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TESTING FOR NONSTATIONARY PARAMETER SPECIFICATIONS IN SEASONAL TIME SERIES MODELS[1]

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Let Y_t be an autoregressive process satisfying $Y_t = \alpha_1 Y_{t-1} + \alpha_2 Y_{t-d} + \alpha_3 Y_{t-d-1} + e_t$, where $\{e_t\}_{t=0}^{\infty}$ is a sequence of iid$(0, \sigma^2)$ random variables and $d \geq 2$. Such processes have been used as parametric models for seasonal time series. Typical values of d are 2, 4, and 12 corresponding to time series observed semi-annually, quarterly, and monthly, respectively. If $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = -1$ then $\Delta_1 \Delta_d Y_t = e_t$, where $\Delta_r Y_t$ denotes $Y_t - Y_{t-r}$. If $(\alpha_1, \alpha_2, \alpha_3) = (1, 1, -1)$ the process is nonstationary and the theory for stationary autoregressive processes does not apply. A methodology for testing the hypothesis $(\alpha_1, \alpha_2, \alpha_3) = (1, 1, -1)$ is presented and percentiles for test statistics are obtained. Extensions are presented for multiplicative processes, for higher order processes, and for processes containing deterministic trend and seasonal components.

1. Introduction. Let the time series $\{Y_t\}$ satisfy the seasonal autoregressive model

$$Y_t = \alpha_1 Y_{t-1} + \alpha_2 Y_{t-d} + \alpha_3 Y_{t-d-1} + e_t, \qquad t = 1, 2, \cdots, \tag{1.1}$$

where the e_t are independent identically distributed random variables with mean zero and variance σ^2, abbreviated iid$(0, \sigma^2)$. It is assumed that $d \geq 2$ and that the initial conditions $(Y_{-d}, Y_{-d+1}, \cdots, Y_0)$ are fixed constants. It is further assumed that $\boldsymbol{\alpha}' = (\alpha_1, \alpha_2, \alpha_3)$ and σ^2 are unknown and that n observations $(Y_1, Y_2, \cdots, Y_n)$ from a realization of $\{Y_t\}$ are available.

If the roots of the associated characteristic equation

$$m^{d+1} - \alpha_1 m^d - \alpha_2 m - \alpha_3 = 0 \tag{1.2}$$

are less than one in modulus then Y_t is, except for the transient effects of the initial conditions, a stationary autoregressive process of order $d + 1$. In the stationary situation the asymptotic theory applicable to the least squares (maximum likelihood when the e_t are normally distributed) estimator of $\boldsymbol{\alpha}$ is well-known. See, for example, Fuller (1976) or Box and Jenkins (1976). However this large sample distributional theory is not applicable when the stationarity conditions are violated.

Several authors including Box and Jenkins (1976) have used nonstationary models for seasonal time series. Typical values of d in such applications are $d = 2$, $d = 4$, or $d = 12$ corresponding to time series observed semi-annually, quarterly, or monthly, respectively.

Under the hypothesis $H_0: \boldsymbol{\alpha}' = (1, 1, -1)$, we have $\Delta_1 \Delta_d Y_t = e_t$, where $\Delta_r Y_t$ denotes $Y_t - Y_{t-r}$. In this situation Y_t is nonstationary and has $d + 1$ characteristic roots with unit modulus. Authors who have previously considered least squares estimation and hypothesis testing for nonstationary autoregressive processes include White (1958), Anderson (1959), Rao (1961, 1978), Stigum (1974), Dickey and Fuller (1979a), Dickey and Fuller (1979b), Hasza and Fuller (1979), Evans and Savin (1981), and Hinkley (1982).

We shall present a testing methodology for the hypothesis $H_0: \boldsymbol{\alpha}' = (1, 1, -1)$. The

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asymptotic distribution of the test statistic is characterized and percentiles are presented. The test studied is the likelihood ratio test when the e_t are normally distributed.

It is noted that model (1.1) may be unduly restrictive and extensions are presented for higher order processes in which $\Delta_1\Delta_d Y_t$ is a stationary time series. Extensions are also presented for a multiplicative autoregressive process in which Y_t satisfies

$$Y_t - \phi_1 Y_{t-1} = \phi_2(Y_{t-d} - \phi_1 Y_{t-d-1}) + e_t. \tag{1.3}$$

In terms of model (1.1) the alternatives to H_0 are assumed to satisfy $\alpha_3 = -\alpha_1\alpha_2$ under model (1.3).

An alternative model to (1.1) with $\boldsymbol{\alpha}' = (1, 1, -1)$ is

$$Y_t = \xi_0 t + \textstyle\sum_{j=1}^{d} \xi_j \delta_{jt} + \alpha_1 Y_{t-1} + \alpha_2 Y_{t-d} + \alpha_3 Y_{t-d-1} + e_t \tag{1.4}$$

where the roots of (1.2) are less than one modulus and the δ_{jt}'s are zero-one dummy variables with $\delta_{jt} = 1$ if $t \bmod(d) = j$. Testing of the hypothesis $H_1: \{\boldsymbol{\alpha}' = (1, 1, -1), \xi_j = 0$ for $j = 1, 2, \cdots, d\}$ is considered.

In Section 2 notation is established and preliminary order in probability results are presented for model (1.1). In Section 3, asymptotic distributions for the least squares estimator of $\boldsymbol{\alpha}$ and the associated test statistics are established under H_0. Extensions to higher order processes and to model (1.4) are presented in Section 4. Tables of percentiles, which permit applications of tests presented earlier, are given in Section 5.

2. Order results. Consider the autoregressive process

$$Y_t = \alpha_1 Y_{t-1} + \alpha_2 Y_{t-d} + \alpha_3 Y_{t-d-1} + e_t, \qquad t = 1, 2, \cdots, \tag{2.1}$$

where the e_t are iid$(0, \sigma^2)$ and $d \geq 2$. The asymptotic results will not depend on the initial conditions and we assume, for convenience, that $Y_{-d} = Y_{-d+1} = \cdots = Y_0 = 0$. It will be convenient to reparametrize model (2.1) as

$$Y_t = \beta_1 Y_{t-1} + \beta_2(Y_{t-1} - Y_{t-d-1}) + \beta_3(Y_{t-d} - Y_{t-d-1}) + e_t \tag{2.2}$$

where $\boldsymbol{\beta}' = (\beta_1, \beta_2, \beta_3) = (\alpha_1 + \alpha_2 + \alpha_3, -\alpha_2 - \alpha_3, \alpha_2)$. The condition that $\Delta_1\Delta_d Y_t = e_t$ is equivalent to $\boldsymbol{\beta}' = (1, 0, 1)$. Define $Z_t = \Delta_1 Y_t = Y_t - Y_{t-1}$ and $W_t = \Delta_d Y_t = Y_t - Y_{t-d}$. The least squares estimator $\boldsymbol{\beta}$ is given by

$$\hat{\boldsymbol{\beta}} = \left(\textstyle\sum_{t=1}^{n} \psi_t\psi_t'\right)^{-1} \textstyle\sum_{t=1}^{n} \psi_t Y_t \tag{2.3}$$

where $\psi_t = (Y_{t-1}, W_{t-1}, Z_{t-d})'$.

Letting $[\cdot]$ denote the greatest integer function we have, under the null hypothesis,

$$\begin{aligned} Z_t &= Z_{t-d} + e_t = \textstyle\sum_{j=0}^{[t/d]} e_{t-dj}, \\ W_t &= W_{t-1} + e_t = \textstyle\sum_{j=1}^{t} e_j = \textstyle\sum_{j=0}^{d-1} Z_{t-j}. \end{aligned} \tag{2.4}$$

Under H_0: $\boldsymbol{\beta}' = (1, 0, 1)$ we have $(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) = \mathbf{H}_n^{-1}\mathbf{h}_n$ where $\mathbf{H}_n = \sum_{t=1}^{n} \psi_t\psi_t'$ and $\mathbf{h}_n = \sum_{t=1}^{n} \psi_t e_t$. The test statistic we shall consider for testing H_0 is analogous to the usual likelihood ratio F test statistic in a fixed normal regression model. Define the test statistic

$$\Phi_{n-3}^{(3)} = (3\hat{\sigma}^2)^{-1}\mathbf{h}_n'\mathbf{H}_n^{-1}\mathbf{h}_n, \tag{2.5}$$

where

$$\hat{\sigma}^2 = (n-3)^{-1} \textstyle\sum_{t=1}^{n} \{e_t - (\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})'\psi_t\}^2.$$

$\Phi_{n-3}^{(3)}$ is the test statistic that would be the output from most standard regression programs.

In the following we shall assume, for convenience, that $n = md$ where m is a positive integer. Define

$$\begin{aligned} \mathbf{e}_i &= (e_i, e_{i+d}, \cdots, e_{i+(m-1)d})', \qquad i = 1, 2, \cdots, d, \\ \mathbf{Z}_i &= (Z_i, Z_{i+d}, \cdots, Z_{i+(m-1)d})', \qquad i = 1, 2, \cdots, d. \end{aligned} \tag{2.6}$$

Therefore, $\mathbf{Z}_i = \mathbf{B}\,\mathbf{e}_i$ for $i = 1, 2, \cdots, d$, where $\mathbf{B}$ is an $m \times m$ lower triangular matrix with $(\mathbf{B})_{ij} = 1$ for $i \geq j$. Let $\mathbf{A} = \mathbf{B}'\mathbf{B}$.

Define $\mathbf{R}_n = (R_{1n}, R_{2n}, \cdots, R_{7n})'$ by

$$\begin{aligned}
&R_{1n} = n^{-1/2} W_n, \qquad R_{2n} = n^{-3/2} Y_n, \qquad R_{3n} = n^{-5/2} \textstyle\sum_{t=1}^{n} Y_t,\\
&R_{4n} = n^{-1} \textstyle\sum_{i=1}^{d} Z_{n-i+1}^2, \qquad R_{5n} = n^{-2} \textstyle\sum_{i=1}^{d} \mathbf{e}_i'\mathbf{A}\mathbf{e}_i, \qquad R_{6n} = n^{-2} \textstyle\sum_{i=1}^{d} \sum_{j=1}^{d} \mathbf{e}_i'\mathbf{A}\mathbf{e}_j,\\
&R_{7n} = n^{-4} \textstyle\sum_{i=1}^{d} \sum_{j=1}^{d} \mathbf{e}_i'\mathbf{A}^2\mathbf{e}_j.
\end{aligned} \tag{2.7}$$

LEMMA 2.1 *Let Y_t satisfy model* (2.2) *with $\boldsymbol{\beta}' = (1, 0, 1)$. Assume the e_t are* iid$(0, \sigma^2)$ *and $d \geq 2$. Let $\mathbf{D}_n = \operatorname{diag}\{n^{-2}, n^{-1}, n^{-1}\}$. Then*

$$\mathbf{D}_n\left(\textstyle\sum_{t=1}^{n} \psi_t\psi_t'\right)\mathbf{D}_n = \begin{pmatrix} dR_{7n} - R_{2n}^2 + 2R_{2n}R_{3n} & (½)dR_{2n}^2 & ½R_{2n}^2 \\ & dR_{6n} & R_{6n} \\ & & R_{5n} \end{pmatrix} + O_p(n^{-1}),$$

$$\mathbf{D}_n\left(\textstyle\sum_{t=1}^{n} \psi_t e_t\right) = (R_{1n}R_{2n} - R_{6n},\ ½R_{1n}^2 - ½\sigma^2,\ ½R_{4n} - ½\sigma^2)' + o_p(1).$$

PROOF. The methods needed to prove the result for each element of $\mathbf{D}_n(\sum_{t=1}^{n}\psi_t\psi_t')\mathbf{D}_n$ and $\mathbf{D}_n(\sum_{t=1}^{n}\psi_t e_t)$ are similar. As examples of the methods employed, the results shall be proved for $n^{-4}\sum_{t=1}^{n} Y_{t-1}^2$ and for $n^{-1}\sum_{t=1}^{n} Z_{t-d}e_t$.

The tth element of $\mathbf{A}\sum_{i=1}^{d}\mathbf{e}_i$ is given by $Y_n - Y_{(t-1)d}$. Therefore

$$\begin{aligned}
\textstyle\sum_{i=1}^{d}\sum_{j=1}^{d} \mathbf{e}_i'\mathbf{A}^2\mathbf{e}_j &= \textstyle\sum_{t=1}^{m} (Y_n - Y_{(t-1)d})^2\\
&= mY_n^2 - 2Y_n \textstyle\sum_{t=1}^{m} Y_{(t-1)d} + \sum_{t=1}^{m} Y^2(t-1)d\\
&= mY_n^2 - 2d^{-1}Y_n \textstyle\sum_{t=1}^{n} Y_t + d^{-1}\sum_{t=1}^{n} Y_t^2 + O_p(n^3).
\end{aligned}$$

It follows that

$$\begin{aligned}
n^{-4}\textstyle\sum_{t=1}^{n} Y_{t-1}^2 &= dn^{-4}\textstyle\sum_{i=1}^{d}\sum_{j=1}^{d} \mathbf{e}_i'\mathbf{A}^2\mathbf{e}_j - n^{-3}Y_n^2 + 2n^{-4}Y_n\sum_{t=1}^{n} Y_t + O_p(n^{-1})\\
&= dR_{7n} - R_{2n}^2 + 2R_{2n}R_{3n} + O_p(n^{-1}).
\end{aligned}$$

Now consider

$$\begin{aligned}
n^{-1}\textstyle\sum_{t=1}^{n} Z_{t-d}e_t &= n^{-1}\textstyle\sum_{t=1}^{n} Z_t e_t - \sigma^2 + o_p(1) = n^{-1}\sum_{i=1}^{d}\mathbf{Z}_i'\mathbf{e}_i - \sigma^2 + o_p(1)\\
&= \frac{n^{-1}}{2}\textstyle\sum_{i=1}^{d}(Z_{n-i+1}^2 + nd^{-1}\sigma^2) - \sigma^2 + o_p(1). = \dfrac{1}{2}R_{4n} - \dfrac{1}{2}\sigma^2 + o_p(1). \qquad \square
\end{aligned}$$

3. Asymptotic distributions. Let $\boldsymbol{\lambda}_m = (\lambda_{1m}, \lambda_{2m}, \cdots, \lambda_{mm})'$ denote the eigenvalues of $\mathbf{A}$ where $\lambda_{1m} > \lambda_{2m} > \cdots > \lambda_{mm}$, and let $\mathbf{X}_{im} = (x_{i1m}, x_{i2m}, \cdots, x_{imm})'$ denote the eigenvector of unit length associated with λ_{im}. We have (see Dickey and Fuller, 1979)

$$\begin{aligned}
\lambda_{im} &= ¼\sec^2\{(2m+1)^{-1}(m+1-i)\pi\},\\
x_{ijm} &= 2(2m+1)^{-1/2}\cos\{(2m+1)^{-1}(2i-1)(j-½)\pi\}.
\end{aligned} \tag{3.1}$$

Define the orthogonal transformation $(\mathbf{e}_1', \mathbf{e}_2', \cdots, \mathbf{e}_d')'$ into $(\mathbf{u}_{1m}', \mathbf{u}_{2m}', \cdots, \mathbf{u}_{dm}')'$, where $\mathbf{u}_{im}' = (u_{i1m}, u_{i2m}, \cdots, u_{imm})'$, by

$$u_{ijm} = \textstyle\sum_{t=1}^{m} x_{jtm}e_{i+(t-1)d}. \tag{3.2}$$

LEMMA 3.1. *Let $\gamma_t = 2\{(2t-1)\pi\}^{-1}(-1)^{t+1}$ for $t = 1, 2, \cdots$. Then under model* (2.2) *with $\boldsymbol{\beta}' = (1, 0, 1)$*

(i) $R_{1n} = 2^{1/2}d^{-1/2}\sum_{i=1}^{d}\sum_{t=1}^{m}\gamma_t u_{itm} + o_p(1)$,

(ii) $R_{2n} = 2^{1/2}d^{-3/2}\sum_{i=1}^{d}\sum_{t=1}^{m}\gamma_t^2 u_{itm} + o_p(1)$,

(iii) $R_{3n} = 2^{1/2}d^{-3/2}\sum_{i=1}^{d}\sum_{t=1}^{m}(\gamma_t^2 - \gamma_t^3)u_{itm} + o_p(1)$,

(iv) $R_{4n} = 2^{-1}d^{-1}\sum_{i=1}^{d}(\sum_{t=1}^{m}\gamma_t u_{itm})^2 + o_p(1)$,

(v) $R_{5n} = d^{-2}\sum_{i=1}^{d}\sum_{t=1}^{m}\gamma_t^2 u_{itm}^2 + o_p(1)$,

(vi) $R_{6n} = d^{-2}\sum_{t=1}^{m}(\sum_{i=1}^{d}\gamma_t u_{itm})^2 + o_p(1)$,

(vii) $R_{7n} = d^{-4}\sum_{i=1}^{d}\sum_{t=1}^{m}\gamma_t^4 u_{itm}^2 + o_p(1)$.

PROOF. Consider $R_{1n} = n^{-1/2}W_n$. The u_{itm} are uncorrelated and W_n is a linear combination of the u_{itm}. Therefore

$$R_{1n} = n^{-1/2}\sigma^{-2}\sum_{i=1}^{d}\sum_{t=1}^{m}\operatorname{Cov}(W_n, u_{itm})u_{itm}$$
$$= n^{-1/2}\sigma^{-2}\sum_{i=1}^{d}\sum_{t=1}^{m}\sum_{s=1}^{n}\operatorname{Cov}(e_s, u_{itm})u_{itm} = d^{-1/2}\sum_{i=1}^{d}\sum_{t=1}^{m}\eta_{tm}u_{itm},$$

where $\eta_{tm} = m^{-1/2}\sum_{s=1}^{m}x_{tsm}$. We have

$$\eta_{tm} = 2(2m+1)^{-1/2}m^{-1/2}\sum_{s=1}^{m}\cos\left\{(2m+1)^{-1}(2t-1)\left(s-\frac{1}{2}\right)\pi\right\}$$
$$= 2^{1/2}m^{-1}\sum_{s=1}^{m}\cos\left\{\left(\frac{2t-1}{2}\right)\frac{s}{m}\pi\right\} + o(1)$$

where $\eta_{tm} \to \eta_t$ as $m \to \infty$, where

$$\eta_t = 2^{1/2}\int_0^1 \cos\left\{\left(\frac{2t-1}{2}\right)\pi y\right\}dy = 2^{3/2}\{(2t-1)\pi\}^{-1}(-1)^{t+1} = 2^{1/2}\gamma_t.$$

We have $\sum_{t=1}^{\infty}\eta_t^2 = 1$ and because the $\mathbf{X}_{im}$ are orthonormal $\sum_{t=1}^{m}\eta_{tm}^2 = 1$ for each m. Let $\varepsilon > 0$ and choose M large enough so that, for $m > M$, $\sum_{t=M+1}^{\infty}\eta_t^2 < \varepsilon$. Then for $m > M$

$$\sum_{t=1}^{m}(\eta_t - \eta_{tm})^2 = \sum_{t=1}^{M}(\eta_t - \eta_{tm})^2 + \sum_{t=M+1}^{m}(\eta_t - \eta_{tm})^2.$$

Because $\eta_{tm} \to \eta_t$ as $m \to \infty$, the first term can be made arbitrarily small, say less than $\frac{1}{4}\varepsilon^2$ by choosing m large enough. For the second term we have

$$\sum_{t=M+1}^{m}(\eta_t - \eta_{tm})^2 \le 2\sum_{t=M+1}^{m}\eta_t^2 + 2\sum_{t=M+1}^{m}\eta_{tm}^2 < 8\varepsilon.$$

Therefore $\sum_{t=1}^{m}(\eta_t - \eta_{tm})^2 \to 0$ as $m \to \infty$. Because the u_{itm} are uncorrelated it follows that $R_{1n} - d^{-1/2}\sum_{i=1}^{d}\sum_{t=1}^{m}\eta_t u_{itm}$ converges in mean square to zero and the first result is established. The other results may be proven in a similar manner. □

Let $\{V_{ij}: i = 1, 2, \cdots, d; j = 1, 2, \cdots\}$ be an array of independent normal random variables with mean zero and variance σ^2. Define $\mathbf{R} = (R_1, R_2, \cdots, R_7)'$ a.e. by

$$(R_1, R_2) = (2^{1/2}d^{-1/2}\sum_{i=1}^{d}\sum_{j=1}^{\infty}\gamma_j V_{ij},\ 2^{1/2}d^{-3/2}\sum_{i=1}^{d}\sum_{j=1}^{\infty}\gamma_j^2 V_{ij}),$$

$$(R_3, R_4) = (2^{1/2}d^{-3/2}\sum_{i=1}^{d}\sum_{j=1}^{\infty}(\gamma_j^2 - \gamma_j^3)V_{ij},\ 2^{-1}d^{-1}\sum_{i=1}^{d}(\sum_{j=1}^{\infty}\gamma_j V_{ij})^2), \tag{3.3}$$

$$(R_5, R_6) = (d^{-2}\sum_{i=1}^{d}\sum_{j=1}^{\infty}\gamma_j^2 V_{ij}^2,\ d^{-2}\sum_{j=1}^{\infty}(\sum_{i=1}^{d}\gamma_j V_{ij})^2),$$

$$R_7 = d^{-4}\sum_{i=1}^{d}\sum_{j=1}^{\infty}\gamma_j^4 V_{ij}^2.$$

In Theorem 3.1 we characterize the asymptotic distribution of $\mathbf{R}_n$ and as corollaries obtain the asymptotic distributions of $\hat{\boldsymbol{\beta}}$ and $\Phi_{n-3}^{(3)}$.

THEOREM 3.1. *Let the assumptions of model* (2.2) *hold with* $\boldsymbol{\beta}' = (1, 0, 1)$. *Then* $\mathbf{R}_n \to_{\mathscr{L}} \mathbf{R}$.

PROOF. Let $K \geq 1$ be a fixed integer. One may then apply the Lindeberg central limit theorem to establish that $(u_{11m}, \cdots, u_{1km}, \cdots, u_{d1m}, \cdots, u_{dkm})' \rightarrow_{\mathscr{L}} N_{dk}(\mathbf{0}, \sigma^2 \mathbf{I}_{dk})$ as $n \rightarrow \infty$. The result may then be established by an application of Lemma 6.3.1 of Fuller (1976). See also Diananda (1953) and Theorem 3.1 of Hasza and Fuller (1979). □

COROLLARY 3.1. *Under the assumptions of Theorem* 3.1

$$\{n^2(\hat{\beta}_1 - 1), n\hat{\beta}_2, n(\hat{\beta}_3 - 1)\}' \rightarrow_{\mathscr{L}} H^{-1}h,$$

where

$$\mathbf{H} = \begin{pmatrix} dR_7 - R_2^2 + 2R_2R_3 & \frac{1}{2}\, dR_2^2 & \frac{1}{2}\, R_2^2 \\ \frac{1}{2}\, dR_2^2 & dR_6 & R_6 \\ \frac{1}{2}\, R_2^2 & R_6 & R_5 \end{pmatrix},$$

$$\mathbf{h} = \left(R_1R_2 - R_6, \frac{1}{2}\, R_1^2 - \frac{1}{2}\, \sigma^2, \frac{1}{2}\, R_4 - \frac{1}{2}\, \sigma^2\right)'.$$

PROOF. The result follows from Theorem 3.1 because $\mathbf{H}$ is nonsingular with probability one.

COROLLARY 3.2. *Let the assumptions of Theorem* 3.1 *hold and let* $\Phi_{n-3}^{(3)}$ *be defined by* (2.5). *Then* $\Phi_{n-3}^{(3)} \rightarrow_{\mathscr{L}} (3\,\sigma^2)^{-1}\mathbf{h}'\mathbf{H}^{-1}\mathbf{h}$.

In Table 5.1 we present the critical points for $\Phi_{n-3}^{(3)}$ for various values of n and d and the limiting values of the critical points as $n \rightarrow \infty$.

Often a seasonal process is modeled as a multiplicative seasonal autoregressive process.

$$Y_t - \phi_1 Y_{t-1} = \phi_2(Y_{t-d} - \phi_1 Y_{t-d-1}) + e_t. \tag{3.4}$$

If $\phi_1 = \phi_2 = 1$ then $\Delta_1\Delta_d Y_t = e_t$. To test the hypothesis that this is so, consider the one-step Gauss-Newton procedure of estimating ϕ_1 and ϕ_2 with initial values $\hat{\phi}_1 = \hat{\phi}_2 = 1$. Under the null hypothesis the procedure results in the regression of e_t on W_{t-1} and Z_{t-d}, where $W_t = \Delta_d Y_{t-1}$ and $Z_{t-d} = \Delta_1 Y_{t-d}$. Let $\Phi_{n-3}^{(2)}$ denote the usual F-type statistic constructed to test $\phi_1 = \phi_2 = 1$. From Theorem 3.1 it follows that $\Phi_{n-2}^{(2)} \rightarrow_{\mathscr{L}} (2\sigma^2)^{-1}\mathbf{h}_1'\mathbf{H}_1^{-1}\mathbf{h}_1$, where $\mathbf{H}_1$ is obtained by deleting the first row and column of $\mathbf{H}$, and $\mathbf{h}_1$ is obtained by deleting the first element of $\mathbf{h}$. Percentiles of this test statistic are given in Table 5.1

4. Extensions. In this section we shall consider processes for which $\Delta_1\Delta_d Y_t$ is a stationary autoregressive process. We consider the model

$$Y_t = \beta_1 Y_{t-1} + \beta_2(Y_{t-1} - Y_{t-d-1}) + \beta_3(Y_{t-d} - Y_{t-d-1}) + \textstyle\sum_{j=1}^{p} \theta_j X_{t-j} + e_t, \tag{4.1}$$

where $X_t = \Delta_1\Delta_d Y_t$ and the e_t are iid$(0, \sigma^2)$ random variables. Note that under H_0: $\boldsymbol{\beta}' = (1, 0, 1)$

$$X_t = \textstyle\sum_{j=1}^{p} \theta_j X_{t-j} + e_t \tag{4.2}$$

and X_t is a pth order autoregressive process. We assume that the associated characteristic equation

$$m^p - \textstyle\sum_{j=1}^{p} \theta_j m^{p-j} = 0 \tag{4.3}$$

has roots $m_1, m_2, \cdots, m_p$ with $|m_i| < 1$ for $i = 1, 2, \cdots, p$.

Let $\boldsymbol{\theta} = (\theta_1, \theta_2, \cdots, \theta_p)(\theta_1, \theta_2, \cdots, \theta_p)'$. The error in the least squares estimator of $(\boldsymbol{\beta}', \boldsymbol{\theta}')'$ is given by

$$\{(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})', (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})'\}' = (\textstyle\sum_{t=1}^{n} \psi_t\psi_t')^{-1} \sum_{t=1}^{n} \psi_t e_t,$$

where $\psi_t = (Y_{t-1}, W_{t-1}, Z_{t-d}, X_{t-1}, X_{t-2}, \cdots, X_{t-P})'$. Because $\{X_t\}$ has characteristic roots less than one in modulus, $\sum_{j=1}^{p} \theta_j \neq 1$. We then let $c = 1 - \sum_{j=1}^{p} \theta_j$ and note that $c \neq 0$.

TABLE 5.1
Empirical percentiles for test statistics

	m	$d=2$ 0.90	$d=2$ 0.95	$d=2$ 0.99	$d=4$ 0.90	$d=4$ 0.95	$d=4$ 0.99	$d=6$ 0.90	$d=6$ 0.95	$d=6$ 0.99	$d=12$ 0.90	$d=12$ 0.95	$d=12$ 0.99
$\Phi^{(3)}_{n-3}$	10	2.52	3.24	5.09	2.46	3.06	4.49	2.42	3.00	4.35	2.37	2.93	4.23
	20	2.48	3.09	4.51	2.44	3.01	4.30	2.41	2.97	4.24	2.37	2.92	4.19
	50	2.45	3.02	4.29	2.44	2.99	4.22	2.41	2.96	4.19	2.37	2.92	4.17
	∞	2.44	2.98	4.20	2.44	2.98	4.19	2.41	2.96	4.17	2.37	2.92	4.16
$\Phi^{(3)}_{n-d-4}$	10	8.33	10.08	14.86	8.95	10.42	13.72	10.09	11.53	14.58	13.49	15.13	18.47
	20	7.36	8.58	11.47	8.53	9.69	12.33	9.72	10.97	13.56	13.40	14.78	17.76
	50	6.97	7.93	9.92	8.26	9.28	11.36	9.54	10.68	12.86	13.24	14.55	17.46
	∞	6.67	7.50	9.17	8.04	8.96	10.90	9.36	10.39	12.43	13.16	14.41	16.93
$\Phi^{(d+4)}_{n-d-4}$	10	5.31	6.37	9.36	4.32	4.96	6.35	3.99	4.46	5.56	3.54	3.84	4.47
	20	4.44	5.09	6.69	3.94	4.44	5.42	3.71	4.08	4.87	3.40	3.65	4.18
	50	4.06	4.54	5.63	3.73	4.12	4.96	3.57	3.90	4.59	3.29	3.52	4.05
	∞	3.81	4.22	5.05	3.60	3.93	4.62	3.45	3.73	4.35	3.22	3.44	3.88
$\Phi^{(2)}_{n-2}$	10	2.61	3.46	5.76	2.58	3.36	5.20	2.53	3.28	5.04	2.46	3.19	4.90
	20	2.59	3.34	5.17	2.54	3.29	4.99	2.50	3.24	4.92	2.45	3.17	4.84
	50	2.55	3.27	4.99	2.53	3.26	4.92	2.49	3.22	4.88	2.45	3.16	4.83
	∞	2.52	3.24	4.93	2.52	3.24	4.90	2.49	3.20	4.87	2.45	3.15	4.82
$\Phi^{(2)}_{n-d-3}$	10	7.94	9.77	14.72	9.84	11.68	15.87	11.75	13.68	17.91	17.37	19.53	24.22
	20	7.43	8.89	12.03	9.63	11.12	14.39	11.57	13.24	16.81	17.29	19.24	23.65
	50	7.23	8.44	11.30	9.41	10.78	13.82	11.46	12.93	16.30	17.21	19.15	23.18
	∞	7.05	8.16	10.48	9.28	10.59	13.26	11.36	12.82	15.72	17.20	19.08	22.61
$\Phi^{(d+3)}_{n-d-3}$	10	4.39	5.30	7.64	3.87	4.47	5.84	3.64	4.11	5.15	3.37	3.68	4.31
	20	3.87	4.50	6.00	3.59	4.05	5.08	3.44	3.84	4.61	3.24	3.50	4.04
	50	3.61	4.14	5.26	3.45	3.86	4.74	3.34	3.68	4.38	3.16	3.40	3.90
	∞	3.45	3.92	4.88	3.35	3.73	4.48	3.26	3.58	4.23	3.11	3.34	3.80

THEOREM 4.1. *Let Y_t satisfy model* (4.1) *with e_t that are* iid$(0, \sigma^2)$. *Let* **H**, **h** *be as defined in Corollary* 3.1. *Then under $H_0: \boldsymbol{\beta}' = (1, 0, 1)$*

(i) $$\{n^2(\hat{\beta}_1 - 1), n\hat{\beta}_2, n(\hat{\beta}_3 - 1)\}' \to_{\mathscr{L}} c\mathbf{H}^{-1}\mathbf{h},$$

(ii) $$n^{1/2}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \to_{\mathscr{L}} N_p(\mathbf{0}, \boldsymbol{\Gamma}^{-1}\sigma^2),$$

where $(\Gamma)_{ij} = lim_{t\to\infty}\mathrm{Cov}(X_t, X_{t+|i-j|})$.

PROOF. Define

$$W_t^\dagger = \textstyle\sum_{j=1}^t e_t,\ Z_t^\dagger = \sum_{j=1}^{[t/d]} e_{t-dj},\ Y_t^\dagger = \sum_{j=1}^t Z_j^\dagger.$$

It is not difficult to show that the asymptotic properties of $(\sum_{t=1}^n \psi_t\psi_t')^{-1}(\sum_{t=1}^n \psi_t e_t)$ are not affected by replacing Y_{t-1}, W_{t-1}, and Z_{t-d} by $c^{-1}Y_{t-1}^\dagger$, $c^{-1}W_{t-1}^\dagger$, and $c^{-1}Z_{t-d}^\dagger$ respectively. Furthermore

$$\textstyle\sum_{t=1}^n Y_{t-1}X_{t-i} = O_p(n^2), \quad \sum_{t=1}^n W_{t-1}X_{t-1} = O_p(n), \quad \sum_{t=1}^n Z_{t-d}X_{t-i} = O_p(n),$$
$$i = 1, 2, \cdots, p.$$

The result then follows from Theorem 3.1 and the well-known asymptotic distributional theory for stationary autoregressive processes. □

Now let $\Phi^{(3)}_{n-p-3}$ denote the test statistic for testing $H_0: \boldsymbol{\beta}' = (1, 0, 1)$ analogous to the usual $F_{3,n-p-3}$ test statistic in a fixed normal regression model.

COROLLARY 4.1. *Let the assumptions of Theorem* 4.1 *hold. Then*

$$\Phi^{(3)}_{n-p-3} \to_{\mathscr{L}} (3\sigma^2)^{-1}\, \mathbf{h}'\mathbf{H}^{-1}\mathbf{h}$$

where **h** *and* **H** *are defined in Theorem* 3.1.

Because of Corollary 4.1 the tables giving the percentage points of $\Phi^{(3)}_{n-3}$ of Corollary 3.1 are also applicable for $\Phi^{(3)}_{n-p-3}$ in large samples.

An alternative to model (2.1) is given by a stationary time series with a deterministic seasonal component and a possible linear time trend. Consider model (1.4) and the hypothesis $H_1: \{\alpha' = (1, 1, -1), \xi_j = 0 \text{ for } j = 1, 2, \cdots, d\}$. The test H_1 can be used to discriminate between the two types of processes: one having nonstationary stochastic trend and seasonality, the other having deterministic trend and seasonality. We denote by $\Phi^{(d+4)}_{n-d-4}$ the usual F type statistic for testing H_1 and denote by $\Phi^{(3)}_{n-d-3}$ the F type statistic for testing $H_2: \alpha' = (1, 1, -1)$. The percentiles for these statistics are presented in Section 5. The characterization of the asymptotic distributions follows along the lines of Section 3. Details are presented in Hasza and Fuller (1982).

5. Percentiles for test statistics. In Table 5.1 estimates of the percentiles of the test statistics discussed in Sections 3 and 4 are presented. For the finite sample sizes the statistics were obtained from samples using the model $\Delta_1 \Delta_d Y_t = e_t$ with zero initial conditions. It is noted that the initial conditions do not affect the asymptotic distributions. The initial conditions will affect the small sample distribution of $\Phi^{(3)}_{n-3}$ but not the distribution of $\Phi^{(3)}_{n-d-4}$ or of $\Phi^{(d+4)}_{n-d-4}$. The e_t were generated as normal (0, 1) random variables using a random number generator discussed in Marsaglia, Ananthanarayanan and Paul (1978).

To estimate the percentiles of the asymptotic distributions Corollary 3.2 was used. The γ_{ij} sequence was truncated at $i = 72$ and adjusted so that the first eight absolute moments of the resulting statistics closely approximated the moments of the R_j's of Theorem 3.1. The estimated standard errors for the estimated percentiles are about 0.7 percent of the table entries for finite values of m and about 0.5 percent of the table entries for the asymptotic distributions. For finite values of m the empirical percentiles are based on 25,000 independent realizations of the sample statistics. For the asymptotic distributions 50,000 independent realizations were used.

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