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TESTING WHETHER NEW IS BETTER THAN USED WITH RANDOMLY CENSORED DATA[1]

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A life distribution F, with survival function $\bar{F} \equiv 1 - F$, is *new better than used* (NBU) if $\bar{F}(x+y) \le \bar{F}(x)\bar{F}(y)$ for all $x, y \ge 0$. We propose a test of $H_0: F$ is exponential, versus $H_1: F$ is NBU, but not exponential, based on a randomly censored sample of size n from F. Our test statistic is $J_n^c = \int\int \bar{F}_n(x+y)\, dF_n(x)\, dF_n(y)$, where F_n is the Kaplan-Meier estimator. Under mild regularity on the amount of censoring, the asymptotic normality of J_n^c, suitably normalized, is established. Then using a consistent estimator of the null standard deviation of $n^{1/2}J_n^c$, an asymptotically exact test is obtained. We also study, using tests for the censored and uncensored models, the efficiency loss due to the presence of censoring.

1. Introduction and summary. Consider a life distribution F, i.e. a distribution function (d.f.) such that $F(x) = 0$ for $x < 0$, with corresponding survival function $\bar{F} \equiv 1 - F$. F is said to be *new better than used* (NBU) if

$$\bar{F}(x+y) \le \bar{F}(x)\bar{F}(y) \quad \text{for} \quad x, y \in [0, \infty). \tag{1.1}$$

(Inequality (1.1) may be interpreted as stating that the probability $\bar{F}(x)$ that a new item will survive to age x is greater than the probability that an unfailed (used) unit of age y will survive an additional time x. That is, a new unit has stochastically greater life than a used unit of any age.) The dual notion of a *new worse than used* (NWU) life d.f. is defined by reversing the inequality in (1.1). The boundary members of the NBU and NWU classes, obtained by insisting on equality in (1.1), are the exponential d.f.'s.

The NBU class of life distributions has proved to be very useful in performing analyses of lifelengths. These d.f.'s provide readily interpretable models for describing wearout, play a fundamental role in studies of replacement policies (Marshall and Proschan, 1972), shock models (Esary, Marshall, and Proschan, 1973), multistate coherent systems (El-Neweihi, Proschan and Sethuraman, 1978), and have desirable closure properties (c.f. Barlow and Proschan, 1975).

Hollander and Proschan (1972), hereafter abbreviated HP (1972), developed a test of $H_0: \bar{F}(x) = \exp(-x/\mu), x \ge 0, \mu > 0$ (μ unspecified) versus $H_1: F$ is NBU, but not exponential, based on a random sample $X_1, \cdots, X_n$ from a continuous life d.f. F. Let $D(x, y) = \bar{F}(x)\bar{F}(y) - \bar{F}(x+y)$. The HP (1972) test is motivated by considering the parameter

$$\gamma(F) = \int_0^\infty \int_0^\infty D(x, y)\, dF(x)\, dF(y) = \frac{1}{4} - \int_0^\infty \int_0^\infty \bar{F}(x+y)\, dF(x)\, dF(y) = 1/4 - \Delta(F),$$

say, as a measure of the deviation of F from exponentiality towards NBU [or NWU] alternatives. The HP (1972) test rejects H_0 in favor of H_1 if $\Delta(G_n)$ is too small, where G_n is the empirical d.f. of $X_1, \cdots, X_n$; H_0 is rejected in favor of $H_1': F$ is NWU, but not exponential, if $\Delta(G_n)$ is too large. For further details about the parameter $\Delta(F)$ see HP

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(1972), and for other discussions of the HP (1972) test see Hollander and Wolfe (1973), Cox and Hinkley (1974), and Randles and Wolfe (1979).

In this paper we consider a randomly censored model where we do not get to observe a complete sample of X's. Let $X_1, X_2, \cdots$ be independent identically distributed (i.i.d.) random variables (r.v.'s) having a common continuous life d.f. F. The X's represent lifelengths of identical items. Let $Y_1, Y_2, \cdots$ be i.i.d. r.v.'s having a common continuous d.f. H. The Y's represent the random times to right-censorship. The censoring d.f. H is unknown and is treated as a nuisance parameter. Throughout we assume the X's and Y's are mutually independent and the pairs (X_1, Y_1), (X_2, Y_2), $\cdots$ are defined on a common probability space $(\Omega, \mathscr{B}, P)$. Further, let $I(A)$ denote the indicator function of the set A, and for $i = 1, \cdots, n$, let $Z_i = \min(X_i, Y_i)$, and $\delta_i = I(X_i \leq Y_i)$. Using the incomplete data set $(Z_1, \delta_1), \cdots, (Z_n, \delta_n)$, we wish to test H_0 against H_1. Due to the censoring, the empirical d.f. G_n corresponding to F cannot be computed. Thus, we propose to reject H_0 in favor of H_1 for small values of

$$\Delta(F_n) =: J_n^c = \int_0^\infty \int_0^\infty \bar{F}_n(x + y)\, dF_n(x)\, dF_n(y) \tag{1.2}$$

where F_n is the *Product Limit Estimator* (PLE) of F, introduced by Kaplan and Meier (1958) and defined by

$$\bar{F}_n(x) = 1 - F_n(x) = \prod_{\{i: Z_{(i)} \leq x\}} \{(n - i)/(n - i + 1)\}^{\delta_{(i)}}, \tag{1.3}$$

where $Z_{(1)} < \cdots < Z_{(n)}$ denote the ordered Z's, and $\delta_{(1)}, \cdots, \delta_{(n)}$ are the δ's corresponding to $Z_{(1)}, \cdots, Z_{(n)}$ respectively. In (1.3), we treat $Z_{(n)}$ as a death (whether or not it actually is) so that $\delta_{(n)} = 1$. Furthermore, although our assumptions preclude the possibility of ties, in practice ties will occur. When censored observations are tied with uncensored observations, our convention, when forming the list of the ordered Z's, is to treat uncensored members of the tie as preceding the censored members of the tie. For strong consistency of the PLE see Peterson (1977) and Langberg, Proschan, and Quinzi (1981) and for strong uniform consistency see Földes and Rejtö (1981).

For computational purposes, it is convenient to write J_n^c as

$$J_n^c = \sum_{i=1}^n \bar{F}_n(2Z_{(i)})\{dF_n(Z_{(i)})\}^2 + 2 \sum_{i<j}^n \bar{F}_n(Z_{(i)} + Z_{(j)})\, dF_n(Z_{(i)})\, dF_n(Z_{(j)}),$$

where $dF_n(Z_{(i)}) = \bar{F}_n(Z_{(i-1)}) - \bar{F}_n(Z_{(i)})$.

In Section 2 we establish the asymptotic normality of the sequence $n^{1/2}\{J_n^c - \Delta(F)\}$ under the following assumptions:

(A.1) The supports of F and H are equal to $[0, \infty)$,
(A.2) $\sup\{[\bar{F}(x)]^{1-\varepsilon}[\bar{H}(x)]^{-1}, x \in [0, \infty)\} < \infty$, for some $0 < \varepsilon < 1$.

Condition (A.2) restricts the amount of censoring allowed in the model. To see this in a simple case, consider the proportional hazards model where $\bar{H} = \bar{F}^\beta$ for some $\beta > 0$. Then $P(X_1 \leq Y_1) = (\beta + 1)^{-1}$, and Condition (A.2) implies that $\beta < 1$. Thus, in the proportional hazards model, the J_n^c test is appropriate only when the expected amount of censoring $P(Y_1 < X_1)$ is less than 0.5.

The null asymptotic mean of J_n^c is ¼, independent of the nuisance parameters μ and H. However, the null asymptotic variance of $n^{1/2}J_n^c$ does depend on μ and H and must be estimated from the data. A consistent estimator, $\hat{\sigma}_n^2$, is given by (2.4). The approximate α-level NBU test rejects H_0 in favor of H_1 if $n^{1/2}\{J_n^c - (¼)\}\hat{\sigma}_n^{-1} \leq -z_\alpha$ where z_α is the upper α-percentile of a standard normal distribution. This asymptotically exact test is, under suitable regularity, consistent against all continuous NBU alternatives. Section 3 considers the loss in efficiency due to the presence of censoring. Section 4 contains an application of the J_n^c test to some survival data.

2. Asymptotic normality of the NBU test statistic. In this section we establish the asymptotic normality of the test statistic J_n^c, defined by (1.2). Let $\bar{K}(t) = \bar{F}(t)\bar{H}(t)$, $t \in (-\infty, \infty)$, and let $\{\phi(t), t \in (-\infty, \infty)\}$ be a Gaussian process with mean zero and covariance kernel given by:

$$(2.1) \qquad E\{\phi(t)\phi(s)\} = \begin{cases} \bar{F}(t)\bar{F}(s) \int_0^s \{\bar{K}(z)\bar{F}(z)\}^{-1} \, dF(z), & 0 \le s \le t < \infty \\ 0, & s < 0 \quad \text{or} \quad t < 0. \end{cases}$$

Further let $Z_{(n)} = \max\{Z_i, i = 1, \cdots, n\}$, and let $W_n(t) = n^{1/2}\{\bar{F}_n(\min(t, Z_{(n)})) - \bar{F}(\min(t, Z_{(n)}))\}$, $n = 1, 2, \cdots$. Unless otherwise specified, all limits are evaluated as $n \to \infty$, and all integrals range over $(-\infty, \infty)$. The main result of this section is Theorem 2.1.

THEOREM 2.1. *Assume that Conditions* (A.1) *and* (A.2), *given in Section* 1, *hold. Then* $n^{1/2}\{J_n^c - \Delta(F)\}$ *converges in distribution to a normal* r.v. *with mean zero and variance* σ^2, *given by*:

$$(2.2) \quad \sigma^2 = \int\int\int\int E[\{\phi(t+s) - 2\phi(t-s)\} \cdot \{\phi(u+v) - 2\phi(u-v)\}] \, dF(t) \, dF(s) \, dF(u) \, dF(v).$$

In the proof of Theorem 2.1 we use the result that our conditions (A.1) and (A.2) imply

(A.3) the processes $\{W_n(t), t \in (-\infty, \infty)\}$ converge to a Gaussian process with mean zero and covariance kernel given by (2.1).

Result (A.3) is a particular case of Gill's (1981) Theorem 2.1 with $h(t) \equiv \bar{F}(t)$. Gill's Condition (2.1) for $h(t) = \bar{F}(t)$ follows easily from (A.1) and (A.2).

To prove Theorem 2.1 we will utilize Lemmas 2.2 and 2.3. Let $M_n(x, y) = \bar{F}_n(x + y) - \bar{F}(x + y)$, $R_n(x, y) = \bar{F}_n(x - y) - \bar{F}(x - y)$ (where we have suppressed, in the notation, the dependence on F and F_n) and note that by integration by parts and change of variables.

$$n^{1/2}\{J_n^c - \Delta(F)\} = B_{n,1} + B_{n,2} - B_{n,3} + B_{n,4},$$

where

$$n^{-1/2}B_{n,1} = \int\int M_n(x, y) \, dF_n(x) \, dF_n(y) - \int\int M_n(x, y) \, dF_n(x) \, dF(y),$$

$$n^{-1/2}B_{n,2} = \int\int M_n(x, y) \, dF_n(x) \, dF(y) - \int\int M_n(x, y) \, dF(x) \, dF(y),$$

$$n^{-1/2}B_{n,3} = \int\int R_n(x, y) \, dF_n(x) \, dF(y) - \int\int R_n(x, y) \, dF(x) \, dF(y),$$

$$n^{-1/2}B_{n,4} = \int\int [\bar{F}_n(x + y) - \bar{F}(x + y) - 2\{\bar{F}_n(x - y) - \bar{F}(x - y)\}] \, dF(x) \, dF(y).$$

Let $\tilde{B}_{n,q}$ be defined as $B_{n,q}$ where $n^{1/2}\{\bar{F}_n(\cdot) - \bar{F}(\cdot)\}$ is replaced by $W_n(\cdot)$, $q = 1, \cdots, 4$. By the continuous mapping theorem (Billingsley, 1968, page 30), the paths-continuity of the process $\{\phi(t), t \in (-\infty, \infty)\}$, the convergence w.p.1. of $Z_{(n)}$ to ∞ and some simple integral evaluations, we obtain

$$p \lim |B_{n,q} - \tilde{B}_{n,q}| = 0, \qquad q = 1, \cdots, 4.$$

Consequently, to prove the result of Theorem 2.1 it suffices, by Slutsky's Theorem (Billingsley, 1968, page 49), to show that $\tilde{B}_{n,1}$, $\tilde{B}_{n,2}$, and $\tilde{B}_{n,3}$ converge in probability to zero, and that $\tilde{B}_{n,4}$ converges in distribution to a normal r.v. with mean zero and variance σ^2, given by (2.2).

To establish the convergence of $\tilde{B}_{n,1}$ through $\tilde{B}_{n,4}$ we introduce some notation. Let D be the set of real valued, bounded, and right-continuous functions defined on $(-\infty, \infty)$, with finite left-hand limits at each $t \in (-\infty, \infty)$, and finite limits at $t = \pm\infty$. Throughout we view D as a metric space with the Skorohod metric (Billingsley, 1968, page 112). Further let Q^1, Q^2, Q_n^1, and Q_n^2 be the probability measures on D induced by the processes: $\{\phi(t), t \in (-\infty, \infty)\}$, $\{\phi(x+y) - 2\phi(x-y), x, y \in (-\infty, \infty)\}$, $\{W_n(t), t \in (-\infty, \infty)\}$, and $\{W_n(x+y) - 2W_n(x-y), x, y \in (-\infty, \infty)\}$, respectively. Finally let S_1, S_2 be two sets, let A be a subset of S_2 and let ξ be a mapping from S_1 to S_2; then $\xi^{-1}(A) = \{s : s \in S_1, \xi(s) \in A\}$.

LEMMA 2.2. *Assume* (A.1) *and* (A.2) *hold. Then* (a) $\tilde{B}_{n,1}$, $\tilde{B}_{n,2}$, *and* $\tilde{B}_{n,3}$ *converge in probability to zero,* (b) $\tilde{B}_{n,4}$ *converges in distribution to the* r.v. $\int\int \{\phi(x+y) - 2\phi(x-y)\}\, dF(x)\, dF(y)$.

PROOF. For $\psi \in D$, and $n = 1, 2, \cdots$ let

$$\xi_{n,1}(\psi) = \int\int \psi(x+y)\, dF_n(x)\, dF_n(y) - \int\int \psi(x+y)\, dF_n(x)\, dF(y),$$

$$\xi_{n,2}(\psi) = \int\int \psi(x+y)\, dF_n(x)\, dF(y) - \int\int \psi(x+y)\, dF(x)\, dF(y),$$

$$\xi_{n,3}(\psi) = \int\int \psi(x-y)\, dF_n(x)\, dF(y) - \int\int \psi(x-y)\, dF(x)\, dF(y),$$

and

$$\xi(\psi) = \int\int \{\psi(x+y) - 2\psi(x-y)\}\, dF(x)\, dF(y).$$

The probabilities Q_n^1 converge weakly to Q^1 by (A.3). By (A.3) and some standard arguments it can be shown that Q_n^2 converges weakly to Q^2, and that the supports of Q^1 and Q^2 coincide with the set of all continuous functions in D. By the definitions of the mappings and the probability measures we have:

$$Q_n^1\xi_{n,q}^{-1}\{(-\infty, x]\} = P(\tilde{B}_{n,q} \le x), \qquad x \in (-\infty, \infty), \qquad q = 1, 2, 3, \qquad n = 1, 2, \cdots,$$

$$Q_n^2\xi^{-1}\{(-\infty, x]\} = P(\tilde{B}_{n,4} \le x), \qquad x \in (-\infty, \infty), \qquad n = 1, 2, \cdots,$$

$$Q^2\xi^{-1}\{(-\infty, x]\} = P\left\{\int\int [\phi(u+v) - 2\phi(u-v)]\, dF(u)\, dF(v) \le x\right\}, \qquad x \in (-\infty, \infty).$$

Thus to obtain the desired results it suffices to show, by the Extended Continuous Mapping Theorem (Billingsley, 1968, page 34, Theorem 5.5), that for every sequence $\psi_n \in D$ that converges to a continuous function $\psi \in D$, $\lim \xi_{n,q}(\psi_n) = 0$ w.p.1 for $q = 1, 2, 3$, and $\lim \xi(\psi_n) = \xi(\psi)$.

We now prove the preceding statements. Let $\psi_n \in D$, $n = 1, 2, \cdots$, and let ψ be a continuous function in D. Assume $\lim \psi_n = \psi$. By a well-known result (Billingsley, 1968, page 112)

$$\lim \sup\{|\psi_n(x) - \psi(x)|, \qquad x \in (-\infty, \infty)\} = 0.$$

Clearly $\lim \xi_{n,q}(\psi) = 0$ w.p.1 for $q = 1, 2, 3$. Consequently, by simple integral evaluations we obtain that $\lim \xi_{n,q}(\psi_n) = 0$ w.p.1 for $q = 1, 2, 3$, and that $\lim \xi(\psi_n) = \xi(\psi)$. □

LEMMA 2.3. *Assume that* (A.1) *and* (A.2) *hold. Then* $\tilde{B}_{n,4}$ *converges in distribution to a normal* r.v. *with mean zero and variance* σ^2, *given by* (2.2).

PROOF. By Lemma 2.2(b) it suffices to show that the r.v. $\int\int \{\phi(x+y) - 2\phi(x-y)\}\, dF(x)\, dF(y)$ is normal with mean zero and variance σ^2. Note that the process $\{\phi(x+y) - 2\phi(x-y), x, y \in (-\infty, \infty)\}$ is Gaussian, and that under (A.2), $\sigma^2 < \infty$. Consequently the desired result follows by the theory of stochastic integration; cf. Parzen (1962), page 78.

TABLE 1
Monte Carlo Properties of $\hat{\sigma}_n^2$ and the Normal Approximation for J^*
$[F = \text{Exponential } (1), H = \text{Exponential } (\lambda)]$

n	Ave. $\hat{\sigma}_n^2$	s	$\hat{P}(J^* \le -z_{.10})$	$\hat{P}(J^* \le -z_{.05})$	$\hat{P}(J^* \le -z_{.01})$
			$\lambda = 1/4$		
100	.0139	.0008	.224	.131	.047
150	.0139	.0007	.185	.117	.035
200	.0139	.0006	.181	.102	.029
			$\lambda = 1/3$		
100	.0150	.0013	.202	.119	.029
150	.0149	.0010	.187	.104	.032
200	.0149	.0009	.186	.098	.019

As indicated in Section 1, in order to perform the test, the null asymptotic variance of $n^{1/2}J_n^c$ needs to be estimated from the data. To do this, first consider the function

$$\sigma^2(\theta) = \int_0^1 f(z)\{\bar{K}(-\theta \ln z)\}^{-1}\, dz, \qquad \theta \in (0, \infty), \tag{2.3}$$

where $f(z) = z^3\{1 + 4 \ln z + 4(\ln z)^2\}/16$, $0 < z \le 1$. Straightforward calculations show that σ^2, given by (2.2), reduces when H_0 is true to the expression $\sigma^2(\mu)$. Thus to obtain a consistent estimator $\hat{\sigma}_n^2$, say, of the null asymptotic variance of $n^{1/2}J_n^c$, in (2.3) we replace $\bar{K}(t)$ by $\bar{K}_n(t) = n^{-1}\sum I(Z_i > t)$ and θ by $\hat{\mu}_n = (\sum \delta_i)^{-1}\cdot\sum Z_i$. Under H_0, $p \lim \hat{\mu}_n = \mu$ and thus under H_0, $\hat{\sigma}_n^2(\hat{\mu}_n) = \hat{\sigma}_n^2$, obtained by making the aforementioned substitutions in (2.3), is a reasonable estimator of σ^2. Chen, Hollander and Langberg (1982), abbreviated as CHL (1982), prove that under H_0, $p \lim \hat{\sigma}_n^2 = \sigma^2$, assuming that $\sigma^2(\theta)$ is finite in an interval that contains μ. Furthermore, assume $\sigma^2(\theta)$ is finite in an interval that contains $\eta = \{P(X_1 \le Y_1)\}^{-1}EZ_1$, that $\mu < \infty$, and that (A.1) and (A.2) hold. Under these conditions CHL (1982) prove that the test which rejects H_0 in favor of H_1 if $J^* = n^{1/2}\{J_n^c - (1/4)\}\hat{\sigma}_n^{-1} < -z_\alpha$, is consistent against all continuous NBU alternatives.

For computation purposes $\hat{\sigma}_n^2$ can be written as

$$\begin{aligned} \hat{\sigma}_n^2 = {} & (128)^{-1} + \textstyle\sum_{i=1}^{n-1} n(n-i+1)^{-1}(n-i)^{-1}\{(128)^{-1} - (32)^{-1}Z_{(i)}(\hat{\mu}_n)^{-1} \\ & + (16)^{-1}Z_{(i)}^2(\hat{\mu}_n)^{-2}\}\exp\{-4Z_{(i)}(\hat{\mu}_n)^{-1}\} - n\{(128)^{-1} - (32)^{-1}Z_{(n)}(\hat{\mu}_n)^{-1} \\ & + (16)^{-1}Z_{(n)}^2(\hat{\mu}_n)^{-2}\}\exp\{-4Z_{(n)}(\hat{\mu}_n)^{-1}\}. \end{aligned} \tag{2.4}$$

Table 1 investigates the accuracy of $\hat{\sigma}_n^2$ as an estimator of σ^2 and the accuracy of the normal approximation in the cases where F is exponential with scale parameter 1 and H is exponential with (censoring) scale parameter λ, for the choices $\lambda = 1/4$ and $\lambda = 1/3$. For these choices, $\sigma^2 = .0137$ and $.0146$, respectively. Column 2 of Table 1 gives the average value of $\hat{\sigma}_n^2$, averaged over 1,000 Monte Carlo replications. Column 3 gives the sample standard deviation s of the 1,000 $\hat{\sigma}_n^2$ values. Columns 4, 5, 6 give estimated probabilities of the events $\{J^* \le -z_\alpha\}$, $\alpha = .10, .05, .01$. In these cases, although $\hat{\sigma}_n^2$ does well as an estimator of σ^2, the convergence to asymptotic normality is very slow. The probability α, assigned to the event $\{J^* \le -z_\alpha\}$ by the normal approximation, is less than the corresponding Monte Carlo estimate $\hat{P}\{J^* \le -z_\alpha\}$. Thus the normal approximation to the NBU test tends to give P values that are less than the true P values. This happens because, although J_n^c is an asymptotically unbiased estimator of $\Delta(F)$, for finite n, it underestimates $\Delta(F)$ because $\bar{F}_n$ underestimates $\bar{F}$. The normal approximation can be improved by defining $\bar{F}_n$ via (1.3) and allowing $\delta_{(n)}$ to be 0 if $Z_{(n)}$ is a censored observation. The Table 1 results are for the case where $\delta_{(n)} \equiv 1$, whether $Z_{(n)}$ is censored or not.

TABLE 2
Asymptotic efficiency of J^c relative to J when H is Exponential with Scale Parameter λ

λ:	3/4	1/2	1/3	1/4	1/10
$e_H(J^c, J)$:	.519	.681	.790	.844	.939
p_H:	.571	.667	.750	.800	.909

3. Efficiency loss due to censoring. Recall that the J_n^c test is a generalization of the HP (1972) test for the uncensored model based on the statistic J_n (see equation (1.5) of HP (1972)). In this section we study the efficiency loss due to the presence of censoring by comparing the power of the J_n test based on n observations in the uncensored model with the power of the $J_{n^*}^c$ test based on n^* observations in the randomly censored model. Let F be a parametric family within the NBU class with F_{γ_0} being exponential with scale parameter 1 (for example, one such family is the Weibull $F_\gamma(x) = 1 - \exp(-x^\gamma)$, $\gamma \geq 1$ and $\gamma_0 = 1$) and assume the randomly censored model with $F = F_\gamma$ and with censoring distribution H. Consider the sequence of alternatives $\gamma_n = \gamma_0 + bn^{-1/2}$, with $b > 0$. Let $\beta_n(\gamma_n)$ be the power of the approximate α-level J_n test based on n observations in the uncensored model and let $\beta_{n^*}(\gamma_n)$ denote the power of the approximate α-level test based on J_n^c for n^* observations in the randomly censored model. Consider $n^* = h(n)$ such that $\lim \beta_n(\gamma_n) = \lim \beta_{n^*}(\gamma_n)$, where the limiting value is strictly between 0 and 1, and let $k = \lim(n/n^*)$. The value of $1 - k$ can be viewed as a measure of the efficiency loss due to censoring. k is adopted from Pitman's (cf. Noether, 1955) measure of asymptotic relative efficiency but the interpretation of k must be modified because J_n and J_n^c are not competing tests which are both applicable in the randomly censored model. Roughly speaking, for large n and NBU alternatives close to the null hypothesis of exponentiality, the J_n^c test requires n/k observations from the randomly censored model to do as well as the J_n test applied to n observations from the uncensored model. Since J_n and J_n^c have the same asymptotic means, k reduces to $k = e_H(J^c, J) = (5/432)/\sigma^2(1)$ where (5/432) is the null asymptotic variance of $n^{1/2}J_n$ and $\sigma^2(1)$, given by (2.3), is the null asymptotic variance of $n^{1/2}J_n^c$. Thus note that k depends only on the censoring distribution H, and not on the parametric family F_γ of NBU alternatives. Hence we use the notation $e_H(J^c, J)$, rather than $e_{F,H}(J^c, J)$.

We consider the case where the censoring distribution is exponential, $\bar{H}(x) = 1$ for $x < 0$, $\bar{H}(x) = \exp(-\lambda x)$, $x \geq 0$. For this choice of H, in order for (A.2) to be satisfied we must impose the restriction $\lambda < 1$. Then using (2.3) we find $e_H(J^c, J) = 5(3 - \lambda)^3/\{27(\lambda^2 - 2\lambda + 5)\}$. Note that, as is to be expected, as λ tends to 0 (corresponding to the case of no censoring), $e_H(J^c, J)$ tends to 1. Values of $e_H(J^c, J)$ are given in Table 2. In order to provide a reference point to the amount of censoring, and thereby facilitate the interpretation of $e_H(J^c, J)$, we also include in Table 2 the value of $p_H = P(X_1 < Y_1) = (1 + \lambda)^{-1}$, the probability of obtaining an uncensored observation when X_1 is exponential with scale parameter 1 and Y_1 is independent of X_1 and has the censoring distribution H.

4. An example. Table 2 of Hollander and Proschan (1979), hereafter abbreviated HP (1979), contains an updated version of data given by Koziol and Green (1976). The data correspond to 211 state IV prostate cancer patients treated with estrogen in a Veterans Administration Cooperative Urological Research Group study. At the March, 1977 closing date there were 90 patients who died of prostate cancer, 105 who died of other diseases, and 16 still alive. Those observations corresponding to deaths due to other causes and those corresponding to the 16 survivors are treated as censored observations (withdrawals). As reported by Koziol and Green (1976), there is a basis for suspecting that had the patients not been treated with estrogen, their survival distribution for deaths from cancer of the prostate would be exponential with mean 100 months.

HP (1979) developed a goodness-of-fit procedure for testing, in the randomly censored model, that F is a certain (completely specified) distribution. They applied their test, and competing procedures of Koziol and Green (1976) and Hyde (1977), to the prostate cancer data. The hypothesized F was taken to be exponential with mean 100. Although Hyde's test and the HP test yielded high P values, those tests are directed to special types of alternatives and are unable to detect certain types of alternatives to the hypothesized distribution. The Koziol-Green test is broader in nature, and its two-sided P value was .14 suggesting that a different model might be more appropriate. Furthermore, Csörgő and Horváth (1981) proposed some goodness-of-fit tests which (for certain alternatives) will be more powerful than the tests of HP (1979), Koziol and Green (1976), and Hyde (1977). Csörgő and Horváth obtained a two-sided P value of 0.0405 for the prostate cancer data, strongly indicating a deviation from the postulated exponential, with mean 100, distribution.

Possible alternative models include an exponential distribution with a mean different than 100, or a distribution, such as an NBU distribution, that could represent "wearout". Chen (1981) tests the composite null hypothesis of exponentiality H_0 using a correlation-type goodness-of-fit test and obtains a two-sided P value of 0.02. To test H_0 against the possibility of an NBU alternative, it is reasonable to apply the test based on J_n^c. Applying the J_n^c test to the prostate cancer data, we obtain $J_{211}^c = 0.193$, $\hat{\sigma}_{211}^2 = 0.105$ and $(211)^{1/2}\{J_{211}^c - (1/4)\}\hat{\sigma}_{211}^{-1} = -2.56$ with a corresponding one-sided P value of 0.0052. The test indicates (despite the optimistic bias of the normal approximation) strong evidence of wearout and suggests that an NBU model is preferable to an exponential model.

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