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# OPTIMAL STOPPING IN THE STOCK MARKET WHEN THE FUTURE IS DISCOUNTED

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In the random walk stock market model, a stock is purchased at price $x$ and is sold at time $t$ for the price $x + S_t$ where $S_t = \sum_0^t X_i$, $X_i$ is the price change during the $i$th epoch, and $X_1, X_2, \cdots$ are i.i.d. random variables with $\mu = E(X_1) > 0$ and finite $\sigma^2 = E(X_1^2) - \mu^2 > 0$. Discounting the future by a factor of $\gamma$ per epoch, $0 < \gamma < 1$, a selling or stopping policy $t$ has expected payoff or utility $u(t) = E\{\gamma^t(x + S_t)\}$. This article determines second order asymptotic properties of the optimal selling policy $s$, the first passage time of $S_n$ across a straight line boundary $c$, whose utility is equal to the value $V(x) = \sup_t u(t)$ of the stock purchased at price $x$. Specifically, as $\gamma \to 1$, renewal theory is utilized to evaluate the limiting distribution of $s$, $E(s)$, $V(x)$, and the first passage boundary $c$ up to second order terms.

**1. Introduction and summary.** In the random walk stock market model, a stock is purchased at price $x$ and is sold at time $t$ for the price $x + S_t$ where $S_t = \sum_0^t X_i$, $X_i$ is the price change during the $i$th epoch ($i = 1, 2, \cdots$), and $X_1, X_2, \cdots$ are i.i.d. random variables possessing mean $\mu > 0$ and finite variance $\sigma^2 > 0$. See Cootner (1967), Mandelbrot and Taylor (1967), Fama (1965), Samuelson (1965, 1967) and Taylor (1967). An increasing sequence of sigma algebras $\{\mathscr{F}_n\}_{n=0}^\infty$ is adapted to $\{X_n\}_{n=1}^\infty$ if, for each $n > 0$, $\mathscr{F}_n$ contains the past $\mathscr{F}\{X_k : k \le n\}$ and is independent of the future $\mathscr{F}\{X_k : k > n\}$, the smallest sigma algebra generated by $\{X_k : k > n\}$. A selling or stopping time $t$ with respect to such $\{\mathscr{F}_n\}_{n=0}^\infty$ must satisfy $\{t \le n\} \in \mathscr{F}_n$ for $n \ge 0$. Here $\mathscr{F}_0$ is the degenerate sigma algebra so that one buys (i.e. $t > 0$) with probability one or zero.

If, as in Dubins and Teicher (1967), the future is discounted by a factor of $\gamma$ per epoch, $0 < \gamma < 1$, then a selling policy $t$ has expected payoff or utility

$$u(t) = E\{\gamma^t(x + S_t)\}$$

where $\gamma^t(x + S_t) \equiv 0$ on $\{t = \infty\}$. Note that for small time epochs the discount factor $\gamma$ is close to one. A selling policy $s$ is optimal if its utility $u(s)$ is equal to the value

$$V(x) = \sup u(t)$$

of the stock purchased at price $x$ where the supremum is taken over all stopping times $t$. The stopping policy $t_a$ sells as soon as the price increases by at least $a$. That is, define the first passage time

$$t_a = \inf\{n \ge 0 : S_n \ge a\}.$$

Dubins and Teicher (1967) showed that if $c = \inf\{x : x = V(x)\}$ then $s = t_{c-x}$ is optimal and, in fact, stops no later than any other optimal policy. Taylor (1972) constructed upper bounds on the value function which were shown to be asymptotically sharp by Finster (1982) even without the assumption of finite variance. Specifically

$$c \sim \mu/(1 - \gamma) \quad \text{and} \quad V(x) \sim \mu/e(1 - \gamma) \tag{1.1}$$

as $\gamma \to 1$. Taylor (1968) also utilized the Markovian structure and Dynkin's (1963) excessive

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characterization of the value function to evaluate explicitly $V(x)$ and $c$ for the analogous continuous time Brownian motion problem. Van Moerbeke (1976) exhibited the exact solution for continuous time by using the infinitesimal generator to formulate the problem as an easily solved Stephan type free boundary differential equation. Continuous time solutions can also be found explicitly by utilizing the fourier analytic approach exhibited by Darling and Siegert (1953). Among others who have worked on discrete time versions of this problem are Albert (1970), Boyce (1970), Darling et al. (1972), and Griffeath and Snell (1974).

In this note, renewal theory is used to develop second order expansions for $c$ and $V(x)$ in terms of $\gamma$, $\mu$ and $\sigma^2$ as the discount factor $\gamma \to 1$. The selling policy $t_a$ is shown to have utility that differs from the value of the stock by terms that are smaller than $o(1)$ provided

$$a = \mu/(1-\gamma) + o([1-\gamma]^{-1/2}).$$

**2. The optimal rule and the value function.** In order to facilitate examination of the optimal selling policy $s$ and of the value function $V(x)$ as $\lambda = -\log \gamma \to 0$, define the normalized stopping time $\hat{s}$ by

$$\hat{s} = \sqrt{\lambda}(s - \lambda^{-1})$$

and the excess $R_s$ of the selling price $x + S_t$ over the selling boundary $c$ by

$$R_s = x + S_t - c.$$

Note that $\gamma \to 1$ forces $c \to \infty$.

The following facts are well known from renewal theory. See, for example, Woodroofe (1981) and Spitzer (1966). As $c \to \infty$, $\hat{s}$ and $R_s$ are asymptotically independent with limiting distribution functions $G$ and $H$ respectively. That is, as $\lambda \to 0$, $P(\hat{s} \le a,\ R_s \le b) \to G(a)H(b)$. Anscombe's Theorem and (1.1) indicates $G \sim N(0, \sigma^2/\mu^2)$ and renewal theory gives $H'(r) = P(S_\tau > r)/E(S_\tau)$, where $\tau = \inf\{n > 0 : S_n \ge 0\}$, provided the distribution of $X_1$ is nonlattice (Feller, 1971, Page 138). Since the variance of $X_1$ is finite, the first moment of $R_s$ and the first two moments of $\hat{s}$ are uniformly integrable (Woodroofe, 1981, pages 2–8 and 4–11). Hence as $\gamma \to 1$

$$E(\hat{s}^2) \to \sigma^2/\mu^2 \tag{2.1}$$

and

$$E(R_s) \to \rho \tag{2.2}$$

where, after an application of Spitzer's formula (cf. Woodroofe, 1981, page 2–15), $H$ has mean

$$\rho = (\mu^2 + \sigma^2)/2\mu + \textstyle\sum_1^\infty k^{-1}E(S_k^-) \tag{2.3}$$

with $S_k^- = \max(-S_k, 0)$. If, in addition, the distribution of $X_1$ is strongly nonlattice (Woodroofe, 1981, page 2–16) then

$$\rho = \mu^2 + \sigma^2/4\mu + \pi^{-1}\int_0^\infty y^{-2}\{\operatorname{Re}\xi(y) + \log(\mu y)\}\,dy$$

with $\phi(t) = E(e^{itX_1})$ and $\xi(t) = \log\{1 - \phi(t)\}^{-1}$ denotes the principal branch of the complex logarithm. If $X_1$ has a lattice distribution with span $d$ then (2.2) holds with

$$\rho = E\{S_\tau(S_\tau + d)\}/2E(S_\tau)$$

as $c \to \infty$ through multiples of $d$. A direct application of Wald's Lemma and (2.2) then yields

$$E(s) = \mu^{-1}(c - x + \rho) + o(1) \quad \text{as} \quad \gamma \to 1. \tag{2.4}$$

Furthermore

$$(2.5)\qquad V(x) = u(s) = E\{e^{-\lambda s}(x + S_s)\} = E\{e^{-\lambda s}(c + R_s)\} = E(e^{-\lambda s})(c + \rho) + o(1).$$

From the definition of $s$ as a first passage time, the Strong Law of Large Numbers, and (1.1), it follows that $\lambda s \to 1$ a.s. Hence a Taylor's expansion of $e^{-\lambda s}$ about 1 gives

$$(2.6)\qquad E(e^{-\lambda s}) = e^{-1} + e^{-1}E(1 - \lambda s) + \tfrac{1}{2}e^{-1}E\{e^{-y}(\lambda s - 1)^2\}$$

where the random variable $y$ lies between $\lambda s$ and 1. Since $e^{-\lambda s} = \gamma^s < 1$ we have $e^{-y} \le e$ and thus the uniform integrability of $\hat{s}^2$ implies the uniform integrability of $e^{-y}(\lambda s - 1)^2$. Since $\lambda s \to 1$ a.s., $y \to 1$ a.s. and (2.1) yields

$$(2.7)\qquad E\{e^{-y}(\lambda s - 1)^2\} = \lambda\sigma^2/\mu^2 + o(\lambda).$$

Substituting (2.4) and (2.7) into (2.6) and then (2.6) into (2.5) and utilizing (1.1) one obtains

$$(2.8)\qquad eV(x) = c + c(1 - \lambda c/\mu) + x + \sigma^2/(2\mu) + o(1).$$

Since (1.1) implies $c = \mu/\lambda + \xi$ where $\xi = o(\lambda^{-1})$, it follows from (2.8) that

$$(2.9)\qquad eV(x) = \mu/\lambda + x + \sigma^2/(2\mu) - \xi^2\lambda/\mu + o(1).$$

By mimicking the above development when $s$ is replaced with the first passage time $t_a$ where $a = \mu/\lambda + o(\lambda^{-1/2})$, the utility

$$u(t_a) = \mu/(e\lambda) + x/e + \sigma^2/(2\mu e) + o(1)$$

is realized and since $u(t_a) \le V(x)$ for all $a$, it must be the case that $\xi = o(\lambda^{-1/2})$ by (2.9).

Collecting the above results, we have proved the following.

THEOREM. *As* $\gamma \to 1$

(1) $V(x) = \mu/(\lambda e) + x/e + \sigma^2/(2\mu e) + o(1)$
(2) $c = \mu/\lambda + o(\lambda^{-1/2})$
(3) $u(t_a) = V(x) + o(1)$ *if* $a = \mu/\lambda + o(\lambda^{-1/2})$
(4) $E(t_a) = (a + \rho)/\mu + o(1)$. *If, in addition,* $a = \mu/\lambda + o(\lambda^{-1/2})$ *then* $E(t_a) = \lambda^{-1} + o(\lambda^{-1/2}) = E(s) + o(\lambda^{-1/2})$.

Note that the best fixed sample procedure sells after $n_0$ days where $n_0$ maximizes $\gamma^n(x + n\mu)$ and hence has utility $\mu/(e\lambda) + x/e + o(1)$. The gain in using the sequential procedure is therefore $\sigma^2/(2\mu e) + o(1)$.

**3. The normal case.** If time is measured in terms of the number of transactions, then Mandelbrot and Taylor (1967) have indicated that stock market prices should follow a random walk with the increments being normally distributed. Hence, in practice, $\mu$ and $\sigma^2$ are easily estimated from the stock's past prices. In this case (2.3) also assumes the particularly tractable and easily estimable form

$$(3.1)\qquad \rho = (\mu^2 + \sigma^2)/(2\mu) - \nu(\sigma/\mu),$$

where

$$\nu(\ell) = \textstyle\sum_1^\infty k^{-1/2}\{\Phi'(\ell^{-1}\sqrt{k}) - \ell^{-1}\sqrt{k}\Phi(\ell^{-1}\sqrt{k})\}$$

and $\Phi$ denotes the standard normal distribution function. Table 1 indicates some representative values of $\nu(\ell)$; this table overlaps slightly with Table 1 of Siegmund (1975), whose $w$ is our $\ell^{-1}$.

**4. Illustrative exact results for exponential right tailed distributions.** The distribution of $X_i$ has an exponential right tail if $x \ge 0$ implies

$$P(X_i > x) = K \exp(-\beta x)$$

TABLE 1
*Values of $\nu(\ell) \times 10^4$ for use in* (3.1)

| $\ell$ | 0 | .1 | .2 | .3 | .4 | .5 | .6 | .7 | .8 | .9 |
|---|---|---|---|---|---|---|---|---|---|---|
| **0** | 0000 | 0000 | 0000 | 0001 | 0020 | 0090 | 0224 | 0418 | 0662 | 0947 |
| **1** | 1264 | 1607 | 1971 | 2353 | 2749 | 3157 | 3575 | 4002 | 4435 | 4875 |
| **2** | 5320 | 5771 | 6225 | 6683 | 7144 | 7607 | 8074 | 8543 | 9014 | 9487 |
| **3** | 9961 | 10437 | 10914 | 11393 | 11873 | 12354 | 12836 | 13319 | 13803 | 14288 |

for some positive constants $K$ and $\beta$. In this case

$$P(s \geq m, R_s > r) = \textstyle\sum_m^\infty P(s \geq n, S_n > a + r) = \sum_m^\infty E\{P(s \geq n, S_n > a + r \mid \mathscr{F}_{n-1})\}$$
$$= \textstyle\sum_m^\infty \displaystyle\int_{s \geq n} K \exp\{-\beta(a + r - S_{n-1})\}\, dP = p_m \exp(-\beta r).$$

Setting $r = 0$ indicates $P(s \geq m) = p_m$ and setting $m = 0$ shows $P(R_s > r) = \exp(-\beta r)$. Hence $s$ and $R_s$ are independent and $\rho = E(R_s) = 1/\beta$. Now proceed as in (2.5) to obtain

$$V(x) = (E\gamma^s)(c + 1/\beta) = \psi(\gamma, x)(c + 1/\beta) \tag{4.1}$$

where

$$\psi(y, x) = E(y^s) \quad \text{for} \quad 0 \leq y \leq 1.$$

Defining the characteristic function $\phi(t) = E\exp(itX_1)$ and utilizing the Fourier methods described in Feller (1971, page 600) one obtains

$$\psi(y, x) = \{1 - \theta(y)/\beta\} \exp\{(x - c)\theta(y)\}, \tag{4.2}$$

where $-i\theta(y)$ is a solution to $y\phi(y) = 1$. Solving

$$c = V(c) = \psi(\gamma, c)(c + 1/\beta)$$

yields

$$c = 1/\theta(\gamma) - \beta^{-1}$$

and substituting this and (4.2) into (4.1) produces

$$V(x) = \{1/\theta(\gamma) - \beta^{-1}\} \exp\{x\theta(\gamma) + \theta(\gamma)/\beta - 1\}.$$

Since $\theta(1) = 0$ and $\theta'(1) = -1/\mu$ we find

$$E(s) = \frac{\partial}{\partial y}\psi(1, x) = \frac{1}{\theta(\gamma)\mu} - \frac{x}{\mu}.$$

For example, if $P(X_1 > 0) = 1$ then $\theta(y) = (1 - y)\beta$, $c = \gamma/\{(1 - \gamma)\beta\}$, $E(s) = (1 - \gamma)^{-1} - x\beta$, and for $x \leq c$, $V(x) = \beta^{-1}(1 - \gamma)^{-1}\gamma \exp\{x(1 - \gamma)\beta - \gamma\}$. In this case the approximations are quite sharp as the exact results for $c$, $V(x)$ and $E(s)$ differ from their approximations by terms that have order of magnitude $O(1 - \gamma)$.

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