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TESTS WITH PARABOLIC BOUNDARY FOR THE DRIFT OF A WIENER PROCESS[1]

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We consider sequential tests with parabolic stopping boundary for the hypothesis that the drift of a Wiener process is positive. Formulas are derived for the power and expected sample size. The performance of minimax parabolic tests is compared with that of other tests considered in the literature.

1. Introduction. Let $Y(t)$ be a Wiener process with unknown drift θ. We consider the problem of testing the hypothesis $H_0: \theta \geq 0$ against the alternative $H_1: \theta < 0$. We shall investigate sequential procedures with parabolic boundary, in which the process is observed up to the stopping time τ,

$$\tau = \inf\{t \mid |Y(t)| = \lambda \sqrt{a - t}\},$$

where $a > 0$ and $\lambda > 0$. This family of tests contains the fixed sample size test and the sequential probability ratio test as special cases corresponding to $\lambda = \infty$ and $\lambda = 0$. We shall be interested in minimizing the maximum expected sample size

$$\sup_\theta E^\theta \tau$$

among all tests with prescribed error probability α at the given values $\theta = \pm\theta_0$:

$$P^{\theta_0}\{Y(\tau) < 0\} = P^{-\theta_0}\{Y(\tau) > 0\} = \alpha.$$

It is well-known that the test of Wald, which accepts H_0 if $Y(\tau) = b$, where $\tau = \inf\{t \mid |Y(t)| = b\}$, has smaller expected sample size at $\pm\theta_0$ than any other test with the same error probabilities at $\pm\theta_0$. But it is also known that the expected sample size is much less favorable for θ between $-\theta_0$ and θ_0. This effect becomes more and more pronounced as the error probability α approaches 0. Indeed, an easy computation shows that, for the Wald test with error probability α at $\theta = \pm\theta_0$, we have, as $\alpha \to 0$,

$$E^{\theta_0}\tau \sim -(\log \alpha)/(2\theta_0^2)$$

but

$$E^0\tau \sim (\log \alpha)^2/(4\theta_0^2).$$

For comparison, the fixed sample size plan with the same error probabilities has sample size

$$t \sim -(2 \log \alpha)/\theta_0^2.$$

Thus while the test of Wald uses asymptotically only a fourth as many observations as the fixed sample size test at $\theta = \theta_0$, it uses $-(\log \alpha)/8$ times as many at $\theta = 0$. These considerations make it plausible that one should be able to modify Wald's test in such a way that a modest sacrifice in performance at $\theta = \pm\theta_0$ leads to appreciable gains for intermediate θ's.

Anderson (1960) attacked this problem by considering trapezoidal continuation regions,

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corresponding to stopping times of the form

$$\tau = t_0 \wedge (\inf\{t \,|\, |Y(t)| = c + dt\}),$$

the hypothesis H_0 being accepted if $Y(\tau) \geq 0$. Using an explicit series expansion for the exit density, he adjusted the three parameters t_0, c, and d numerically so as to minimize $E^0\tau = \sup_\theta E^\theta\tau$ subject to the condition $P^{\theta_0}\{x(\tau) < 0\} = \alpha$.

The problem has been taken up again by Lai (1973), who showed that, as $\alpha \to 0$, the asymptotic form of the continuation region with minimax expected sample size is essentially triangular.

It would be of interest to have information on the performance of other test regions. Unfortunately, for most regions it is at present impossible to compute the probabilities and expectations of interest other than by simulation or brute force numerical approximation.

In the present paper we carry out these computations for parabolic regions of the type described above. We do this by obtaining numerically convenient expansions of $P^\theta\{Y(\tau) > 0\}$ and $E^\theta\tau$ in terms of certain Hermite functions. The plan of the paper is as follows. In Section 2 we briefly recapitulate the needed properties of the Hermite functions which we will use. In Section 3 we define a family of martingales associated with these functions and obtain from them explicit formulas for $E^0W^n(\tau)$ and $E^0\tau^{n/2}$. Applying these formulas to the Radon-Nikodym derivative dP^θ/dP^0 then yields the desired series expansions of $P^\theta\{Y(\tau) > 0\}$ and $E^\theta\tau$ (Section 4). These expansions are then applied in Section 5 to the construction of sequential tests with parabolic boundary, whose performance is compared numerically with the fixed sample size test, the test of Wald, and the trapezoidal test of Anderson.

NOTE ADDED IN PROOF. Daniels (1982) has used the method of images to construct regions of the same general shape as those considered here for which the exit density can be given explicitly. His numerical results are quite similar to those for parabolas.

2. A class of Hermite functions. In this section we give a simple probabilistic definition of certain Hermite functions and briefly recall some of their properties. The treatment is similar to the classical treatment of Hermite functions (see Lebedev, 1965) and proofs will be omitted.

DEFINITION 1. Let Z be a standard $N(0, 1)$ random variable. For $n = 0, 1, \cdots$ let

$$k_n(x) = E(Z + x)^n, \quad \ell_n(x) = E\{(Z + x)^n \tfrac{1}{2}\operatorname{sgn}(Z + x)\}. \tag{1}$$

Notice that k_n and ℓ_n are of opposite parity: k_{2m} and ℓ_{2m+1} are even, while k_{2m+1} and ℓ_{2m} are odd.

EXAMPLES.

$$\begin{aligned}
k_0(x) &= 1, & \ell_0(x) &= \Phi(x) - \frac{1}{2},\\
k_1(x) &= x, & \ell_1(x) &= x\left\{\Phi(x) - \frac{1}{2}\right\} + \varphi(x),\\
k_2(x) &= x^2 + 1, & \ell_2(x) &= (x^2 + 1)\left\{\Phi(x) - \frac{1}{2}\right\} + x\varphi(x),\\
k_3(x) &= x^3 + 3x, & \ell_3(x) &= (x^3 + 3x)\left\{\Phi(x) - \frac{1}{2}\right\} + (x^2 + 2)\varphi(x).
\end{aligned} \tag{2}$$

Here φ and Φ are the standard normal density and distribution functions, $\varphi(x) = \exp(-x^2/2)/\sqrt{2\pi}$, $\Phi(x) = \int_{-\infty}^x \varphi(u)\,du$.

Directly from the definitions, we find the generating functions for k_n and ℓ_n.

PROPOSITION 1.

$$(3) \quad \sum_{n=0}^{\infty} k_n(x)\frac{s^n}{n!} = \exp(xs + s^2/2), \quad \sum_{n=0}^{\infty} \ell_n(x)\frac{s^n}{n!} = \left\{\Phi(x+s) - \frac{1}{2}\right\}\exp(xs + s^2/2)$$

for $-\infty < s < \infty$.

Multiplying both sides by $\exp(x^2/2)$, differentiating n times with respect to s, and setting $s = 0$, we get alternative expressions for k_n and ℓ_n.

PROPOSITION 2.

$$(4) \qquad k_n(x) = e^{-x^2/2}\frac{d^n}{dx^n}e^{x^2/2}, \quad \ell_n(x) = e^{-x^2/2}\frac{d^n}{dx^n}\left[\left\{\Phi(x) - \frac{1}{2}\right\}e^{x^2/2}\right].$$

Comparing these expressions with those for the classical Hermite functions

$$H_n(x) = (-1)^n\frac{d^n}{dx^n}e^{-x^2}, \quad H_{-n-1}(x) = \frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}\left\{e^{x^2}\int_x^\infty e^{-u^2}\,du\right\}$$

we find that

$$k_n(x) = \frac{1}{(i\sqrt{2})^n}H_n\left(\frac{ix}{\sqrt{2}}\right), \quad \ell_n(x) = k_n(x) + \frac{(-1)^n 2^{n/2} n!}{\sqrt{\pi}}e^{-x^2/2}H_{-n-1}\left(\frac{x}{\sqrt{2}}\right).$$

Differentiating the generating functions and expanding, we get the important recurrence relations for k_n and ℓ_n.

PROPOSITION 3.

$$(5) \qquad \begin{aligned} k_{n+1}(x) &= xk_n(x) + nk_{n-1}(x), \\ \ell_{n+1}(x) &= x\ell_n(x) + n\ell_{n-1}(x), \quad n \geq 1. \end{aligned}$$

COROLLARY 1. *Let* m_n, $n \geq 0$ *be the sequence of polynomials defined by* $m_0(x) = 0$, $m_1(x) = 1$,

$$m_{n+1}(x) = xm_n(x) + nm_{n-1}(x), \quad n \geq 1.$$

Then

$$\ell_n(x) = k_n(x)\left\{\Phi(x) - \frac{1}{2}\right\} + m_n(x)\varphi(x).$$

It is interesting to note that $m_n(x)/k_n(x)$ is the nth partial quotient in the well-known continued fraction expansion of $\{1 - \Phi(x)\}/\varphi(x)$,

$$\frac{1|}{|x} + \frac{1|}{|x} + \frac{2|}{|x} + \cdots + \frac{n-1|}{|\ x} = \frac{m_n(x)}{k_n(x)};$$

see Wall (1948, page 358).

The recurrence relations (5) are important for purposes of practical computation.

3. Moments of the exit time. It is possible to obtain eigen-function expansions for the exit density from parabolic regions of the kind we are considering (see Sweet and Hardin, 1970), but these are not suitable for practical computation. On the other hand, it has long been known that one can obtain explicit formulas for the moments of the exit time from parabolas (see Shepp, 1967). In this section we give a simple direct derivation of such formulas using martingales associated with the Hermite functions. While the present paper was in preparation we learned from Prof. A. A. Novikov that this idea had already been employed by him (Novikov, 1971), but our approach is different.

In this section and the following it will be simplest to consider parabolas $|y| = \lambda\sqrt{-t}$, $t \le 0$, with vertex at the origin, and to let the Wiener process under study start at a general point (t_0, y_0), $t_0 < 0$. We consider only the Wiener process without drift. The path functions will be denoted by $W(t)$, $t \ge t_0$ and the associated probability measure by $P_{(t_0,y_0)}$. The σ-algebra generated by $\{X(s) \mid t_0 \le s \le t\}$ will be denoted as usual by F_t.

We begin with a simple method for obtaining martingales of the Wiener process.

LEMMA. *Let Z be a standard $N(0, 1)$ random variable and let g be a measurable function such that $E|g(\mu + \sigma Z)| < \infty$ for all μ, σ. Define $G(\mu, \sigma) = Eg(\mu + \sigma Z)$. Then*

$$M(t) = G(W(t), \sqrt{-t}), \quad t_0 \le t \le 0$$

is a $P_{(t_0,y_0)}$-martingale.

PROOF. For the Wiener process starting at (t, y), $t_0 \le t \le 0$, the random variable $W(0)$ has the same distribution as $y + \sqrt{-t}\,Z$. Therefore we have, using the Markov property,

$$E_{(t_0,y_0)}\{g(W(0)) \mid F_t\} = E_{(t, W(t))}g(W(0)) = G(W(t), \sqrt{-t}).$$

This shows that $M(t)$, $t_0 \le t \le 0$, is a uniformly integrable $P_{(t_0,y_0)}$-martingale. □

We only need the Lemma in the simple cases $g(x) = x^n$ and $g(x) = x^n(\operatorname{sgn} x)/2$.

PROPOSITION 4. *The stochastic processes*

$$K_n(t) = (-t)^{n/2}k_n(W(t)/\sqrt{-t}), \quad L_n(t) = (-t)^{n/2}\ell_n(W(t)/\sqrt{-t})$$

with $t_0 \le t \le 0$, $n = 0, 1, \cdots$, are $P_{(t_0,y_0)}$-martingales.

PROOF. Take $g(x) = x^n$ in the Lemma. Then

$$G(\mu, \sigma) = E(\mu + \sigma Z)^n = \sigma^n E\left(\frac{\mu}{\sigma} + Z\right)^n = \sigma^n k_n\left(\frac{\mu}{\sigma}\right),$$

so $K_n(t) = G(W(t), \sqrt{-t})$ is a martingale. In the same way, taking $g(x) = x^n(\operatorname{sgn} x)/2$, we see that $L_n(t)$ is a martingale. □

Now we come to the main result of this section. Consider the parabola $|y| = \lambda\sqrt{-t}$, $\lambda > 0$. Let (t, y) be a point inside the parabola, $|y| < \lambda\sqrt{-t}$, and let τ be the first exit time for the Wiener process starting at (t, y),

$$\tau = \inf\{s \mid t \le s, \ |Y(s)| = \lambda\sqrt{-s}\}.$$

THEOREM 1. *Let $\mu = y/\sqrt{-t}$ with $|\mu| < \lambda$. The following hold for $n = 0, 1, \cdots$:*

(i) $$E_{(t,y)}(-\tau)^n = (-t)^n k_{2n}(\mu)/k_{2n}(\lambda)$$

(ii) $$E_{(t,y)}(-\tau)^{n+1/2} = (-t)^{n+1/2}\ell_{2n+1}(\mu)/\ell_{2n+1}(\lambda)$$

(iii) $$E_{(t,y)}(-\tau)^n \operatorname{sgn} W(\tau) = (-t)^n \ell_{2n}(\mu)/\ell_{2n}(\lambda)$$

(iv) $$E_{(t,y)}(-\tau)^{n+1/2} \operatorname{sgn} W(\tau) = (-t)^{n+1/2} k_{2n+1}(\mu)/k_{2n+1}(\lambda).$$

PROOF. Since the stopping time τ is bounded, $\tau \le 0$, we can apply the optional stopping theorem $E_{(t,y)}M(\tau) = M(t)$. For the martingale $K_m(s)$ we get

$$E_{(t,y)}(-\tau)^{m/2}k_m(W(\tau)/\sqrt{-\tau}) = (-t)^{m/2}k_m(y/\sqrt{-t}).$$

Using $W(\tau)/\sqrt{-\tau} = \lambda \operatorname{sgn} W(\tau)$ and $k_m(-x) = (-1)^m k_m(x)$ we see that $k_m(W(\tau)/\sqrt{-\tau}) = k_m(\lambda)\{\operatorname{sgn} W(\tau)\}^m$. This proves (i) and (iv). The same argument applied to $L_m(s)$, using $\ell_m(-x) = (-1)^{m+1}\ell_m(x)$, proves (ii) and (iii). □

REMARK. Formula (i) has an interesting geometrical interpretation in the case $n = 1$. We have

$$E_{(t,y)}(-\tau) = \frac{y^2 - t}{1 + \lambda^2},$$

or, for the elapsed time $(\tau - t)$

$$E_{(t,y)}(\tau - t) = \frac{\lambda^2(-t) - y^2}{1 + \lambda^2}.$$

But the horizontal distance $(t^* - t)$ from (t, y) to the parabola, that is, the elapsed time until the moving boundary crosses the starting point, is just

$$t^* - t = \frac{\lambda^2(-t) - y^2}{\lambda^2}.$$

Thus for all points inside the parabola we have

$$E_{(t,y)}(\tau - t) = \frac{\lambda^2}{1 + \lambda^2}(t^* - t).$$

In other words, for all points inside the parabola, the expected time until exit for a Wiener process is a constant multiple of the time until exit for a motionless point.

Since $W(\tau) = \lambda\sqrt{-\tau} \operatorname{sgn} W(\tau)$ the following corollary follows immediately from the theorem.

COROLLARY 2.

(a) $$E_{(t,y)}W^n(\tau) = \lambda^n(-t)^{n/2}k_n(\mu)/k_n(\lambda)$$

(b) $$E_{(t,y)}W^n(\tau) \operatorname{sgn} W(\tau) = \lambda^n(-t)^{n/2}\ell_n(\mu)/\ell_n(\lambda).$$

PROOF. If n is even, then $W^n(\tau) = \lambda^n(-\tau)^{n/2}$; if odd, then $W^n(\tau) = \lambda^n(-\tau)^{n/2} \operatorname{sgn} W(\tau)$. In the first case (a) follows from (i), in the second from (iv). Similarly (b) follows from (ii) and (iii). □

Since $1_{\{W(\tau)>0\}} = (1 + \operatorname{sgn} W(\tau))/2$, we get the following result, which we record for future use.

COROLLARY 3.

$$E_{(t,y)}W^n(\tau)1_{\{W(\tau)>0\}} = \frac{\lambda^2}{2}(-t)^{n/2}\left\{\frac{k_n(\mu)}{k_n(\lambda)} + \frac{\ell_n(\mu)}{\ell_n(\lambda)}\right\}.$$

4. Expectations for the Wiener process with drift. Now we consider a Wiener process $Y(t)$ with drift θ. To evaluate tests with parabolic boundary we need to be able to compute $P^{\theta}_{(t,y)}\{Y(\tau) > 0\}$ and $E^{\theta}_{(t,y)}\tau$. We do this by applying the results of the previous section to the Radon-Nikodym derivative dP^{θ}/dP^0.

It is well known that, for any stopping time $\tau, P^{\theta}_{(t,y)}$ is absolutely continuous with respect to $P^0_{(t,y)}$ on the trace of the σ-algebra F_τ on the event $\{\tau < \infty\}$, and that the Radon-Nikodym derivative is given by

$$\frac{dP^{\theta}_{(t,y)}}{dP^{0}_{(t,y)}} = \frac{\exp\{Y(\tau)\theta - \tau\theta^2/2\}}{\exp(y\theta - t\theta^2/2)}; \tag{6}$$

see Robbins and Siegmund (1973), and Liptser and Shiryayev (1977, Section 7.2). In our case, since τ is bounded, (6) holds on all of F_τ. In particular, if $f(s, x)$ is measurable, with $E_{(t,y)}|f(\tau, Y(\tau))| < \infty$, then we can compute $E^{\theta}_{(t,y)} f(\tau, Y(\tau))$ as follows:

$$\begin{aligned} E^{\theta}_{(t,y)} f(\tau, Y(\tau)) &= E^{0}_{(t,y)} f(\tau, Y(\tau))\, dP^{\theta}_{(t,y)}/dP^{0}_{(t,y)} \\ &= \exp(-y\theta + t\theta^2/2) E^{0}_{(t,y)} f(\tau, Y(\tau)) \exp\{Y(\tau)\theta - \tau\theta^2/2\}. \end{aligned}$$

Using this makes it straightforward to compute the desired expectations.

By a happy coincidence, dP^{θ}/dP^{0} can be expanded in the same functions which occur in the moments of τ.

LEMMA.

$$\exp\{Y(\tau)\theta - \tau\theta^2/2\} = \sum_{n=0}^{\infty} \frac{k_n(\lambda)}{n!} \frac{\theta^n Y^n(\tau)}{\lambda^n}.$$

PROOF. By (3)

$$\exp\left[\left\{\frac{Y(\tau)}{\sqrt{-\tau}}\right\}(\theta\sqrt{-\tau}) + (\theta\sqrt{-\tau})^2/2\right] = \sum_{n=0}^{\infty} k_n\left(\frac{Y(\tau)}{\sqrt{-\tau}}\right)\frac{(\theta\sqrt{-\tau})^n}{n!}$$

$$= \sum_{n=0}^{\infty} k_n(\lambda)\{\operatorname{sgn} Y(\tau)\}^n \frac{\theta^n \lambda^n (-\tau)^{n/2}}{\lambda^n n!} = \sum_{n=0}^{\infty} \frac{k_n(\lambda)}{n!}\frac{\theta^n Y^n(\tau)}{\lambda^n}. \qquad \square$$

THEOREM 2. *Let τ be as above, $t < 0$, $\mu = y/\sqrt{-t}$, $|\mu| < \lambda$. Then*

$$P^{\theta}_{(t,y)}\{Y(\tau) > 0\} = \frac{1}{2} + \frac{1}{2}\exp(-y\theta + t\theta^2/2) \sum_{n=0}^{\infty} \frac{k_n(\lambda)\,\ell_n(\mu)}{\ell_n(\lambda)} \frac{(\theta\sqrt{-t})^n}{n!}. \tag{7}$$

In particular

$$P^{\theta}_{(t,0)}\{Y(\tau) > 0\} = \frac{1}{2} + \frac{1}{2\sqrt{2\pi}}\exp(t\theta^2/2) \sum_{n=0}^{\infty} \frac{k_{2n+1}(\lambda)}{\ell_{2n+1}(\lambda)} \frac{2^n n!}{(2n+1)!}(\theta\sqrt{-t})^{2n+1}. \tag{8}$$

PROOF.

$$\begin{aligned} P^{\theta}_{(t,y)}\{Y(\tau) > 0\} &= E^{\theta}_{(t,y)} 1_{\{Y(\tau)>0\}} \\ &= \exp(-y\theta + t\theta^2/2) E^{0}_{(t,y)} \exp\{Y(\tau)\theta - \tau\theta^2/2\} 1_{\{Y(\tau)>0\}}. \end{aligned} \tag{9}$$

By the Lemma, Corollary 3 and (3) we have

$$\begin{aligned} E^{0}_{(t,y)} \exp\{Y(\tau)\theta - \tau\theta^2/2\} 1_{\{Y(\tau)>0\}} &= \sum_{n=0}^{\infty} \frac{k_n(\lambda)}{n!}\frac{\theta^n}{\lambda^n} E^{0}_{(t,y)} Y^n(\tau) 1_{\{Y(\tau)>0\}} \\ &= \frac{1}{2}\sum_{n=0}^{\infty} \frac{k_n(\lambda)}{n!}\theta^n(-t)^{n/2}\frac{1}{2}\left\{\frac{k_n(\mu)}{k_n(\lambda)} + \frac{\ell_n(\mu)}{\ell_n(\lambda)}\right\} \\ &= \frac{1}{2}\exp(\mu\theta\sqrt{-t} - t\theta^2/2) + \frac{1}{2}\sum_{n=0}^{\infty} \frac{k_n(\lambda)\,\ell_n(\mu)}{\ell_n(\lambda)}\frac{(\theta\sqrt{-t})^n}{n!}. \end{aligned}$$

TABLE 1a
Parameters of tests with
$P^{1}_{(t,0)}\{Y(\tau) < 0\} = .05.$

λ	$-t$	$E^0_{(t,0)}(\tau - t)$
0	∞	216.7
.5	1009.2	201.8
.9268	417.0	192.7
1.4	303.6	210.0
2.0	275.5	220.4
∞	270.6	270.6

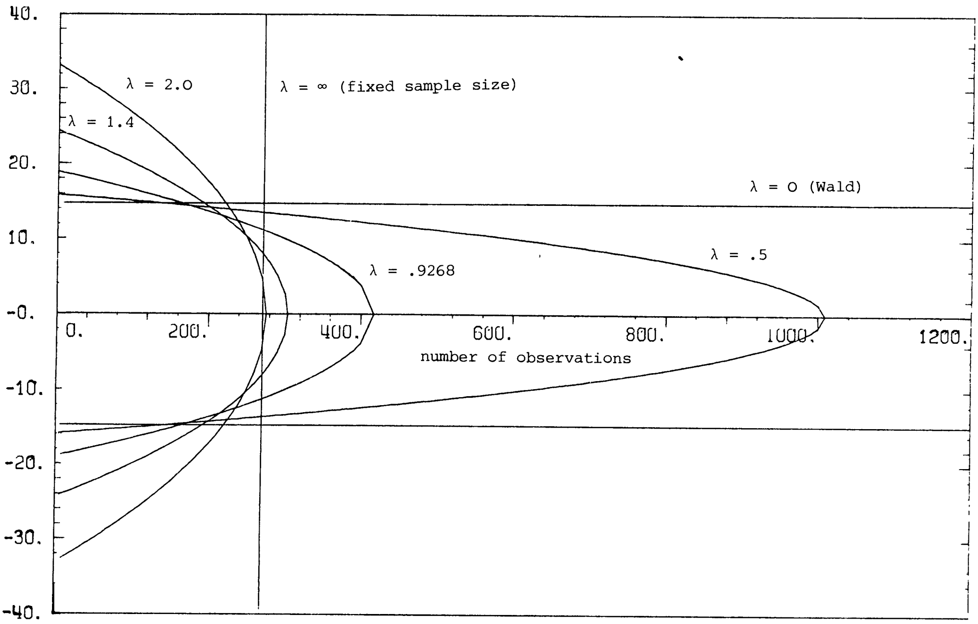

FIG. 1a: *Boundaries of tests with 5% error probability at $\theta = \pm .1$*

Substituting this into (9) we get (7). When $y = 0$ so that $\mu = 0$, we have $\ell_{2n}(\mu) = 0$. From (5) we see by induction that $\ell_{2n+1}(0) = 2^n n!/\sqrt{2\pi}$. Thus (8) follows from (7). □

THEOREM 3.

$$E^\theta_{(t,y)}(-\tau) = (-t)\exp(-y\theta + t\theta^2/2)\sum_{n=0}^{\infty}\frac{k_n(\lambda)k_{n+2}(\mu)}{k_{n+2}(\lambda)}\frac{(\theta\sqrt{-t})^n}{n!}. \tag{10}$$

In particular

$$E^\theta_{(t,0)}(-\tau) = (-t)\exp(t\theta^2/2)\sum_{n=0}^{\infty}\frac{k_{2n}(\lambda)}{k_{2n+2}(\lambda)}\frac{(2n+1)}{2^n n!}(\theta\sqrt{-t})^n. \tag{11}$$

PROOF.

$$E^\theta_{(t,y)}(-\tau) = \exp(-y\theta + t\theta^2/2)E^0_{(t,y)}(-\tau)\exp\{Y(\tau)\theta - \tau\theta^2/2\}.$$

As in the proof of Theorem 2,

$$E^0_{(t,y)}(-\tau)\exp\{Y(\tau)\theta - \tau\theta^2/2\} = \sum_{n=0}^{\infty}\frac{k_n(\lambda)}{n!}\frac{\theta^n}{\lambda^n}E^0_{(t,y)}Y^n(\tau)(-\tau).$$

TABLE 2a
Error probabilities $P^{\theta}_{(t,0)}\{Y(\tau) < 0\}$, as function of θ for the tests in Table 1a.

θ	λ					
	0	.5	.9268	1.4	2.0	∞
.00	.500	.500	.500	.500	.500	.500
.05	.187	.193	.200	.204	.205	.205
.10	.050	.050	.050	.050	.050	.050
.15	.012	.011	.009	.007	.007	.007
.20	.003	.002	.0013	.0007	.0005	.0005

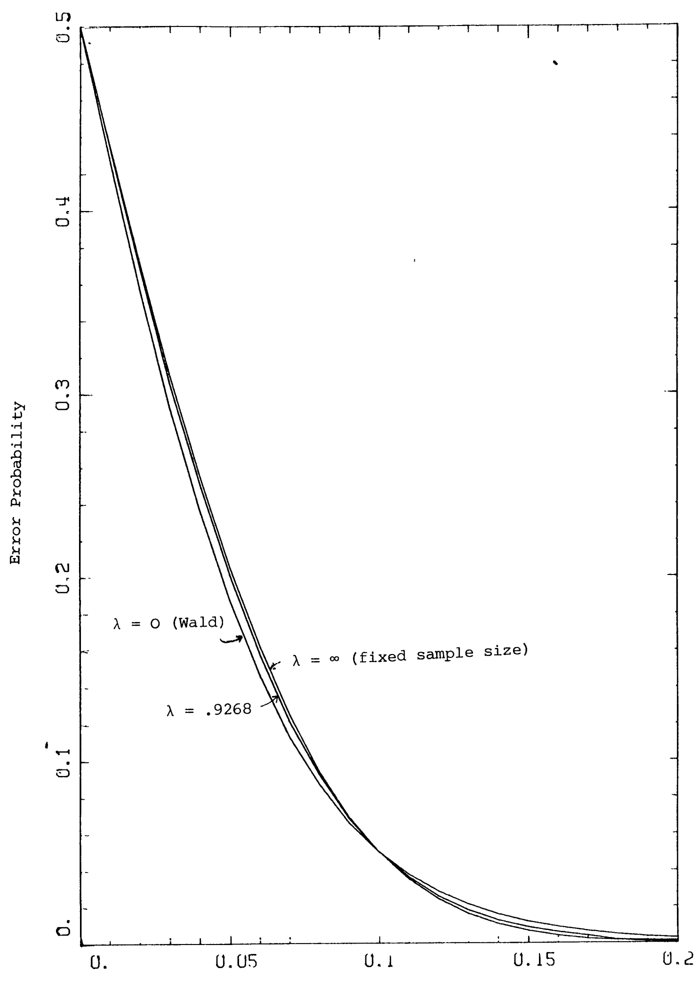

FIG. 2a: *Error probability as function of θ for tests with* 5% *error at* $\theta = \pm .1$

But

$$E^0_{(t,y)} Y^n(\tau)(-\tau) = E^0_{(t,y)}\{Y^{n+2}(\tau)/\lambda^2\} = \lambda^n(-t)^{n/2+1}\frac{k_{n+2}(\mu)}{k_{n+2}(\lambda)}.$$

Combining, we get (10). Setting $\mu = 0$ and using $k_{2n+2}(0) = (2n+2)!/2^{n+1}(n+1)!$, we get (11). □

TABLE 3a
Expected sample size $E^{\theta}_{(t,0)}(\tau - t)$ as function of θ for the tests in Table 1a.

θ	λ					
	0	.5	.9268	1.4	2.0	∞
.00	216.7	201.9	192.7	201.0	220.4	270.6
.05	184.6	177.6	175.2	186.2	207.3	270.6
.10	132.5	133.2	138.9	153.9	177.6	270.6
.15	95.8	98.4	106.6	122.6	146.9	270.6
.20	73.2	76.0	84.2	99.4	122.5	270.6

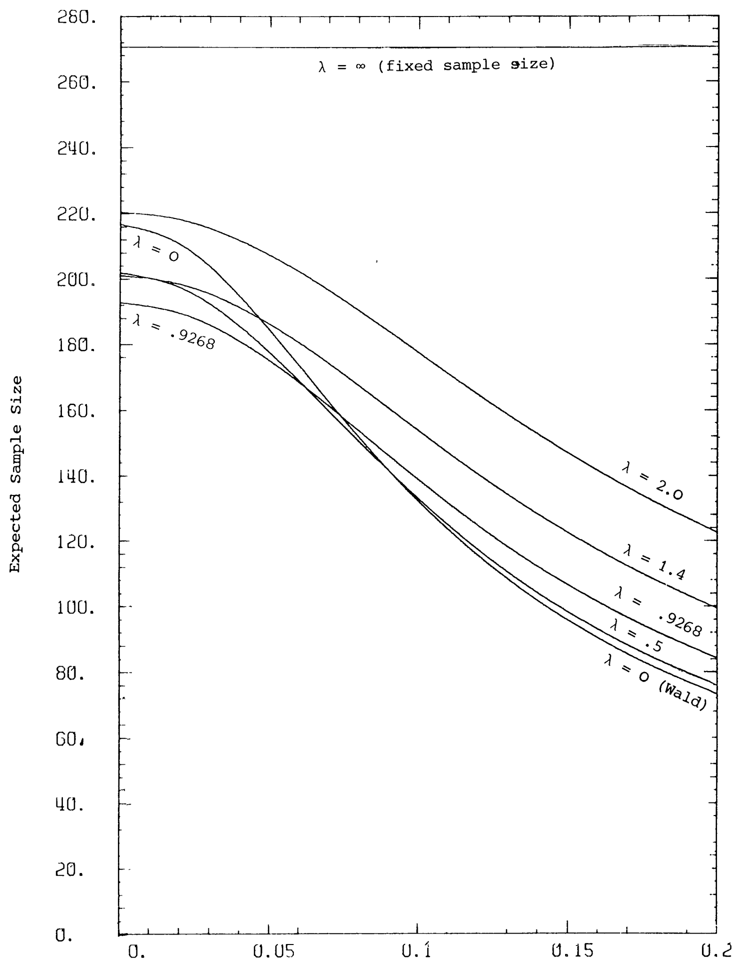

FIG. 3a: *Expected sample size as function of θ for tests with* 5% *error at $\theta = \pm .1$*

5. Tests with parabolic boundary—numerical results. We now apply the theory developed in the preceding sections to our original problem of deciding whether the drift of a Weiner process is positive or negative. We consider a Wiener process with unknown drift θ starting at $(t, 0)$, $t < 0$. We test the hypothesis $H_0: \theta \geq 0$ against the alternative $H_1: \theta < 0$ by observing the process up to the stopping time τ,

$$\tau = \inf\{s \mid t \leq s, \mid Y(s) \mid = \lambda\sqrt{-s}\},$$

TABLE 1b
Parameters of tests with
$P^{.1}_{(t,0)}\{Y(\tau) < 0\} = 0.01$

λ	$-t$	$E^{0}_{(t,0)}(\tau - t)$
0	∞	527.9
.5	2350.8	470.9
1.188	691.2	404.5
1.5	596.5	413.0
2.0	552.0	441.6
∞	541.2	541.2

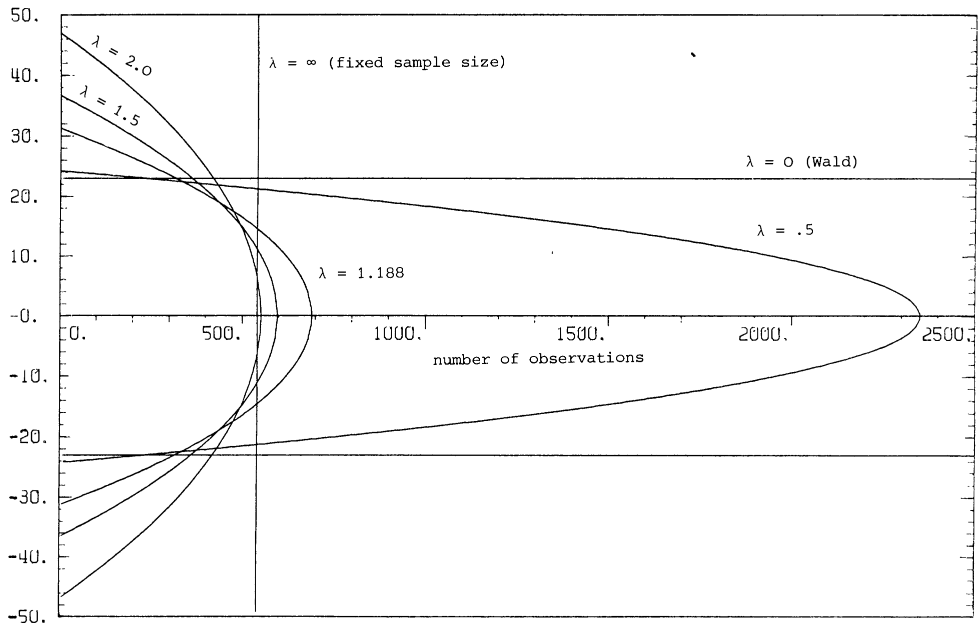

FIG. 1b: *Boundaries of tests with* 1% *error probability at* $\theta = \pm .1$

where $\lambda > 0$, and accepting H_0 iff $Y(\tau) \geq 0$. We determine the parameters t and λ of the test in such a way that the error probability at $\theta = \pm\theta_0$ is equal to a prescribed level α,

$$P^{\theta_0}_{(t,0)}\{Y(\tau) < 0\} = P^{-\theta_0}_{(t,0)}\{Y(\tau) > 0\} = \alpha. \tag{12}$$

Among all such tests, we seek the one whose maximum expected sample size, as a function of θ, is as small as possible:

$$\sup_\theta E^{\theta}_{(t,0)}(\tau - t) = \min ! \tag{13}$$

As expected, the computations using Theorem 3 confirm that the supremum in (13) is assumed at $\theta = 0$. By the Remark following Theorem 1 we see that

$$E^{0}_{(t,0)}(\tau - t) = \frac{\lambda^2}{1 + \lambda^2}(-t).$$

Thus the computational program can be carried out as follows. Given λ, determine t using Theorem 2 so that the error condition (12) is satisfied. Then vary λ until $\lambda^2(-t)/(1 + \lambda^2)$ has been minimized.

The computations have been carried out for the traditional significance levels $\alpha = .05$ and $\alpha = .01$. The parameter θ_0 was taken to be $\theta_0 = .1$. This value is arbitrary, since the

TABLE 2b
Error probabilities $P^{\theta}_{(t,0)}\{Y(\tau) < 0\}$, as function of θ for the tests in Table 1b.

θ	λ					
	0	.5	1.188	1.5	2.0	∞
.00	.500	.500	.500	.500	.500	.500
.05	.091	.098	.117	.120	.122	.122
.10	.010	.010	.010	.010	.010	.010
.15	.0010	.0009	.0004	.0003	.0002	.0002
.20	.0001	—	—	—	—	—

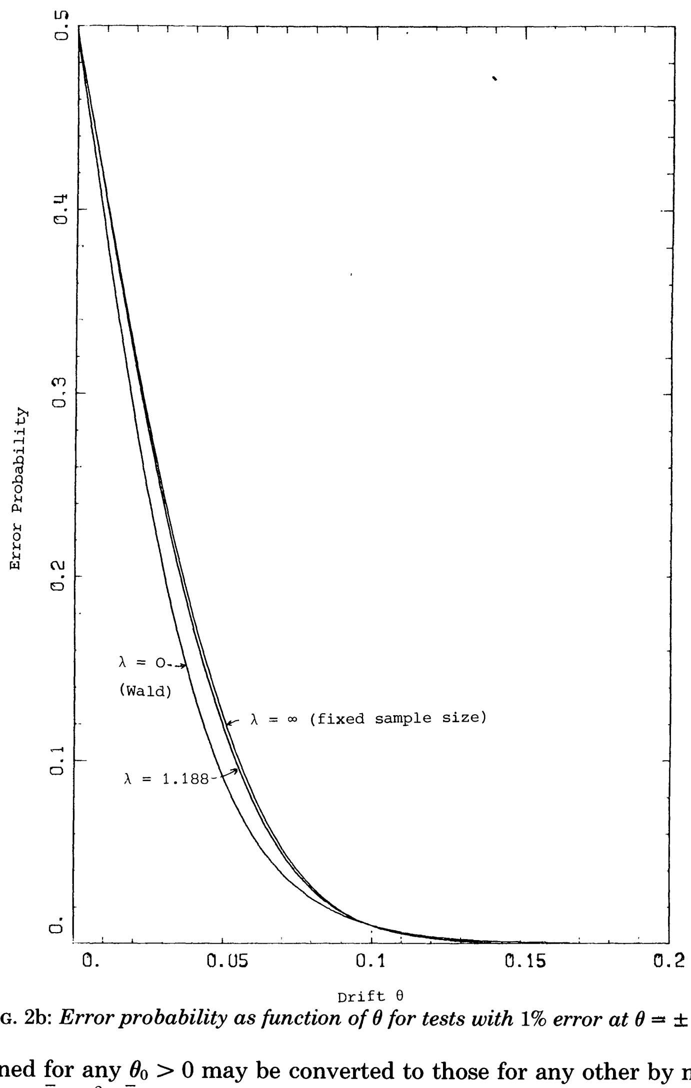

FIG. 2b: *Error probability as function of θ for tests with* 1% *error at $\theta = \pm .1$*

results obtained for any $\theta_0 > 0$ may be converted to those for any other by means of the rescaling $\bar{y} = cy$, $\bar{t} = c^2 t$, $\bar{\theta}_0 = \theta_0 / c$.

Some of the tests with error probability $\alpha = .05$ are shown in Table 1a and Figure 1a. (In the figure the continuation regions have been shifted so that the starting points for the Wiener process coincide with the origin.) The limiting cases $\lambda = 0$ and $\lambda = \infty$ denote the Wald and fixed sample size tests. The maximum expected sample size assumes its minimum value, 193, when λ is .93 and $(-t)$, the maximum sample size, is 417.

TABLE 3b
Expected sample size $E^{\theta}_{(t,0)}(\tau - t)$ as function of θ for the tests in Table 1b.

θ	λ					
	0	.5	1.188	1.5	2.0	∞
.00	527.9	470.2	404.5	413.0	441.6	541.2
.05	375.6	358.8	343.4	358.4	392.7	541.2
.10	225.2	225.8	244.5	265.0	304.0	541.2
.15	152.9	155.9	178.6	198.4	235.4	541.2
.20	114.9	118.1	139.4	156.9	189.8	541.2

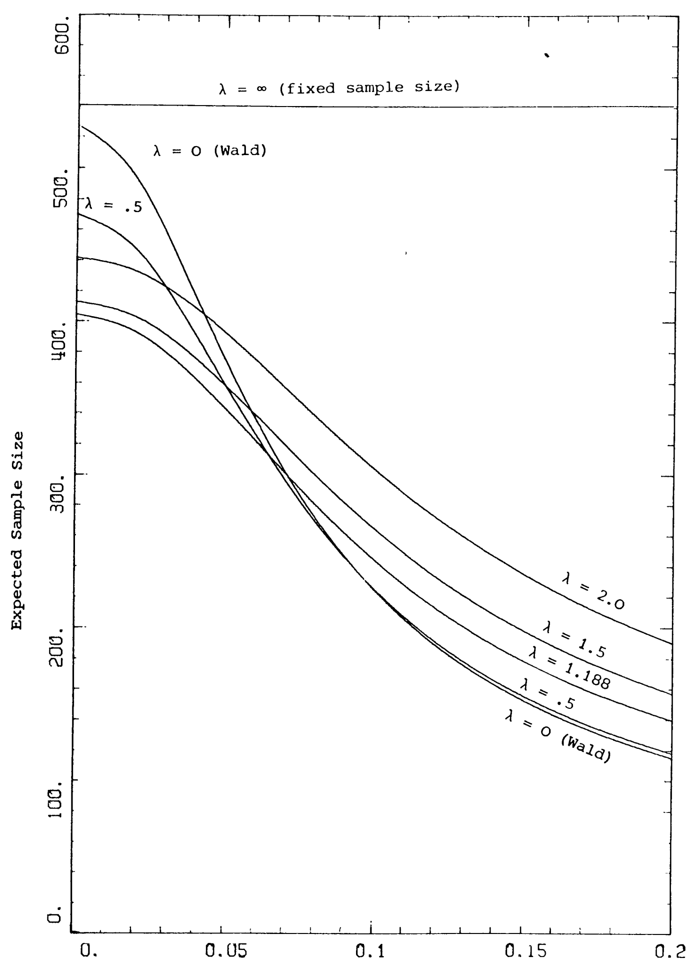

FIG. 3b: *Expected sample size as function of θ for tests with* 1% *error at $\theta = \pm .1$*

Table 2a shows the error probabilities of the tests in Table 1a as a function of θ. The graphs of three of these are shown in Figure 2a. It will be seen that the error curves are remarkably similar considering the very different shapes of the continuation regions.

Table 3a and Figure 3a show the expected sample size as a function of θ. Here great differences are apparent among the tests. The fixed sample size test ($\lambda = \infty$) is by far the worst, with a sample size of 271. As λ decreases to the minimax value, the expected sample

TABLE 4
Comparison of sample size for minimax parabolas and trapezoids.

		expected sample size		maximum sample size
		$\theta = 0$	$\theta = .1$	
	parabola	192.7	138.9	417.0
$\alpha = .05$	trapezoid	192.2	139.2	600.2
$\alpha = .01$	parabola	404.5	244.5	691.2
	trapezoid	402.1	249.4	783.2

size decreases monotonically for all θ in the range considered ($0 \leq \theta \leq .2$). As λ continues to decrease beyond the minimax value, the expected sample size rises for $\theta = 0$ but continues to fall for $\theta = \theta_0 = .1$, reaching its minimum when $\lambda = 0$, the Wald test.

Table 1b and Figure 1b show tests with error probability $\alpha = .01$ at $\theta = \pm.1$. The minimax parabola is now somewhat squatter in shape, with $\lambda = 1.19$ instead of $\lambda = .93$, and over half again as long, with $(-t) = 691$ as compared to $(-t) = 417$. The maximum expected sample size is twice as great, 405 as compared to 193.

Table 2b and Figure 2b show the error probability of the $\alpha = .01$ tests as function of θ. As before, the functions are all similar, though the differences are somewhat greater than when $\alpha = .05$.

Table 3b and Figure 3b show the expected sample size as a function of θ. The behavior of the curves as λ decreases from ∞ to 0 is qualitatively the same as when $\alpha = .05$, but the poor performance of Wald's test for $\theta = 0$ which was discussed in the introduction is beginning to make itself apparent.

We conclude with a comparison of the minimax parabolic regions described above with the minimax trapezoidal regions found by Anderson (1960). Anderson's regions are described by the three parameters c, d, t_0; they consist of all points (s, x) with $0 \leq s \leq t_0$, $|x| \leq c + ds$. The minimax regions found by Anderson are given by $c = 19.9$, $t_0 = 600.2$, $c + dt_0 = 0$ for $\alpha = .05$; and $c = 35.5$, $t_0 = 783.2$, $c + dt_0 = .1c$, for $\alpha = .01$. Thus in both cases the regions are nearly triangular. In Table 4 below we compare the expected sample size for $\theta = 0$ and $\theta_0 = .1$ and the maximum sample size of the trapezoids and parabolas. It will be seen that both for $\alpha = .05$ and $\alpha = .01$ the expected sample size is slightly less for the trapezoid at $\theta = 0$ and slightly greater at $\theta = .1$.

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