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MINIMAL COMPLETE CLASSES OF TESTS OF HYPOTHESES WITH MULTIVARIATE ONE-SIDED ALTERNATIVES

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In this paper we present minimal complete class theorems for testing problems with simple null hypotheses and multivariate one-sided alternative hypotheses. The results extend previous ones by not requiring that the tests be based on exponentially distributed random variables or that the null and alternative parameter spaces be topologically separated.

1. Introduction. We are interested in characterizing admissibility in a variety of problems testing

(1.1) $$H_0: \theta = 0 \quad \text{versus} \quad H_A: \theta \in \Theta_A \equiv \Theta - \{0\}$$

based on observing X whose distribution depends on the parameter $\theta \in \Theta \subseteq \mathbb{R}^p$. We require that Θ be one-sided in the sense that $\Theta \subseteq V$ where

(1.2) V is a closed convex cone in $\mathbb{R}^p$ with vertex $\{0\}$ such that $v \in V$ and $v \neq 0$ implies that $-v \notin V$.

Section 2 contains minimal complete classes of tests of (1.1) under conditions general enough to apply to the one-sided combination problems and invariance-reduced multivariate normal problems described below. It is assumed that the range space of X, $\mathscr{X}$, is an open convex subset of $\mathbb{R}^m$, and that X has a density $f(x; \theta)$ with respect to Lebesgue measure μ on $\mathbb{R}^m$ which is positive and jointly continuous in (x, θ).

In a combination problem there are p independent statistics $X_1, \cdots, X_p$ such that it is appropriate to test $\theta_i = 0$ versus $\theta_i > 0$ based on X_i, and an overall test of (1.1), where $\theta = (\theta_1, \cdots, \theta_p)$, is desired. Examples include combining tests of means based on $X_i \sim N(\theta_i, 1)$ (see Birnbaum, 1955); combining tests on noncentrality parameters θ_i based on X_i being noncentral χ^2 (Marden, 1982), F (Marden and Perlman, 1981) or t; combining tests of $\sigma_i^2 = 1$ versus $\sigma_i^2 > 1$ or versus $\sigma_i^2 < 1$ based on $X_i \sim \sigma_i^2\chi^2$, where $\theta_i = 1 - \sigma_i^{-2}$ or $\sigma_i^{-2} - 1$; combining tests of $\sigma_{i1}^2 = \sigma_{i2}^2$ versus $\sigma_{i1}^2 > \sigma_{i2}^2$ based on $X_i \sim (\sigma_{i1}^2/\sigma_{i2}^2)\ F$ where $\theta_i = 1 - \sigma_{i2}^2/\sigma_{i1}^2$; and combining tests of $\rho_i = 0$ versus $\rho_i > 0$ (or $\rho_i^2 = 0$ versus $\rho_i^2 > 0$) based on X_i being a sample correlation coefficient from a bivariate normal with correlation ρ_i (or based on the square of such). In the normal mean and $\sigma_i^2 < 1$ cases, the density of X is exponential with natural parameter θ and natural statistic X, and Θ equals the nonnegative orthant. Hence the results of Birnbaum (1954 and 1955), Matthes and Truax (1967) and Eaton (1970) can be used to show that the minimal complete class of tests consists of all tests equal a.e. $[\mu]$ to one with an acceptance region A which is convex and decreasing, i.e., if $x \in A$ and $y_i \leq x_i$ for all i, then $y \in \mathscr{X}$ implies $y \in A$. A similar result can be shown for the noncentral χ^2 problem even though it is not in an exponential framework. The other problems need our theorem. See Marden (1980a) for details.

Often, a testing problem on multivariate normal parameters can be reduced by invari-

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ance to one of the form (1.1) where θ and X represent the ordered characteristic roots of certain matrix valued parameters and statistics. In such cases, Θ is contained in the ordered nonnegative cone. The minimal complete classes of invariant tests in the multivariate analysis of variance and generalized multivariate analysis of variance problems can be found using the results of Section 2. See Marden (1980b). Other applications are made in Marden (1980a) to invariant tests in testing $\sum = I$ versus $\sum > I$ or versus $\sum < I$ based on a Wishart ($\sum$), in testing $\sum_1 = \sum_2$ versus $\sum_1 > \sum_2$ based on two independent Wisharts, and in the canonical correlation problem.

Our theorem builds on the work of Farrell (1968) and Ghia (1976). Farrell considers testing $H_0 : \theta \in \Theta_0$ versus $H_A : \theta \in \Theta_A$ based on X exponential when Θ_0 is compact and $\Theta_0 \cap \bar{\Theta}_A = \varnothing$. Note that these conditions imply that Θ_0 and Θ_A are topologically separated. If for some V as in (1.2), $\Theta_A \subseteq V$ and $V - \Theta_A$ is bounded, then a minimal complete class of tests consists of all tests of the form

$$\phi(x) = \begin{cases} 1 & \text{if} \quad x \notin C \\ 1 & \text{if} \quad \int_{\bar{\Theta}_A} f(x;\theta)\nu(d\theta) > \int_{\Theta_0} f(x;\theta)\rho(d\theta) \\ 0 & \text{otherwise} \quad \text{a.e. } [\mu], \end{cases} \tag{1.3}$$

where ν is a σ-finite measure on $\bar{\Theta}_A$, ρ is a finite measure on Θ_0, and C is a convex and decreasing $[V]$ subset of $\mathscr{X}$, the partial ordering $\cdot \leq \cdot [V]$ being defined by

$$\begin{aligned} x \leq y[V] \quad &\text{if} \quad x'\theta \leq y'\theta \quad \text{for all} \quad \theta \in V, \\ x < y[V] \quad &\text{if} \quad x'\theta < y'\theta \quad \text{for all} \quad \theta \in V - \{0\}. \end{aligned} \tag{1.4}$$

Ghia treats the same problem but does not require that f be exponential. For a fairly large class of densities f he defines a partial ordering on $\mathscr{X}$ based on properties of the tails of f. The minimal complete classes contain tests (1.3) where the sets C, which he calls "truncation sets," are decreasing in the new ordering. He also considers cases for which $\bar{\Theta}_A$ is compact and contained in Θ, and shows that the sets C can be eliminated from (1.3).

Unfortunately, in our problems the condition that $\{0\} \cap \bar{\Theta}_A = \varnothing$ does not hold, hence we cannot directly apply the above results. Farrell and Ghia need to separate the null and alternative spaces lest ν and ρ in (1.3) turn out to be the same measure. We modify the middle term in (1.3) to solve that difficulty, which admits locally optimal tests into the class, but we restrict the null space to be simple and the alternative to be one-sided. The densities we allow are more general than exponential but not as general as Ghia's. In Section 4 we show how our results apply to "almost exponential" and other densities.

Problems in which Θ_A is not restricted to be one-sided are being attacked by Professors R. Farrell and L. Brown, and the author. These anticipated results will apply to combining the two-sided tests with alternatives $\sigma_i^2 \neq 1$ or $\sigma_{i1}^2 \neq \sigma_{i2}^2$, and to the invariance-reduced problems with alternatives $\sum \neq I$ or $\sum_1 \neq \sum_2$.

2. The minimal complete class theorem. Consider problem (1.1) as in the first paragraph of Section 1. Define

$$R_\theta(x) = h(\theta)f(x;\theta)/f(x;0), \tag{2.1}$$

where the function h will be described later. We make the following assumption.

LOCAL ASSUMPTION. There exist real-valued functions $\ell_1(x), \cdots, \ell_p(x)$ such that for each x,

$$R_\theta(x) = 1 + \ell'\theta + o(\sigma) \quad \text{as} \quad \sigma \to 0, \tag{2.2}$$

where $\sigma \equiv \sigma(\theta) = \sum_{i=1}^{p} |\theta_i|$, and $\ell = (\ell_1, \cdots, \ell_p)$. Also, there exists an $a > 0$ such that for

$\Theta_a \equiv \{\theta \in \Theta_A \mid \sigma \leq a\}$,

$$\int_{\mathscr{X}} [\sup_{\theta \in \Theta_a} \mid R_\theta - 1 \mid / \sigma] f(x; 0)\, dx < \infty. \tag{2.3}$$

Define the class $\mathscr{C}$ of truncation sets to be all sets $C \subseteq \mathscr{X}$ for which there is a sequence $\{\pi_n\}$ of proper measures on Θ_A with

$$C = \text{closure } \{x \mid \text{limsup}_{n \to \infty} \int_{\Theta_A} R_\theta(x) \pi_n(d\theta) < \infty\}. \tag{2.4}$$

Throughout this paper the topological terms (closure, interior, boundary, etc.) will refer to the relative topologies on $\mathscr{X}$, and on $\mathscr{W}$ in Case B below. For Θ, the topology will be the full one on $\mathbb{R}^p$.

We treat two different cases.

CASE A. Suppose Θ is convex and bounded, and $\Theta_a = \{\theta \in V \mid 0 < \sigma \leq a\}$ for V in (1.2). Assume the function h in (2.1) can be chosen bounded away from zero so that R_θ can be extended to a continuous, finite and positive function (also called R_θ) on $\mathscr{X} \times \bar{\Theta}$. In addition, for each $\theta \in \bar{\Theta} - \Theta$, there exists a sequence $\{\theta_n\} \subseteq \Theta_A$, $\theta_n \to \theta$, such that for every x,

$$R_{\theta_n}(x) \uparrow R_\theta(x). \tag{2.5}$$

In this case, for each x, R_θ is bounded away from zero and infinity as a function of θ. Thus C in (2.4) is either $\mathscr{X}$ or $\varnothing$ as the limit superior of $\pi_n(\Theta_A)$ is finite or infinite, i.e.,

$$\mathscr{C} = \{\mathscr{X}, \varnothing\} \quad \text{in Case A.} \tag{2.6}$$

Note that if this case does hold then $h(\theta) = 1/f(x^*; \theta)$ will work for any $x^* \in \mathscr{X}$. The function h is necessary when the density f is identically zero in x for $\theta \in \bar{\Theta} - \Theta$ because we need R_θ to remain positive.

CASE B. Suppose $\Theta = V(1.2)$ and take $h = 1$ in (2.1). There exists a function w from $\mathscr{X}$ onto $\mathscr{W}$, a convex subset of $\mathbb{R}^p$, for which the following hold.

(i) $\mathscr{C} = \{w^{-1}(C_w) \mid C_w \in \mathscr{C}_w\}$, where $\mathscr{C}_w$ is the class of closed, convex and decreasing $[V]$ (1.4) subsets of $\mathscr{W}$. Furthermore, for each $C_w \in \mathscr{C}_w$,

$$\text{int } w^{-1}(C_w) = w^{-1}(\text{int } C_w), \tag{2.7}$$

$$\mu[\text{boundary } w^{-1}(C_w)] = 0, \tag{2.8}$$

and if $y \in \text{int } C_w$, then there exists $y' \in \text{int } C_w$ such that

$$y < y'[V]. \tag{2.9}$$

(ii) Take $C \in \mathscr{C}$ as in (2.4). If $x \notin C$, then on a subsequence of $\{n\}$,

$$\lim_{n \to \infty} \int R_\theta(x') \pi_n(d\theta) = \infty \quad \text{for all} \quad x' \quad \text{such that} \quad w(x') \geq w(x)[V]. \tag{2.10}$$

(iii) If $w(x') > w(x)[V]$, then

$$\lim_{i \to \infty} \sup_{\sigma \geq i} [R_\theta(x)/R_\theta(x')] = 0. \tag{2.11}$$

(iv) There exists a positive function $\alpha(\theta)$ such that for any $\xi \in V - \{0\}$ and t real,

$$\lim_{s \to \infty} \alpha(s\xi) \exp(-st) R_{s\xi}(x) = \begin{cases} \infty & \text{if } \xi' w(x) > t \\ 0 & \text{if } \xi' w(x) < t \end{cases} \tag{2.12}$$

and

$$\int_{\{x \mid \xi' w(x) < t\}} [\sup_{s > 0} \alpha(s\xi) \exp(-st) R_{s\xi}(x)] f(x; 0)\, dx < \infty. \tag{2.13}$$

Recall that a test is a measurable function from $\mathscr{X}$ to [0, 1]. We define the class Φ of tests for problem (1.1) to consist of all tests of the form

$$\phi(x) = \begin{cases} 1 & \text{if } x \notin C, \\ 1 & \text{if } d(x; \lambda, \pi_a, \pi_b) > c, \\ 0 & \text{otherwise a.e. } [\mu], \end{cases} \tag{2.14}$$

where $C \in \mathscr{C}$,

$$d(x; \lambda, \pi_a, \pi_b) = \ell' \lambda + \int_{\Theta_a} [(R_\theta - 1)/\sigma]\pi_a(d\theta) + \int_{\bar{\Theta}_b} R_\theta \pi_b(d\theta), \tag{2.15}$$

$\lambda \in V$, π_a is a finite measure on Θ_a (see the Local Assumption), π_b is a locally finite measure [i.e., it assigns finite mass to any compact set] on $\bar{\Theta}_b$, $\Theta_b \equiv \Theta_A - \Theta_a$, $|c| < \infty$, and

$$|d(x; \lambda, \pi_a, \pi_b)| < \infty \quad \text{for} \quad x \in \operatorname{int}(C). \tag{2.16}$$

In Case A the description of Φ can be made simpler. The set C is trivial by (2.6). Since $\bar{\Theta}_b$ is compact, π_b is a finite measure, and since $\sigma \geq a$ for $\theta \in \bar{\Theta}_b, \int_{\bar{\Theta}_b} \sigma^{-1}\pi_b(d\theta) < \infty$. Thus for any (π_a, π_b, c) as above we can find a finite measure π' on $\bar{\Theta} - \{0\}$ and a $|c'| < \infty$ such that

$$d(x; \lambda, \pi_a, \pi_b) - c = \ell'\lambda + \int_{\bar{\Theta}-\{0\}} [(R_\theta - 1)/\sigma]\pi'(d\theta) - c',$$

and vice versa.

Here is the main result.

THEOREM 2.1. *Suppose the Local Assumption holds and either Case* A *or* B *obtains. If for any* $(\lambda, \pi_a, \pi_b, c)$ *as above, not all zero,*

$$\mu(\{x \mid d(x; \lambda, \pi_a, \pi_b) = c\}) = 0, \tag{2.17}$$

then Φ *is the minimal complete class of tests for problem* (1.1).

3. Proof of Theorem 2.1. The proof is broken into two parts. In Part I we show that Φ is essentially complete. In Part II we show that if $\phi \in \Phi$ and ψ is essentially different from ϕ, then

$$r_\theta(\phi) < r_\theta(\psi) \quad \text{for some} \quad \theta \in \Theta, \tag{3.1}$$

where r_θ is the risk function $r_\theta(\phi) = E_0(\phi)$ if $\theta = 0$ and $r_\theta(\phi) = 1 - E_\theta(\phi)$ if $\theta \in \Theta_A$. The parts together imply that Φ is minimal complete: Part II immediately shows that every ϕ in Φ is admissible. Suppose ϕ is admissible. Part I guarantees that there exists $\psi \in \Phi$ such that $r_\theta(\phi) = r_\theta(\psi)$ for all $\theta \in \Theta$. By Part II, $\phi = \psi$ a.e. $[\mu]$. But Φ is defined only up to null sets, hence $\phi \in \Phi$.

PART I. Suppose ϕ is a weak* limit of a sequence $\{\phi_n\}$ of proper Bayes tests, i.e.,

$$\int_B \phi_n(x)\, dx \to \int_B \phi(x)\, dx \quad \text{for all compact} \quad B \subseteq \mathscr{X}. \tag{3.2}$$

Wald's (1950) Theorem 5.8 implies that the set of all such ϕ constitutes an essentially complete class. Hence we need to show that $\phi \in \Phi$. Let ϕ_n be Bayes with respect to $(\nu_n, \rho_n) \neq (0, 0)$, where ν_n is a finite measure on Θ_A and $\rho_n \geq 0$ represents the mass at $\theta = 0$, so that $\phi_n = 1$ (0) a.e. $[\mu]$ when $\int_{\Theta_A} f(x; \theta)\nu_n(d\theta) >(\leq) \rho_n f(x; 0)$. This last expression with ">" can be rewritten as

$$q_n \int_{\Theta_a} [(R_\theta - 1)/\sigma]\pi_{an}(d\theta) + \int_{\Theta_b} R_\theta \pi_{bn}(d\theta) > c_n, \tag{3.3}$$

where $q_n \geq 0$, $q_n + |c_n| = 1$, π_{an} is a probability measure on Θ_a, and π_{bn} is a finite measure on Θ_b. If $q_n = 0$, then π_{an} can be chosen arbitrarily. Since $\bar{\Theta}_a$ is compact, $\{\pi_{an}\}$ can be considered a sequence of probability measures on a compact space. Clearly $\{(q_n, c_n)\}$ ranges over a compact space. Thus there exist (q, c), $q \geq 0$ and $q + |c| = 1$, and a probability measure π_a^* on $\bar{\Theta}_a$ such that on a subsequence

$$(q_n, c_n) \to (q, c) \quad \text{and} \quad \pi_{an} \to \pi_a^* \text{ weakly.} \tag{3.4}$$

Choose a sequence $\{\epsilon_j\}$, $\epsilon_j > 0$, such that $\epsilon_j \downarrow 0$ and $\pi_a^*(\{\theta \mid \sigma\epsilon_j\}) = 0$. Let

$$\gamma_{nj} \equiv \int_{\sigma \leq \epsilon_j} [\theta/\sigma]\pi_{an}(d\theta) \in V, \tag{3.5}$$

the inclusion following because V is a convex one. We will show that along subsequences of $\{n\}$ and $\{j\}$, as $n \to \infty$ and then $j \to \infty$,

$$\begin{aligned} \int_{\Theta_a} [(R_\theta - 1)/\sigma]\pi_{an}(d\theta) &= \ell'\gamma_{nj} + \int_{\sigma \leq \epsilon_j} [(R_\theta - 1 - \ell'\theta)/\sigma]\pi_{an}(d\theta) \\ &\quad + \int_{\epsilon_j < \sigma \leq a} [(R_\theta - 1)/\sigma]\pi_{an}(d\theta) \\ &\to \ell'\gamma + \int_{\Theta_a} [(R_\theta - 1)/\sigma]\bar{\pi}_a(d\theta), \end{aligned} \tag{3.6}$$

finite for some $\gamma \in V$, where $\bar{\pi}_a$ is π_a^* restricted to Θ_a and

$$(\gamma, \bar{\pi}_a) \neq (0, 0). \tag{3.7}$$

Now (2.2) and the fact that for each x, R_θ is bounded in $\theta \in \Theta_a$ show that the integrands in the last two terms to the right of the equality in (3.6) are bounded in θ for fixed x, hence the limits as $n \to \infty$ are found by replacing π_{an} by π_a^*. Then as $j \to \infty$, the final term goes to the final term in the line below it, and the penultimate term vanishes since its integrand is zero when $\theta = 0$. Since $\sigma(\gamma_{nj}) \leq 1$, there exist subsequences of $\{n\}$ and $\{j\}$ on which $\lim_{j\to\infty}\lim_{n\to\infty}\gamma_{nj} = \gamma$, where $\gamma \in V$ because V is closed. Also, since V is one-sided (1.2), there exists $\delta > 0$ such that $|\sum \theta_i/\sigma| \geq \delta$ for all $\theta \in V - \{0\}$. Hence $|\sum \gamma_{nji}| \geq \delta\pi_n(\sigma \leq \epsilon_j)$ for each (n, j), implying that $|\sum \gamma_i| > \delta\pi_a^*(\theta = 0)$. Equation (3.7) then follows since $\pi_a^*(\bar{\Theta}_a) = 1$, which proves (3.6).

Define C as in (2.4) with $\pi_n = \pi_{bn}$. Consider first Case A. By the discussion above (2.6), either $C = \mathscr{X}$ or limsup $\pi_{bn}(\Theta_b) = \infty$, in which case there exists a subsequence such that $\lim_{n\to\infty} \int_{\Theta_b} R_\theta(x)\pi_{bn}(d\theta) = \infty$ for all x. For fixed x, the first term on the left-hand side of (3.3) is bounded in n, as is c_n. Thus since ϕ is the weak* limit of $\{\phi_n\}$,

$$\phi = 1 \text{ a.e. } [\mu] \quad \text{for} \quad x \notin C. \tag{3.8}$$

When $C = \mathscr{X}$, $\pi_{bn}(\Theta_b)$ is bounded, so that there is a finite (hence locally finite) measure π_b on $\bar{\Theta}_b$ which is a weak limit of a subsequence of $\{\pi_{bn}\}$. By the boundedness of R_θ in θ for each x we have

$$\int_{\Theta_b} R_\theta(x)\pi_{bn}(d\theta) \to \int_{\bar{\Theta}_b} R_\theta(x)\pi_b(d\theta) < \infty \quad \text{for} \quad x \in \text{int}(C). \tag{3.9}$$

Next assume Case B obtains. The development here is similar to Ghia's (1976) Theorem 4.4. We have $C = w^{-1}(C_w)$ for some $C_w \in \mathscr{C}_w$. If $C = \mathscr{X}$ then (3.8) holds trivially. Suppose $C \neq \mathscr{X}$, and take $z \notin C$. By Case B (ii),

$$\int_{\Theta_b} R_\theta(x)\pi_{bn}(d\theta) \to \infty \quad \text{for all} \quad x \in B_z \tag{3.10}$$

along a subsequence, where $B_z \equiv \{x \mid w(x) \geq w(z)[V]\}$, so that $\phi = 1$ a.e. $[\mu]$ on B_z. Let $\{z_i\}$ be a countable set in C^c such that $\{w(z_i)\}$ is dense in C^c_w. The above discussion indicates that $\phi = 1$ a.e. $[\mu]$ on $\cup_i B_{z_i}$. But since $\{w(z_i)\}$ is dense and V is one-sided, $C^c_w = \cup_i\{y \mid y \geq w(z_i)[V]\}$. Hence (3.8) holds.

Next we show (3.9). Take $\mathrm{int}(C)$ nonempty, and $x_0 \in \mathrm{int}(C)$. Equation (2.7) shows that $w(x_0) \in \mathrm{int}(C_w)$. Since $R_\theta(x_0)$ is bounded from below in θ for any compact set $\Omega \subseteq V$, (2.3) implies that $\limsup \pi_{bn}(\Omega) < \infty$. Thus there exists a locally finite measure π_b on $\bar{\Theta}_b$ which is a vague limit of a subsequence of $\{\pi_{bn}\}$, i.e., $\pi_{bn}(\Omega) \to \pi_b(\Omega)$ for all compact $\Omega \subseteq \bar{\Theta}_b$. Take $x \in \mathrm{int}(C)$. Since $\{\theta \mid a \leq \sigma \leq i\}$ is compact and π_b is a vague limit,

$$\lim_{n\to\infty} \int_{a<\sigma\leq i} R_\theta(x)\pi_{nb}(d\theta) = \int_{a\leq\sigma\leq i} R_\theta(x)\pi_b(d\theta) \tag{3.11}$$
$$\leq \limsup_{n\to\infty} \int_{\Theta_b} R_\theta(x)\pi_{bn}(d\theta) < \infty,$$

the inequality holding by (2.4). Hence (3.11) and the Monotone Convergence Theorem show that

$$\lim_{i\to\infty} \int_{a<\sigma\leq i} R_\theta(x)\pi_b(d\theta) = \int_{\bar{\Theta}_b} R_\theta(x)\pi_b(d\theta) < \infty. \tag{3.12}$$

Now

$$\left| \int_{\Theta_b} R_\theta(x)\pi_{bn}(d\theta) - \int_{\Theta_b} R_\theta(x)\pi_b(d\theta) \right|$$
$$\leq \left| \int_{a<\sigma\leq i} R_\theta(x)\pi_{bn}(d\theta) - \int_{a\leq\sigma\leq i} R_\theta(x)\pi_b(d\theta) \right| \tag{3.13}$$
$$+ \int_{\sigma>i} R_\theta(x)\pi_{bn}(d\theta) + \int_{\sigma>i} R_\theta(x)\pi_b(d\theta).$$

Take the limit superior of (3.13) as $n \to \infty$. The absolute value on the right-hand side goes to zero by (3.11). Next let $i \to \infty$. The final term in (3.13) vanishes by (3.12). Consider

$$\lim_{i\to\infty}\lim_{n\to\infty} \int_{\sigma>i} R_\theta(x)\pi_{bn}(d\theta). \tag{3.14}$$

By (2.9), there exists $x' \in \mathrm{int}(C)$ such that $w(x') > w(x)[V]$. Clearly

$$\limsup_{n\to\infty} \int_{\sigma>i} R_\theta(x)\pi_{bn}(d\theta) \leq \sup_{\sigma>i}[R_\theta(x)/R_\theta(x')]\limsup_{n\to\infty} \int_{\sigma>i} R_\theta(x')\pi_{bn}(d\theta). \tag{3.15}$$

Since $x' \in \mathrm{int}(C)$, (2.4) shows that the final limit superior in (3.15) is finite. Case B (iii) then implies by (3.15) that the value of (3.14) is zero. Thus the right-hand side of (3.13) vanishes, proving (3.9).

For both cases we have shown by (3.6) and (3.9) that the left-hand side of (3.3) approaches $d(x;\lambda, \pi_a, \pi_b)$ of (2.15) and (2.16) where $\lambda = q\gamma$ and $\pi_a = q\bar{\pi}_a$. Thus

$$\phi = 1(0) \quad \text{as} \quad d(x; \lambda, \pi_a, \pi_b) > (<)\ c \quad \text{a.e. } [\mu] \text{ on } \mathrm{int}(C). \tag{3.16}$$

Furthermore, $q + |c| = 1$ and (3.7) imply that $(\lambda, \pi_a, c) \neq (0, 0, 0)$, so that (2.17) holds. Finally, (2.8), (2.17), (3.8) and (3.16) yield (2.14), which completes the proof of Part I.

PART II. Take $\phi \in \Phi$ and ψ essentially different from ϕ. If $r_0(\phi) \neq r_0(\psi)$, then (3.1) holds because the risk function is continuous in θ and Θ_A contains points arbitrarily close to 0. Assume $r_0(\phi) = r_0(\psi) \equiv \alpha$. We will present a sequence of finite measures $\{\pi_n\}$ on Θ_A such that

$$\lim_{n\to\infty} \int_{\Theta_A} [r_\theta(\psi) - r_\theta(\phi)]\pi_n(d\theta) > 0, \tag{3.17}$$

which implies (3.1). The main step in the proof is to show that for any test ϕ_1 with $r_0(\phi_1) = \alpha$,

$$\begin{aligned} r(\phi_1) &\equiv \lim_{n\to\infty}\left\{\int_{\Theta_A} r_\theta(\phi_1)\pi_n(d\theta) - (1-\alpha)\int_{\Theta_a} h^{-1}(\theta)\pi_n(d\theta)\right\} \\ &= \begin{cases} \infty & \text{if } \mu(\{x \mid \phi(x) < 1\} \cap C^c) > 0 \\ \int_{\mathscr{X}} (1-\phi_1(x))\, d(x; \lambda, \pi_a, \pi_b) f(x; 0)\, dx & \text{otherwise,} \end{cases} \end{aligned} \tag{3.18}$$

where the lower expression is bounded from below but may be $+\infty$. By definition (2.14), $r(\phi) \le c$, hence the limit in (3.17) equals $r(\psi) - r(\phi)$. If $r(\psi) = \infty$, then (3.17) holds immediately. If $r(\psi) < \infty$, then by (3.18)

$$r(\psi) - r(\phi) = \int_C (\phi - \psi)\, d(x; \lambda, \pi_a, \pi_b) f(x; 0)\, dx < \infty. \tag{3.19}$$

For this latter possibility to obtain, it must be that $\psi = 1$ a.e. $[\mu]$ on C^c. Hence, since $r_0(\phi) = r_0(\psi)$,

$$\int_C \phi(x) f(x; 0)\, dx = \int_C \psi(x) f(x; 0)\, dx.$$

Thus $r(\psi) - r(\phi) > 0$ by the Neyman-Pearson Lemma and Equations (3.18) and (2.17), which proves (3.1).

It remains to show (3.18). Let π_n be the finite measure on Θ_A defined by

$$\begin{aligned} \pi_n(d\theta) = h(\theta)[n\delta(d\theta; \lambda/n) &+ \sigma^{-1}\pi_a(d\theta) I_{\{1/n<\sigma\le a\}} \\ &+ \pi_b(d\theta) I_{\{a\le\sigma<n\}} + \textstyle\sum m_{in}\delta(d\theta; \theta_{in})], \end{aligned} \tag{3.20}$$

where $\delta(d\theta; \theta')$ represents the measure putting mass one at θ', and the θ_{in}'s, which will be explained below, represent θ's on the outskirts of the parameter space. We consider the two cases separately.

First look at Case A. If C is empty, then ϕ is admissible immediately. Thus by (2.6) we can take $C = \mathscr{X}$. Since π_b is finite and Θ is convex, there are at most a countable number of points in $\bar{\Theta} - \Theta$ which enjoy positive π_b mass. Call these points $\{\theta_i\}$, $i \in I$, and for each i let $\{\theta_{in}\}$ be a sequence such that $\theta_{in} \to \theta_i$ and (2.5) holds. Let $m_{in} = \pi_b(\{\theta_i\})$. Write

$$\begin{aligned} r(\phi_1) = \lim_{n\to\infty} \int_{\mathscr{X}} (1-\phi_1)\{n(R_{\lambda/n}(x) - 1) &+ \int_{1/n<\sigma\le a} [(R_\theta(x) - 1)/\sigma]\pi_a(d\theta) \\ &+ \int_{a\le\sigma<n} R_\theta \pi_b(d\theta) + \textstyle\sum_i m_{in} R_{\theta_{in}}(x)\} f(x; 0)\, dx. \end{aligned} \tag{3.21}$$

Interchanging the limit and integral above is valid. Use the Dominated Convergence Theorem and (2.3) for the first two terms and the Monotone Convergence Theorem and (2.5) for the others, and note that only the positive terms may have an infinite integral. Thus (3.21) implies (3.18).

Next take Case B. Let C_w be the set in Case B (i). Since C_w is convex and decreasing $[V]$, there exist sequences $\{\xi_i\} \subseteq V$ and $\{t_i\} \subseteq R$ indexed by $i \in I$, a subset of the whole numbers, such that

$$C_w = \cap_{i\in I} \{y \in \mathscr{W} \mid \xi_i' y \le t_i\}.$$

In (3.21), take $\theta_{in} = n\xi_i$ and

$$m_{in} = \alpha(n\xi_i)\exp(-nt_i)2^{-i}. \tag{3.22}$$

By (2.12) and (3.22),

$$\lim_{n\to\infty} m_{in} R_{n\xi_i}(x) = \begin{cases} \infty & \text{for any } \; x \notin w^{-1}(C_w) \equiv C \\ 0 & \text{for any } \; x \in \operatorname{int}(w^{-1}(C_w)) \equiv \operatorname{int}(C). \end{cases} \tag{3.23}$$

Equation (2.3) shows that Fatou's Lemma can be used to find a lower bound for the limit inferior of the integral in (3.21), hence (3.23) gives that the upper part of (3.18) is true. If $\phi_1 = 1$ a.e. $[\mu]$ on C^c, then we can interchange the limit and integral in (3.18) as for Case A, using (2.13) and the Dominated Convergence Theorem on the final term in the braces, which completes the validation of (3.18).

The proof is finished.

4. Remarks on the cases. The following problems mentioned in the Introduction are Case A: combining scaled χ^2 tests with alternatives $\sigma_i^2 > 1$, combining scaled F tests, combining tests on correlation coefficients, the invariance-reduced problems with alternative $\sum > I$ or $\sum_1 > \sum_2$, and the canonical covariates problem. The other problems are Case B.

A useful generalization of the exponential density for Case B is the "almost exponential" density which has $h = 1$, $\Theta = V$, and

$$R_\theta(x) = a(\theta)b(x;\theta)\exp\{\theta' w(x)\} \tag{4.1}$$

for continuous functions a and b where

$$0 < i(x) \equiv \inf_{\theta\in\Theta} b(x;\theta) \le \sup_{\theta\in\Theta} b(x;\theta) \equiv s(x) < \infty. \tag{4.2}$$

When $b = 1$, f is exponential. By (4.1) and (4.2),

$$\begin{aligned} &\lim_{n\to\infty} \int_{\Theta_A} R_\theta(x)\pi_n(d\theta) = 0\ (\infty) \quad \text{if and only if} \\ &\lim_{n\to\infty} \int_{\Theta_A} a(\theta)\exp\{\theta' w(x)\}\pi_n(d\theta) = 0\ (\infty). \end{aligned} \tag{4.3}$$

Thus C in (2.4) equals closure $(w^{-1}(C'_w))$, where

$$\begin{aligned} C'_w &= \left\{ y \in \mathscr{W} \,\middle|\, \limsup_{n\to\infty} \int_{\Theta_A} a(\theta)\exp(\theta' y)\pi_n(d\theta) < \infty \right\} \\ &= \cup_{m=1}^\infty \cap_{k=1}^\infty \cap_{n=k}^\infty \left\{ y \in \mathscr{W} \,\middle|\, \int_{\Theta_A} a(\theta)\exp(\theta' y)\pi_n(d\theta) < m \right\}. \end{aligned} \tag{4.4}$$

Clearly the final set in braces is convex and decreasing $[V]$ for each (n, m), and since these properties persist through intersections and increasing unions, closure $(C'_w) \in \mathscr{C}_w$. Thus if every such C'_w satisfies closure $(w^{-1}(C'_w)) = w^{-1}$ (closure (C'_w)), then Case B(i) holds. To show (ii), take x such that $x \notin C$, and x' with $w(x') \ge w(x)[V]$. By (4.3), $\lim_{n\to\infty} \int_{\Theta_A} R_\theta(x)\pi_n(d\theta) = \infty$ on a subsequence, hence (4.3) and the definition (1.4) implies that the same holds for x', proving $x' \notin C$. For (iii), take x and x' with $w(x') > w(x)[V]$. By (4.2),

$$\sup_{\sigma>\iota} R_\theta(x)/R_\theta(x') \le [i(x)/s(x')]\sup_{\sigma>\iota}\exp[-\theta'\{w(x') - w(x)\}]. \tag{4.5}$$

Now $\inf_{\sigma>\iota}\theta'(w(x') - w(x)) \ge \inf_{\sigma>\iota}\sigma(\theta)\, \inf_{\sigma(\gamma)=1}\gamma'(w(x') - w(x)) = i\delta$ for $\delta > 0$ since $\{\gamma \in V \mid \sigma(\gamma) = 1\}$ is compact and the inequality (1.4) is strict. Hence (2.11) follows from (4.5). Equation (2.12) holds with $\alpha^{-1}(\theta) = a(\theta)$ by (4.3), and (2.13) is satisfied if $\int s(x)f(x;0)\,dx < \infty$.

The problems on combining independent noncentral χ^2 or F tests fall into the above framework. Other examples include the invariance-reduced problem in Marden and Perlman (1980), which tests whether the mean vector of a normal is zero when it is known that some of its components are zero; the problem in Eaton and Kariya (1975) which tests the same hypotheses but has extra observations on the unknown components; and an invariance-reduced problem which tests for a certain type of multiplicative interaction in the two-way analysis of variance; see Johnson and Graybill (1972) for the set-up. Another interesting application is to the problem of testing $\rho = 0$ versus $\rho > 0$ based on a sample $\{X_i\}$ of independent bivariate normal variables with means 0, variances 1 and correlation ρ. Though this is an exponential situation, the Birnbaum-Matthes-Truax-Eaton results cannot be invoked since the natural parameter is two-dimensional but ranges over only a one-dimensional curve in $\mathbb{R}^2$. However, the density does satisfy (4.1) and (4.2) with $\theta = \rho^2/(1-\rho^2)$ and $w(\{x_i\}) = \sum_i (x_{i1} - x_{i2})^2/2$, hence is a one-dimensional "almost exponential" density. See Marden (1981).

The following type of density arises in some invariance-reduced problems, such as the multivariate analysis of variance problem and that with alternative $\sum < I$. Let $\Theta = V = \{\theta \in \mathbb{R}^p \mid \theta_1 \geq \cdots \geq \theta_p \geq 0\}$, $\mathcal{O}$ be the group of $p \times p$ orthogonal matrices, ν be Haar probability measure on $\mathcal{O}$, and $\Delta(z)$ be the $p \times p$ diagonal matrix with $z \in \mathbb{R}^p$ containing the diagonal elements. Suppose there exist continuous functions a and b such that

$$R_\theta(x) = a(\theta) \int_{\mathcal{O}} b(x; \theta, \Gamma) \exp\{\Delta(w(x))\Gamma\Delta(\theta)\Gamma'\} d\Gamma,$$

where (4.3) holds with b there replaced by $\dot{b}$ here, and the infimum and supremum are taken over $\theta \in \Theta$ and $\Gamma \in \mathcal{O}$. Then, as in the "almost exponential" situation, it can be shown that Case B obtains. See Marden (1980a).

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