}'.$$

PROOF. It is clear that the problem consists in determining a MDLUE of $\boldsymbol{\theta} = \mathbf{H}\tau$ in a Gauss-Markoff model of the form

$$\left\{\begin{pmatrix}\mathbf{U}_1\\ \mathbf{U}_2\end{pmatrix}, \begin{pmatrix}\mathbf{H}\\ \mathbf{H}\end{pmatrix}\tau, \begin{pmatrix}\boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12}\\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22}\end{pmatrix}\right\}.$$

The proof follows similarly as in the case of Theorem 1 apart from the fact that Lemma 2, with

$$\mathbf{F} = \begin{pmatrix}\mathbf{I} & \mathbf{O}\\ \mathbf{I} & -\mathbf{I}\end{pmatrix} \in \mathscr{M}_{2k,2k}, \tag{4.3}$$

is used to calculate the product of the form $\mathbf{X}'\mathbf{G}^-\mathbf{X}$ involved in (2.1). □

Analogues to Corollaries 1.1 and 1.2 for the problem of combining two unbiased estimators are the following.

COROLLARY 2.1. *If in Theorem 2* $\boldsymbol{\Sigma} \in \mathscr{M}_{2k}^{>}$, *then the best estimator of* $\boldsymbol{\theta} \in \mathscr{H}$ *in the class* $\mathscr{U}_{\mathscr{H}}$ *is*

$$\mathbf{U}_{\mathscr{H}}^0 = \mathbf{H}(\mathbf{H}'\boldsymbol{\Sigma}_0^{-1}\mathbf{H})^-\mathbf{H}'\boldsymbol{\Sigma}_0^{-1}\mathbf{U}^0,$$

where

$$\boldsymbol{\Sigma}_0 = \boldsymbol{\Sigma}_{11} - (\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12})\boldsymbol{\Sigma}_*^{-1}(\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{21})$$

and

$$\mathbf{U}^0 = (\boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{21})\boldsymbol{\Sigma}_*^{-1}\mathbf{U}_1 + (\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12})\boldsymbol{\Sigma}_*^{-1}\mathbf{U}_2, \tag{4.4}$$

with $\boldsymbol{\Sigma}_*$ *as defined in* (4.1). *The dispersion matrix of* $\mathbf{U}_{\mathscr{H}}^0$ *is*

$$\mathrm{D}(\mathbf{U}_{\mathscr{H}}^0) = \mathbf{H}(\mathbf{H}'\boldsymbol{\Sigma}_0^{-1}\mathbf{H})^-\mathbf{H}'.$$

COROLLARY 2.2. *If in Theorem 2 the parameter space equals $\mathscr{R}^k$, then a best estimator of $\boldsymbol{\theta}$ in the class $\mathscr{U}_{\mathscr{R}^k}$ is* $\mathbf{U}^*$ *as defined in* (4.2). *The dispersion matrix of* $\mathbf{U}^*$ *is*

$$D(\mathbf{U}^*) = \boldsymbol{\Sigma}_{11} - (\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12})\boldsymbol{\Sigma}_*^-(\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{21}),$$

with $\boldsymbol{\Sigma}_*$ *as defined in* (4.1).

According to the authors' knowledge, the problem of combining two unbiased estimators of a vector parameter has not hitherto been considered in the literature under such general frameworks as those adopted in Theorem 2 and Corollaries 2.1 and 2.2. A solution of this problem given by Lewis and Odell (1971, page 69) is valid for the case wherein $\mathscr{H} = \mathscr{R}^k$ and $\boldsymbol{\Sigma}_* \in \mathscr{M}_k^>$. It can be noticed that their formula for the best estimator of $\boldsymbol{\theta}$ is identical with the formula (4.4) of the present paper.

To conclude this section note that the nonsingularity of $\boldsymbol{\Sigma}_*$ enabled us to transform the statistic $\mathbf{U}^*$ defined in (4.2) to a symmetric form revealed in (4.4). It seems interesting to ask whether such a form is available also in the case of a singular $\boldsymbol{\Sigma}_*$. But it would be so if and only if

$$\mathbf{I} - (\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12})\boldsymbol{\Sigma}_*^- = (\boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{21})\boldsymbol{\Sigma}_*^-,$$

or, on account of the definition of $\boldsymbol{\Sigma}_*$, if and only if $\boldsymbol{\Sigma}_*\boldsymbol{\Sigma}_*^- = \mathbf{I}$, thus implying the negative answer to the question.

5. Improving upon an unbiased estimator. This section can be viewed as an extension of Section 4 in the sense that the second of the available two statistics is now assumed to be an unbiased estimator of a part of the vector parameter $\boldsymbol{\theta}$. Obviously, there is no loss of generality in specifying this part as $\boldsymbol{\theta}_2$ when $\boldsymbol{\theta}$ is decomposed as $\boldsymbol{\theta} = (\boldsymbol{\theta}_1' \,\vdots\, \boldsymbol{\theta}_2')'$.

THEOREM 3. *Let* $\mathbf{W}_1 \in \mathscr{R}^k$ *and* $\mathbf{W}_2 \in \mathscr{R}^{k_2}$, *where* $k = k_1 + k_2$, *be given statistics such that*

$$E(\mathbf{W}_1) \equiv E\begin{pmatrix}\mathbf{W}_1^{(1)}\\ \mathbf{W}_1^{(2)}\end{pmatrix} = \begin{pmatrix}\boldsymbol{\theta}_1\\ \boldsymbol{\theta}_2\end{pmatrix} \equiv \boldsymbol{\theta} \in \mathscr{H}, \qquad E(\mathbf{W}_2) = \boldsymbol{\theta}_2,$$

and

$$D\begin{pmatrix}\mathbf{W}_1^{(1)}\\ \mathbf{W}_1^{(2)}\\ \mathbf{W}_2\end{pmatrix} = \begin{pmatrix}\mathbf{S}_{11}^{(1,1)} & \mathbf{S}_{11}^{(1,2)} & \mathbf{S}_{12}^{(1,2)}\\ \mathbf{S}_{11}^{(2,1)} & \mathbf{S}_{11}^{(2,2)} & \mathbf{S}_{12}^{(2,2)}\\ \mathbf{S}_{21}^{(2,1)} & \mathbf{S}_{21}^{(2,2)} & \mathbf{S}_{22}^{(2,2)}\end{pmatrix} \equiv \mathbf{S} \in M_{k+k_2}^{\geqq},$$

where $\boldsymbol{\theta}$ is an unknown parameter, the parameter space $\mathscr{H}$ is a given subspace of $\mathscr{R}^k$, and $\mathbf{S}$ *is known apart from a positive scalar multiplier. Moreover, let* $\mathbf{H} = (\mathbf{H}_1' \,\vdots\, \mathbf{H}_2')'$ *be any matrix such that* $\mathscr{C}(\mathbf{H}) = \mathscr{H}$, *and let*

$$\mathbf{P} = \begin{pmatrix}\mathbf{H}_1\\ \mathbf{H}_2\end{pmatrix}\mathbf{Z}(\mathbf{H}_1' \,\vdots\, \mathbf{H}_2') + \begin{pmatrix}\mathbf{S}_{11}^{(1,1)} & \mathbf{S}_{11}^{(1,2)}\\ \mathbf{S}_{11}^{(2,1)} & \mathbf{S}_{11}^{(2,2)}\end{pmatrix} - \begin{pmatrix}\mathbf{S}_{11}^{(1,2)} - \mathbf{S}_{12}^{(1,2)}\\ \mathbf{S}_{11}^{(2,2)} - \mathbf{S}_{12}^{(2,2)}\end{pmatrix}\mathbf{S}_*^-\,(\mathbf{S}_{11}^{(2,1)} - \mathbf{S}_{21}^{(2,1)} \,\vdots\, \mathbf{S}_{11}^{(2,2)} - \mathbf{S}_{21}^{(2,2)}),$$

where $\mathbf{Z}$ *is any nonnegative definite matrix for which* $\mathscr{C}(\mathbf{H}) \subseteq \mathscr{C}(\mathbf{P})$, *and*

$$\mathbf{S}_* = \mathbf{S}_{11}^{(2,2)} - \mathbf{S}_{12}^{(2,2)} - \mathbf{S}_{21}^{(2,2)} + \mathbf{S}_{22}^{(2,2)}.$$

Then a best estimator of $\boldsymbol{\theta}$ in the class

$$\mathscr{W}_{\mathscr{H}} = \{\mathbf{W} = \mathbf{M}_1\mathbf{W}_1 + \mathbf{M}_2\mathbf{W}_2 : \mathbf{M}_1 \in \mathscr{M}_{k,k}, \mathbf{M}_2 \in \mathscr{M}_{k,k_2}, E(\mathbf{W}) = \boldsymbol{\theta}\ \forall\ \boldsymbol{\theta} \in \mathscr{H}\}$$

is expressible in the form

$$\mathbf{W}_{\mathscr{H}}^* = \mathbf{H}(\mathbf{H}'\mathbf{P}^-\mathbf{H})^-\mathbf{H}'\mathbf{P}^-\mathbf{W}^*,$$

with

$$\mathbf{W}^* = \begin{pmatrix}\mathbf{W}_1^{(1)}\\ \mathbf{W}_1^{(2)}\end{pmatrix} - \begin{pmatrix}\mathbf{S}_{11}^{(1,2)} - \mathbf{S}_{12}^{(1,2)}\\ \mathbf{S}_{11}^{(2,2)} - \mathbf{S}_{12}^{(2,2)}\end{pmatrix}\mathbf{S}_*^-(\mathbf{W}_1^{(2)} - \mathbf{W}_2). \tag{5.1}$$

The dispersion matrix of $\mathbf{W}^*_{\mathscr{H}}$ *is*

$$D(\mathbf{W}^*_{\mathscr{H}}) = \mathbf{H}(\mathbf{H}'\mathbf{P}^-\mathbf{H})^-\mathbf{H}' - \mathbf{HZH}'.$$

PROOF. The proof follows similarly as that of Theorem 2, apart only from the fact that

$$\mathbf{F} = \begin{pmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & -\mathbf{I} \end{pmatrix}$$

is used instead of $\mathbf{F}$ specified in (4.3). □

The simplifications of Theorem 3 to the case of $\mathbf{S} \in M^{\geq}_{k+k_2}$ and to the case of $\mathscr{H} = \mathscr{R}^k$ are obtainable similarly as the corollaries in Sections 2 and 3, and thus are not stated explicitly here. On the other hand, it can be observed that an equivalent form of (5.1) is

$$\mathbf{W}^* = \begin{pmatrix} \mathbf{W}_1^{(1)} - (\mathbf{S}_{11}^{(1,2)} - \mathbf{S}_{12}^{(1,2)})\mathbf{S}_*^-(\mathbf{W}_1^{(2)} - \mathbf{W}_2) \\ [\mathbf{I} - (\mathbf{S}_{11}^{(2,2)} - \mathbf{S}_{12}^{(2,2)})\mathbf{S}_*^-]\mathbf{W}_1^{(2)} + (\mathbf{S}_{11}^{(2,2)} - \mathbf{S}_{12}^{(2,2)})\mathbf{S}_*^-\mathbf{W}_2 \end{pmatrix},$$

wherefrom it is seen that the first component of $\mathbf{W}^*$ is actually a covariance adjustment of $\mathbf{W}_1^{(1)}$ by $\mathbf{W}_1^{(2)} - \mathbf{W}_2$, while its second component is a best linear combination of two unbiased estimators of $\boldsymbol{\theta}_2$, $\mathbf{W}_1^{(2)}$ and $\mathbf{W}_2$.

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