The Annals of Statistics
1983, Vol. 11, No. 1, 240–246

CONSTRUCTION OF OPTIMAL BALANCED INCOMPLETE BLOCK DESIGNS FOR CORRELATED OBSERVATIONS[1]

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Some methods for the construction of equineighbored balanced incomplete block designs introduced by Kiefer and Wynn (1981) are presented. An algorithm for constructing designs with $k = 3$ is developed. Kiefer and Wynn's result for $k = 3$ is difficult to implement in practice. Our algorithm provides a practical solution and makes use of the decomposition of complete graphs into disjoint Hamiltonian cycles. The construction of designs with $k = v - 1$ and $v - 2$ is also completely solved. The neighbor designs proposed for use in serology are useful for the construction of equineighbored balanced incomplete block designs. Several infinite families of equineighbored balanced incomplete block designs are listed.

1. Introduction. Balanced incomplete block designs (BIBD) have been shown to be optimal for the elimination of one-way heterogeneity under homoscedastic and additive models. Kiefer and Wynn (1981) initiated the study of optimal designs under some "nearest neighbor" correlation models. In an experiment for the comparison of v treatments in b blocks of size k with $k < v$, the position of an observation in a block becomes important when the observations are assumed to be correlated. Let y_{jr} be the observation taken at the rth position in the jth block. One covariance structure considered by Kiefer and Wynn (1981) assumes

$$\text{Cov}(y_{jr}, y_{j'r'}) = \begin{cases} \sigma^2 & \text{if } j = j', \quad r = r', \\ \rho\sigma^2, & \text{if } j = j', \quad |r - r'| = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{1.1}$$

Let $g(j, r)$ be the treatment number of the rth observation in the jth block, A_i the set of blocks in which treatment i occurs, and define

$$e_{ii'} = \#\{j : j \in A_i \cap A_{i'}, g(j, 1) = i \quad \text{or} \quad g(j, k) = i\}$$
$$+ \#\{j : j \in A_i \cap A_{i'}, g(j, 1) = i' \quad \text{or} \quad g(j, k) = i'\};$$

i.e., $e_{ii'}$ is the number of blocks in $A_i \cap A_{i'}$ in which i occurs at an end plus the number where i' occurs at an end. Also define

$$N_{ii'} = \#\{j : g(j, r) = i, g(j, s) = i', |r - s| = 1\};$$

i.e., $N_{ii'}$ is the number of times i and i' are adjacent in a block. Then Kiefer and Wynn showed that under the nearest neighbor model (1.1), a BIBD in which all the quantities $e_{ii'} + kN_{ii'}$ $(i \neq i')$ are equal possesses strong optimality properties in the set of BIBD's. This induces the interesting problem of constructing BIBD's with the additional condition. Since the $kN_{ii'}$'s usually are much bigger than the $e_{ii'}$'s, Kiefer and Wynn suggest that it is also useful to look for designs with all $N_{ii'}$ equal. They called a BIBD with all $N_{ii'}$ equal an equineighbored BIBD (EBIBD).

Kiefer and Wynn showed that a necessary condition for the existence of a BIBD with

Received August 1981; revised August 1982.

[1] Research sponsored by National Science Foundation Grant No. MCS-79-09502 and the Committee on Research, University of California, Berkeley.

AMS 1970 *subject classifications*. Primary 62K10; secondary 05B05.

Key words and phrases. Equineighbored balanced incomplete block designs, Hamiltonian cycles neighbor designs.

all $e_{ii'} + kN_{ii'}$ equal is

$$k \mid 4\lambda, \tag{1.2}$$

where $\lambda = bk(k-1)/v(v-1)$. When k is odd, (1.2) is equivalent to

$$k \mid \lambda. \tag{1.3}$$

They also showed that if an EBIBD exists, then

$$k \mid 2\lambda. \tag{1.4}$$

Of course, this condition is stronger than (1.2). Again, (1.4) is equivalent to (1.3) if k is odd.

The present paper is concerned with the construction, for given k and v, of EBIBD's and BIBD's with all $e_{ii'} + kN_{ii'}$ equal, for which b is as small as possible. For example, Kiefer and Wynn showed that for $k = 3$, the smallest BIBD with all $e_{ii'} + kN_{ii'}$ equal has $b = v(v-1)$ when v is even and $b = v(v-1)/2$ when v is odd. Such designs also have all $N_{ii'}$ equal. But their proof is based on the theory of distinct representatives. No simple algorithm for construction is available. In the next section, for $k = 3$, we shall present an explicit construction of BIBD's with all $e_{ii'} + kN_{ii'}$ equal. This provides a practical solution. The construction makes use of the decomposition of complete graphs into Hamiltonian cycles. In Sections 3 and 4, some methods of construction for other values of k are presented. These methods may or may not yield designs with smallest possible b. As an application, the construction of EBIBD's with $k = v - 1$ and $v - 2$ is completely solved. The neighbor designs introduced by Rees (1967) turn out to be useful for the construction of Kiefer-Wynn type designs.

For given v and k, we call an EBIBD with the smallest possible value of b a *minimum* EBIBD.

2. An algorithm for $k = 3$: construction via graph theory. Kiefer and Wynn showed in their Theorem 5.2 that for $k = 3$, a necessary and sufficient condition for the existence of a BIBD with all $e_{ii'} + kN_{ii'}$ equal is

$$k = 3, \quad \lambda = 3m, \quad b = mv(v-1)/2, \quad r = 3m(v-1)/2, \tag{2.1}$$
$$\text{with } m \text{ even or } m \text{ and } v \text{ odd.}$$

Thus, for $k = 3$, the smallest BIBD with all $e_{ii'} + kN_{ii'}$ equal has $b = v(v-1)/2$ when v is odd and $b = v(v-1)$ when v is even. Bigger designs can be obtained by making copies of this smallest one. In this section, we shall present an explicit method of construction which is useful from a practical point of view.

To fix the idea, assume that v is odd. We need to construct a design with $v(v-1)/2$ blocks. Firstly, we pick the BIBD with $k = 2$ and $b = v(v-1)/2$, i.e., all the possible pairs of v treatments. The problem is how to add one treatment to each block so that the resulting design is a BIBD and all the $e_{ii'} + kN_{ii'}$ are equal.

If we are able to partition the $v(v-1)/2$ blocks of size two into v groups $B_1, B_2, \cdots, B_v$ of $(v-1)/2$ blocks each such that for each i, the ith treatment does not appear in B_i and each other treatment appears in B_i exactly once, then inserting treatment i in the middle of each block in B_i will produce a design desired. In fact, since each treatment other than the ith appears exactly once in B_i, in the $(v-1)/2$ blocks of size 3 constructed from B_i, the ith treatment is adjacent to any other treatment exactly once, which makes all the $N_{ii'}$ equal in the new design. This together with the fact that the original $v(v-1)/2$ blocks of size 2 constitute a BIBD imply that the new design is a BIBD with all the $e_{ii'}$'s also equal. For even v, we need to take *two* copies of the BIBD with $k = 2$ and $b = v(v-1)/2$ and partition the $v(v-1)$ blocks into v groups $B_1, B_2, \cdots, B_v$ of $v - 1$ blocks each such that for each i, the ith treatment does not appear in B_i, and each other treatment appears in B_i twice. In what follows, we shall show by direct construction that the above partition always exists. The construction turns out to be closely related to the partition of complete graphs into disjoint cycles. Thus we need some terminology from graph theory.

In a graph with v vertices, a *cycle* is a closed path, i.e., a path with the same starting and ending vertices. A Hamiltonian cycle is a cycle covering all the v vertices, i.e., a cycle with length v. For convenience, the complete graph with v vertices (i.e., a graph in which any two vertices are connected) is denoted by K_v. An edge connecting vertices x and y is denoted by (x, y). For another application of Hamiltonian cycles to design construction see, e.g., Cheng and Wu (1981).

In the rest of this section, we shall split the discussion into two cases, according as v is even or odd.

2(a) *v odd.* As we mentioned earlier, we need to partition the blocks of a BIBD with $k = 2$ and $b = v(v-1)/2$ into v groups $B_1, B_2, \cdots, B_v$ of $(v-1)/2$ blocks each such that for each i, treatment i does not appear in B_i and each other treatment appears exactly once in B_i. Now consider the v treatments as the v vertices in a graph and consider each block of size two as an edge connecting the two treatments in the block. Write $v = 2n + 1$ and label the v treatments by $0, 1, 2, \cdots, 2n$. For each i, $1 \le i \le n$, let C_i be the Hamiltonian cycle $(0, i, i+1, i-1, i+2, \cdots, i+n, 0)$, where all the components except 0 are taken as the positive integers $1, 2, \cdots, 2n$ mod $2n$. Note that C_{i+1} is obtained from C_i by adding 1 mod $2n$ to each component except 0. By a well-known result in graph theory (see e.g., Berge, 1973), $C_1, C_2, \cdots, C_n$ together cover each edge in K_v once.

In each C_i, let $f(i)$ and $g(i)$ be the two middle vertices; then we have $f(i) - g(i) \equiv n \pmod{2n}$. For example, when $n = 3$, $C_1 = (0, 1, 2, 6, 3, 5, 4, 0)$, $f(1) = 6$, $g(1) = 3$. The $2n + 1$ edges in C_i except $(f(i), g(i))$ can be partitioned into two sets C_i' and C_i'' of n edges each such that C_i' covers all the vertices except $f(i)$ and C_i'' covers all the vertices except $g(i)$. For example, when $n = 3$, we can take $C_1' = \{(1, 2), (3, 5), (4, 0)\}$ and $C_1'' = \{(0, 1), (2, 6), (5, 4)\}$. Let $B_{f(i)}$ ($B_{g(i)}$, respectively) consist of the n blocks of size 2 defined by C_i' (C_i'', respectively). Since $f(i) - g(i) \equiv n \pmod{2n}$, $f(i+1) \equiv f(i) + 1$, and $g(i+1) \equiv g(i) + 1$, we have $\{f(1), g(1), \cdots, f(n), g(n)\} = \{1, 2, \cdots, 2n\}$. Therefore for each i, $1 \le i \le 2n$, we have already constructed a collection B_i of $n = (v-1)/2$ blocks of size 2 such that treatment i does not appear in B_i and each other treatment appears exactly once. Finally, let B_0 consist of the n blocks $\{(f(1), g(1)), \cdots, (f(n), g(n))\}$. Then treatment 0 does not appear in B_0 and each other treatment appears once. By the way C_i' and C_i'' are constructed, $B_0 \cup B_1 \cup \cdots \cup B_n$ is exactly the BIBD with $k = 2$ and $b = v(v-1)/2$. Now by the observation in the beginning of this section, inserting treatment i in the middle of each block in B_i produces a BIBD with all $e_{ii'} + kN_{ii'}$ equal. This design actually has all $N_{ii'}$ equal and all $e_{ii'}$ equal.

EXAMPLE 1. For $v = 7$, $n = 3$. We have $C_1 = (0, 1, 2, 6, 3, 5, 4, 0)$, $C_2 = (0, 2, 3, 1, 4, 6, 5, 0)$, and $C_3 = (0, 3, 4, 2, 5, 1, 6, 0)$. So $B_1 = \{(2, 3), (4, 6), (5, 0)\}$, $B_2 = \{(3, 4), (5, 1), (6, 0)\}$, $B_3 = \{(0, 1), (2, 6), (5, 4)\}$, $B_4 = \{(3, 1), (6, 5), (0, 2)\}$, $B_5 = \{(4, 2), (1, 6), (0, 3)\}$, $B_6 = \{(1, 2), (3, 5), (4, 0)\}$, $B_0 = \{(1, 4), (2, 5), (3, 6)\}$. Thus for $k = 3$, $v = 7$, a minimum BIBD with all $e_{ii'} + kN_{ii'}$ equal is (213), (416), (510), (324), (521), (620), (031), (236), (534), (341), (645), (042), (452), (156), (053), (162), (365), (460), (104), (205), (306).

2(b) *v even.* The case that v is even is a little bit more complicated. It suffices to partition two copies of K_v into v cycles $A_1, A_2, \cdots, A_v$ of length $v - 1$ such that A_i does not pass through the ith vertex. Then each A_i defines $v - 1$ blocks of size two which can be taken as B_i. In other words, we only have to construct, for each i, a cycle A_i of length $v - 1$ which does not pass through the ith vertex such that $A_1, A_2, \cdots, A_v$ together cover each edge in K_v twice.

The detailed procedure goes as follows:

(1) Write $v = 2n + 2$, and label the v treatments by $0, 1, 2, \cdots, 2n$, and ∞. Consider the v treatments as the v vertices in a graph. For each i, $1 \le i \le n$, let C_i be the Hamiltonian cycle $(0, i, i+1, i-1, i+2, \cdots, \infty, \cdots, i+n, 0)$, where ∞ is put in the middle and all the components except 0 and ∞ are taken as the positive integers $1, 2, \cdots, 2n$ mod $2n$.

(2) For each i, $1 \le i \le n$, let A_i be the cycle of length $v - 1$ obtained by deleting the two edges $(0, i)$ and $(i, i + 1)$ from and adding the edge $(0, i + 1)$ to C_i, i.e., $A_i = (C_i \setminus \{(0, i), (i, i + 1)\}) \cup \{(0, i + 1)\}$. Then A_i does not pass through vertex i.

(3) In each A_i, $1 \le i \le n$, make the following transformation of the vertices:

$$n + i \to i \quad \text{for} \quad 1 \le i \le n, \quad i \to n + i + 1 \quad \text{for} \quad 0 \le i \le n - 1, \quad n \to \infty, \quad \infty \to 0. \tag{2.2}$$

Then A_i is transformed to a cycle of length $v - 1$ which does not pass through vertex $n + 1 + i$, $1 \le i \le n$, where $2n + 1$ is interpreted as ∞. Denote this cycle by A_{n+1+i}. Then n more cycles $A_{n+2}, A_{n+3}, \cdots, A_{2n}, A_\infty$ of length $v - 1$ are constructed.

(4) Let A_0 be the cycle $(1, 2, 3, \cdots, 2n, \infty, 1)$ and A_{n+1} be the cycle $(\infty, 0, 1, n + 2, 2, n + 3, 3, \cdots, 2n, n, \infty)$. Then A_0 and A_{n+1} have length $v - 1$ and do not pass through vertices 0 and $n + 1$, respectively. Now we have constructed $v = 2n + 2$ cycles $A_0, A_1, \cdots, A_{2n}, A_\infty$ of length $v - 1$. Using the well-known fact (see Berge, 1973) that $C_1, C_2, \cdots$, and C_n are mutually disjoint and together cover all the edges in K_v except $(0, \infty)$, $(1, n + 1)$, $(2, n + 2)$, $\cdots$, and $(n, 2n)$, one can easily show that $A_0, A_1, A_2, \ldots, A_{2n}, A_\infty$ together cover each edge in K_v twice.

(5) For each i, $i = 0, 1, \cdots, 2n, \infty$, let B_i be the $v - 1$ blocks of size two obtained from A_i by considering each edge as a block.

(6) Put treatment i in the middle of each block in B_i. Then the resulting $v(v - 1)$ blocks of size three constitute a BIBD with all $e_{ii'} + kN_{ii'}$ equal. In fact, for this design, all the $N_{ii'}$ are also equal.

EXAMPLE 2. For $v = 8$, we have $C_1 = (0, 1, 2, 6, \infty, 3, 5, 4, 0)$, $C_2 = (0, 2, 3, 1, \infty, 4, 6, 5, 0)$, and $C_3 = (0, 3, 4, 2, \infty, 5, 1, 6, 0)$. Then the eight cycles of length seven are $A_1 = (026\infty3540)$, $A_2 = (031\infty4650)$, $A_3 = (042\infty5160)$, $A_4 = (\infty015263\infty)$, $A_5 = (4630\infty214)$, $A_6 = (4\infty501324)$, $A_\infty = (41602534)$, $A_0 = (123456\infty1)$. Note that A_1, A_2, and A_3 are obtained from C_1, C_2, and C_3 by deleting vertices 1, 2, and 3, respectively, and A_5, A_6, and A_∞ are obtained from A_1, A_2, and A_3 by making the transformation $4 \to 1$, $5 \to 2$, $6 \to 3$, $0 \to 4$, $1 \to 5$, $2 \to 6$, $3 \to \infty$, $\infty \to 0$. Thus, for $v = 8$, $k = 3$, a minimum BIBD with all $e_{ii'} + kN_{ii'}$ equal is (012), (216), (61∞), $(\infty13)$, (315), (514), (410), (023), (321), (12∞), $(\infty24)$, (426), (625), (520), (034), (432), (23∞), $(\infty35)$, (531), (136), (630), $(\infty40)$, (041), (145), (542), (246), (643), (34∞), (456), (653), (350), (05∞), $(\infty52)$, (251), (154), (46∞), $(\infty65)$, (560), (061), (163), (362), (264), $(4\infty1)$, $(1\infty6)$, $(6\infty0)$, $(0\infty2)$, $(2\infty5)$, $(5\infty3)$, $(3\infty4)$, (102), (203), (304), (405), (506), (60∞), $(\infty01)$.

3. Construction via neighbor designs. *Neighbor designs* were introduced by Rees (1967) for use in serology. These designs are very similar to those considered by Kiefer and Wynn. The major difference is that in neighbor designs, the treatments are arranged in circular blocks. A neighbor design is a collection of circular blocks in which any pair of treatments appear as neighbors equally often. In the *NN* structure considered by Kiefer and Wynn, the two ends in a block are not considered as neighbors. So, there is one extra pair of neighbors per block in neighbor designs. Despite the difference, neighbor designs are useful for the construction of Kiefer-Wynn type designs. We have the following:

THEOREM 3.1. *If there exists a* BIBD *d with parameters v, b, r, k, λ which is also a neighbor design, then there is an* EBIBD *d^* with parameters $v^* = v$, $b^* = kb$, $r^* = (k - 1)r$, $k^* = k - 1$, $\lambda^* = (k - 2)\lambda$.*

PROOF. By construction. From each circular block of d, say $(x_1 x_2 \cdots x_k)$ with x_1 and x_k also considered as neighbors, we now construct k linear blocks of size $k - 1$ by cyclically permuting $x_1, x_2, \cdots, x_k$: $(x_1, x_2, \cdots, x_{k-1})$, $(x_2, x_3, \cdots, x_k)$, $\cdots$, $(x_k, x_1, \cdots, x_{k-2})$. The resulting design is an EBIBD. □

Note that the design d^* constructed in Theorem 3.1 have all $N_{ii'}$ equal, but may not have all $e_{ii'}$ equal. With proper choice of d, d^* can be a minimum EBIBD. For example, consider Sprott's (1954) Series B. This is a series of BIBD's with parameters $v = 2mk + 1$, $b = mv$, $r = mk$, k, $\lambda = (k-1)/2$, where v is a prime power. These designs are obtained by adding $1, 2, \cdots, v-1 \pmod v$ to the m initial blocks $(x^i, x^{i+2m}, \cdots, x^{i+4\lambda m})$, $i = 0, 1, \cdots, m-1$, where x is a primitive element of the Galois field $GF(v)$. It was shown by Lawless (1971) that these BIBD's are neighbor designs. Thus, by our Theorem 3.1, there is an EBIBD with parameters $v^* = 2mk + 1$, $b^* = mkv$, $r^* = mk(k-1)$, $k^* = k-1$, $\lambda^* = (k-1)(k-2)/2$ provided that v is a prime power. For $m = 1$ and $v \equiv 3 \pmod 4$, this produces an EBIBD with v, $b = v(v-1)/2$, $k = (v-3)/2$, $r = (v-1)(v-3)/4$, and $\lambda = (v-3)(v-5)/8$. We claim that this is a minimum EBIBD. If there exists an EBIBD with parameters v, $\bar{b}$, $\bar{r}$, $k = (v-3)/2$, and $\bar{\lambda}$, then by (1.4), $k \mid 2\bar{\lambda}$. On the other hand, $(v-1)\bar{\lambda} = (k-1)\bar{r} \Rightarrow (k-1) \mid 2(k+1)\bar{\lambda} \Rightarrow (k-1) \mid 2\bar{\lambda}$ since $k-1$ is odd. Therefore $k(k-1) \mid 2\bar{\lambda}$ and hence the smallest possible value of $\bar{\lambda}$ is $k(k-1)/2$, which is $(v-3)(v-5)/8$. In summary, we have the following theorem.

THEOREM 3.2. *If $v \equiv 3 \pmod 4$ is a prime power and $k = (v-3)/2$, then there is an* EBIBD *with b blocks of size k if and only if $v(v-1) \mid 2b$.*

For an odd integer $v = 2n + 1$, let $C_1, C_2, \cdots, C_n$ be the n Hamiltonian cycles constructed in Section 2(a). Each Hamiltonian cycle can be considered as a circular block of size v. Using the same scheme as in Theorem 3.1, from each C_i, we can construct v linear blocks of size $v-1$ by cyclic permutation. This yields an EBIBD with $b = v(v-1)/2$ and $k = v-1$. By (1.4) the following can easily be proved:

THEOREM 3.3. *If v is odd, then there exists an* EBIBD *with b blocks of size $v-1$ if and only if $v(v-1) \mid 2b$.*

4. Construction via an ordinary BIBD. Kiefer and Wynn introduced a method for making a single block equineighbored. Given any positive integer k, the $k \times k$ square whose (j, ℓ) cell $(1 \le j, \ell \le k)$ contains the treatment

$$\sum_{r=1}^{j} (-1)^r (r-1) + \sum_{r=1}^{\ell} (-1)^r (r-1) \tag{4.1}$$

reduced (mod k) is such that the k rows for k odd, or the first $k/2$ rows for k even provide an equineighbored complete block design with k treatments. Kiefer and Wynn (1981, page 754) only applied the method, however, to the initial blocks of the difference set method. Applying the method to *every* block of a BIBD, we obtain the following.

THEOREM 4.1. *If there exists a* BIBD *with parameters v, b, r, k, λ, and k is odd (even, respectively), then there is a* BIBD *with parameters v, kb, kr, k, $k\lambda$, (v, $bk/2$, $kr/2$, k, $k\lambda/2$, respectively) in which all $e_{ii'}$ are equal and all $N_{ii'}$ are equal.*

This is a simple device which can be used to expand an ordinary BIBD to an EBIBD. If the initial BIBD has $\lambda = 1$, then the resulting design is a minimum EBIBD. For odd k, it even achieves the minimum possible value of b for the equality of all $e_{ii'} + kN_{ii'}$. Finite planes provide examples of BIBD's with $\lambda = 1$. The following two corollaries are obtained by applying this device to finite Euclidean and projective planes:

COROLLARY 4.2. *Let $v = s^2$ and $k = s$, where s is an odd prime power. Then there exists a* BIBD *with b blocks such that all the $e_{ii'} + kN_{ii'}$ are equal if and only if $s^2(s+1) \mid b$.*

COROLLARY 4.3. *Let $v = s^2 + s + 1$ and $k = s + 1$, where s is an odd prime power. Then there exists an EBIBD with b blocks if and only if $(s+1)(s^2+s+1) \mid 2b$.*

One can easily write down similar results for v being a power of 2. Applying Theorem 4.1 to Sprott's series with $m = 1$, we obtain the following result.

COROLLARY 4.4. *If $v \equiv 3 \pmod 4$ is a prime power and $k = (v-1)/2$, then there is a* BIBD *with b blocks such that all $e_{ii'} + kN_{ii'}$ are equal if and only if $v(v-1) \mid 2b$. Such a design can be constructed to have all $N_{ii'}$ equal and all $e_{ii'}$ equal.*

Note that Kiefer and Wynn (1981) have a similar result for $k = (v+1)/2$, but for v a prime only (see their Theorem 5.4).

As a final application, when v is even, consider the symmetric BIBD with $b = v$, and $k = v - 1$. By Theorem 4.1, there is a BIBD with parameters v, $b = v(v-1)$, and $k = v - 1$, which has all $N_{ii'}$ equal and all $e_{ii'}$ equal. This design is a minimum EBIBD and also achieves the minimum possible value of b for the equality of all $e_{ii'} + kN_{ii'}$. We state this as

COROLLARY 4.5. *If v is even, then there is an* EBIBD *(or a* BIBD *with all $e_{ii'} + kN_{ii'}$ equal) with b blocks of size $k = v - 1$ if and only if $v(v-1) \mid b$. Such a design can be constructed to have all $N_{ii'}$ equal and all $e_{ii'}$ equal.*

Note that Corollary 4.5 and Theorem 3.3 together solve the construction of EBIBD's with $k = v - 1$.

5. Construction of designs with $k = v - 2$. We shall conclude the paper with a complete solution of the EBIBD's with $k = v - 2$:

THEOREM 5.1. *There exists an* EBIBD *with b blocks of size $v - 2$ if and only if $v(v-1) \mid 2b$.*

PROOF. By (1.4), it is easy to see that if there is an EBIBD with b blocks of size $v - 2$, then $v(v-1) \mid 2b$. We shall show that an EBIBD with $v(v-1)/2$ blocks of size $v - 2$ always exists.

For odd v, we know that K_v can be decomposed into $(v-1)/2$ disjoint Hamiltonian cycles. Each Hamiltonian cycle can be considered as a circular block of size v. From each of these Hamiltonian cycles, we can construct v linear blocks of size $v - 2$ by the method of cyclic permutation similar to that employed in Theorem 3.1. The resulting design is obviously an EBIBD with $v(v-1)/2$ blocks of size $v - 2$.

Designs of even v can be constructed from array (4.1). Consider the $v \times v$ square $\mathbf{A}$ whose (j, ℓ)th cell $(1 \le j, \ell \le v)$ is given by $a_{j\ell} = \sum_{r=1}^{j} (-1)^r (r-1) + \sum_{r=1}^{\ell} (-1)^r (r-1)$ reduced $(\bmod v)$. Each of the v rows of $\mathbf{A}$ contains $v - 1$ pairs of adjacent treatments. A block of size $v - 2$ can be obtained by deleting any of such pairs while keeping the order of the other treatments. For example, from the jth row $(a_{j1}, a_{j2}, \cdots, a_{jv})$, $v - 1$ blocks $B_{j1} \equiv (a_{j3}, a_{j4}, \cdots, a_{jv})$, $B_{j2} \equiv (a_{j1}, a_{j4}, \cdots, a_{jv})$, $\cdots$, and $B_{j,v-1} \equiv (a_{j1}, a_{j2}, \cdots, a_{j,v-2})$ of size $v - 2$ can be constructed. We claim that the $v(v-1)/2$ blocks of size $v - 2$ thus constructed from the first $v/2$ rows of $\mathbf{A}$ constitute an equineighbored BIBD.

As indicated earlier, the first $v/2$ rows of $\mathbf{A}$ define an equineighbored complete block design with v treatments. It follows immediately that the $v(v-1)/2$ blocks $\{B_{j\ell}\}_{1 \le j \le v/2, 1 \le \ell \le v-1}$ constitute a BIBD. It remains to show that this BIBD is equineighbored. Clearly, array $\mathbf{A}$ is symmetric with respect to the two diagonals, i.e.,

$$a_{j\ell} = a_{\ell j} = a_{v+1-j, v+1-\ell} = a_{v+1-\ell, v+1-j}, \ \forall\, j, \ell.$$

Thus it suffices to show that in the $v(v-1)$ blocks $\{B_{j\ell}\}_{1 \le j \le v, 1 \le \ell \le v-1}$ constructed from all the v rows of $\mathbf{A}$, each pair of treatments appear as neighbors equally often.

Originally, the jth row of $\mathbf{A}$ has $v - 1$ pairs of adjacent treatments (a_{j1}, a_{j2}), (a_{j2}, a_{j3}),

$\cdots$, $(a_{j,v-1}, a_{jv})$. Each $B_{j\ell}$ only contains $v-3$ adjacent pairs. The two pairs (a_{j1}, a_{j2}) and (a_{j2}, a_{j3}) (respectively, $(a_{j,v-2}, a_{j,v-1})$ and $(a_{j,v-1}, a_{jv})$) are missing from B_{j1} (respectively, $B_{j,v-1}$), while in each $B_{j\ell}$ with $\ell \neq 1$, $v-1$, the three pairs $(a_{j,\ell-1}, a_{j,\ell})$, $(a_{j,\ell}, a_{j,\ell+1})$, and $(a_{j,\ell+1}, a_{j,\ell+2})$ are missing and there is one extra pair $(a_{j,\ell-1}, a_{j,\ell+2})$. So it is enough to show that $\{(a_{j1}, a_{j2})\}_{1\le j\le v}$, $\{(a_{j,v-1}, a_{jv})\}_{1\le j\le v}$, and $\{(a_{j\ell}, a_{j,\ell+3})\}_{1\le j\le v, 1\le \ell\le v-3}$ together cover each of the $v(v-1)$ pairs $\{(i,j)\}_{1\le i\neq j\le v}$ exactly once. This follows immediately from the fact that for each j,

$$\{a_{j2}-a_{j1}, a_{jv}-a_{j,v-1}, a_{j4}-a_{j1}, a_{j5}-a_{j2}, \cdots, a_{jv}-a_{j,v-3}\} = \{1, 2, \cdots, v-1\} \pmod v$$

and that each of the v treatments appears exactly once in each column of $\mathbf{A}$. □

EXAMPLE 3. For $v = 6$, array $\mathbf{A}$ is

$$\begin{matrix} 6 & 1 & 5 & 2 & 4 & 3 \\ 1 & 2 & 6 & 3 & 5 & 4 \\ 5 & 6 & 4 & 1 & 3 & 2 \\ 2 & 3 & 1 & 4 & 6 & 5 \\ 4 & 5 & 3 & 6 & 2 & 1 \\ 3 & 4 & 2 & 5 & 1 & 6. \end{matrix}$$

Therefore a minimum EBIBD with $v = 6$ and $k = 4$ is (5243), (6243), (6143), (6153), (6152), (6354), (1354), (1254), (1264), (1263), (4132), (5132), (5632), (5642), (5641).

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