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A BAYES BUT NOT CLASSICALLY SUFFICIENT STATISTIC[1]

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In a Borel setting, every classically sufficient statistic is Bayes sufficient, but not vice versa. The example is a hypothesis testing problem in which Bayesians, but not classicists, can achieve zero error probabilities.

Let X be a random variable whose distribution P_θ depends on the parameter θ, and let Y be a function of X. According to Fisher (1922), the statistic Y is sufficient if the distribution of the observation X given Y does not depend on the parameter θ. Another concept of sufficiency, introduced by Kolmogorov (1942), is that Y is sufficient if for every prior distribution of θ the posterior distribution of θ given X depends on Y only. We shall call the Fisher concept *classical sufficiency* and the Kolmogorov concept *Bayes sufficiency*.

Classical sufficiency implies Bayes sufficiency; a short proof is sketched below. And it follows easily from the results of Halmos and Savage (1949) that in the dominated case—all P_θ absolutely continuous with respect to a single measure—Bayes sufficiency implies classical sufficiency; one has to check just that pairwise Bayes sufficiency implies pairwise classical sufficiency. The purpose of this note is to give an example of a Bayes sufficient statistic Y that is not classically sufficient.

To see that classical sufficiency implies Bayes, note that if Y is classically sufficient, then for any prior distribution of θ the triple θ, Y, X is a Markov chain. Since a Markov chain is Markov in reverse, the distribution of θ given Y and X (which is obviously also the distribution of θ given X) depends on Y only, so that Y is Bayes sufficient.

We first describe our example as a hypothesis testing problem, and then relate it to sufficiency. The statistic Y is a sequence $Y_1, Y_2, \cdots$ of 0-1 variables, and the parameter set Θ is the set of all distributions θ of Y under which $\{Y_n\}$ converges in probability to 0 or 1:

$$P_\theta\{Y_n = 1\} \to L(\theta) = 0 \text{ or } 1 \quad \text{for each} \quad \theta \in \Theta.$$

We observe Y and want to test the hypothesis $H_0 : L(\theta) = 0$ against the alternative $H_1 : L(\theta) = 1$. What are the smallest error probabilities we can attain?

Any Bayesian can attain (in his opinion) zero error probabilities of both kinds. For if m is any prior distribution on θ that gives positive probability to both H_0 and H_1, and P_0, P_1 are the conditional distributions of Y under H_0, H_1 respectively, the problem reduces to testing the simple hypothesis P_0 against the simple alternative P_1. Since

$$P_i\{Y_n = 1\} = \int_{H_i} P_\theta\{Y_n = 1\}\, dm(\theta)/m(H_i),$$

converges to i (bounded convergence), i.e. Y converges in probability to i under P_i, Y has a subsequence Z that converges with probability 1 to i under P_i. So lim Z indicates the correct hypothesis with probability 1, for $i = 0$ or 1.

On the other hand, a classical test of H_0 vs. H_1 is a (Borel) function f on the sample space of all infinite sequences of 0's and 1's with values 0 (accept H_0) and 1 (accept H_1). To have zero error probabilities under both H_0 and H_1 it would satisfy

$$P_\theta\{f(Y) = i\} = 1 \qquad \text{for all} \qquad \theta \in H_i, \quad i = 0, 1$$

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i.e. it would identify, with probability 1, the limit of every sequence Y of 0-1 variables that converges in probability to a constant limit. It has been shown recently (Blackwell, 1980) that no such f (called a Borel SPLIF) exists. Thus there is no classical test of H_0 vs. H_1 that achieves zero error probabilities, but every Bayesian is certain of being right.

To relate our example to sufficiency, introduce an additional 0-1 observation Z that tells us whether H_0 or H_1 is true: $Z = L(\theta)$, and let X be the pair (Y, Z). Then Y is Bayes sufficient since, as we have already seen, any Bayesian can compute Z with probability 1 from Y. But Y is not classically sufficient, since a classical statistician can test H_0 vs. H_1 with zero errors using X, but not using Y alone.

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