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UNBIASED ESTIMATION FOR SOME NON-PARAMETRIC FAMILIES OF DISTRIBUTIONS[1]

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This paper is concerned with the theory of unbiased estimation for nonparametric families of distributions subject to "generalised moment" restrictions. Necessary and sufficient conditions under which symmetric statistics are unique minimum variance unbiased estimators of their expectations are obtained, and some new boundedly complete families of distributions are exhibited.

1. Introduction. In 1946, Halmos published the first paper which gave formal justification of the heuristic principle that statistics which are symmetric functions of a random sample are to be preferred (in terms of minimum variance and unbiasedness) when estimating parameters of some general families of probability distributions. This was accomplished by showing that if a given parameter θ admits an unbiased estimator of finite variance, then there exists a unique symmetric unbiased estimator of θ with smaller variance than all other (asymmetric) unbiased estimators based on the same sample. In essence, Halmos' paper introduced the concept of "completeness" as it pertains to estimators (or more formally, to families of distributions), as it is shown that any symmetric unbiased estimator of zero must be identically zero.

Halmos' results were confined to families of discrete distributions comprising all distributions concentrated on finite subsets of a given set $\mathbf{X}$. Subsequently, Fraser (1954, 1957) showed that similar results were true for some general families of continuous distributions (e.g. all distributions on the real line which have probability density functions). Fraser's discussion was in terms of the order statistic of a random sample, since a function which is symmetric in its arguments is a function of the order statistic, and conversely. Thus, for the nonparametric families considered by Halmos and Fraser, the order statistic is a complete sufficient statistic. The uniqueness of symmetric statistics of finite variance as uniform minimum variance unbiased estimators (UMVUE's) is then a consequence of the Rao-Blackwell theorem, and the class of all parameters θ admitting UMVUE's is obtained by computing the expectations of all the symmetric statistics of finite variance.

Fraser's results were extended to general probability measure spaces by Bell, Blackwell and Breiman (1960).

Hoeffding (1977a, b) considered situations similar to those studied by Halmos and Fraser, but in which a certain amount of information is (assumed) known about the distributions: each distribution P in the family of interest satisfies the k "generalised moment" conditions

$$\int u_i(x)\, dP(x) = c_i, \quad 1 \le i \le k, \tag{1.1}$$

for some constants $c_1, \cdots, c_k$. (For example, the first k moments of each P may be fixed at the common values $c_1, \cdots, c_k$ by choosing $u_i(x) = x^i$, $1 \le i \le k$). He showed that when the restrictions (1.1) are imposed (thus reducing the size of the family of distributions), it

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is no longer necessarily the case that a symmetric unbiased estimator of θ is unique as such. (That is, the order statistic is no longer complete, although it is still sufficient.) For cases of non-uniqueness, Hoeffding established the general form

$$\sum_{i=1}^{k} \sum_{j=1}^{n} \{u_i(x_j) - c_i\} h_i(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n) \tag{1.2}$$

(where $h_1, \cdots, h_k$ are arbitrary symmetric integrable functions) for all symmetric unbiased estimators of zero. On the other hand, if the restrictions possess a certain "unboundedness" property, then the uniqueness of a *bounded* symmetric statistic as an unbiased estimator is still true. (The order statistic is then boundedly complete.) Formal statements of Hoeffding's results are given in Section 2.

At this point, two general questions arise:

(a) are the sort of results obtained by Hoeffding for families of distributions satisfying the "linear" restrictions (1.1) also true for a broader class of "nonlinear" restrictions of the form

$$\int \cdots \int u(x_1, \cdots, x_k)\, dP(x_1) \cdots dP(x_k) = c \tag{1.3}$$

for symmetric u and real c?

(b) do there exist parameters θ possessing unique symmetric UMVUE's for the families of distributions considered by Hoeffding, and for other such families arising from (a), and if so, how may they be characterized?

Concerning (a), only the case $k = 2$, with the condition

$$\int\int u(x_1, x_2)\, dP(x_1)\, dP(x_2) = c \tag{1.4}$$

is considered herein. For certain types of u it is possible to deduce the analogue

$$\sum \sigma_{2,n-2}^{1,\cdots,n} \{u(x_{i_1}, x_{i_2}) - c\} h(x_{i_3}, \cdots, x_{i_n}), \quad \text{arbitrary symmetric integrable } h \tag{1.5}$$

of (1.2) for a symmetric unbiased estimator of zero, and also the condition (essentially, that u be unbounded) under which the order statistic is boundedly complete. (See the end of this section for definition of $\sigma_{2,n-2}^{1,\cdots,n}$.) It is not clear how to obtain these results for general k, because the nature of their derivation is, in part, peculiar to the case $k = 2$. On the other hand, it is clear that condition (1.4) does not always imply that every symmetric unbiased estimator of zero must have the form (1.5), contrary to what one might expect from the results in Hoeffding (1977a).

Concerning (b), it is possible to characterize the class of parameters θ possessing unique symmetric UMVUE's for the families of distributions considered by Hoeffding, and also for those families of distributions satisfying (1.4) for which symmetric unbiased estimators of zero can be shown to be of the form (1.5). An implication of these results is that the price of imposing restrictions on the class of distributions under study is a (possibly considerable) reduction in the class of functions θ which admit unique symmetric UMVUE's.

The answers to (a) and (b), taken together, provide a set of results parallel to those of Halmos and Fraser, and useful in situations where some information is known about such nonparametric families. They throw new light on the structure of unbiased estimators, and lead to new classes of boundedly complete families of distributions.

Section 2 contains some examples which illustrate the preceding remarks, and Section 3 contains formal statements of Hoeffding's results and of the new results presented in this paper. Section 4 contains some discussion of the conditions on $u(x_1, x_2)$ under which the results are applicable. Sections 5 and 6 sketch the derivation of (1.5) and the bounded completeness property of the order statistic; the full details are very technical and are available from the author upon request. Section 7 gives the proof of the characterization of the class of UMVUE's.

NOTATION. The following notations and abbreviations are used throughout the paper. $\Delta(r, s)$ is the set of all s-tuples of specified indices $r_1, \cdots, r_s$ such that $r_1 \geq 0, \cdots, r_s \geq 0$, $r_1 + \cdots + r_s = r$; $\sigma_{r_1, \cdots, r_k}^{i_1, \cdots, i_s}$ is the set of distinct partitions of $\{i_1, \cdots, i_s\}$ into k parts, with the ith part consisting of precisely r_i elements, $1 \leq i \leq k$, and in particular $\sigma_s^{i_1, \cdots, i_s}$ is the set of all permutations of $i_1, \cdots, i_s$. $(n; r_1, \cdots, r_s)$ denotes $n!/(r_1! \cdots r_s!)$, $r_1 + \cdots + r_s = n$.

$I(A)$ is the indicator function of set A. R^1, R_+^1 and C are respectively the real line, positive real line and complex plane.

We write u_{ij} for $u(x_i, x_j)$ $g_{i_1 \cdots i_n}$ for $g(x_{i_1}, \cdots, x_{i_n})$, and $g^*_{j_1 \cdots j_N}$ for $g(x_1, \cdots, x_1, \cdots, x_N, \cdots, x_N)$ where x_i occurs j_i times, $i = 1, \cdots, N$.

2. Examples. For convenience of exposition, we shall assume (without loss of generality) that the constants $c_1, \cdots, c_k$ in (1.1) and c in (1.4) are zero. Thus (1.1), (1.4) and (1.5) reduce respectively to

$$\int u_i(x) \, dP(x) = 0, \quad 1 \leq i \leq k \tag{2.1}$$

$$\int \int u(x_1, x_2) \, dP(x_1) \, dP(x_2) = 0 \tag{2.2}$$

and

$$\sum \sigma_{2, n-2}^{1, \cdots, n} u(x_{i_1}, x_{i_2}) h(x_{i_3}, \cdots, x_{i_n}). \tag{2.3}$$

EXAMPLE 1. Let $\lambda(x)$ be a continuous, strictly increasing function on the real line such that $\lambda(-x) = -\lambda(x)$. If $F(x)$ is a continuous distribution function such that

$$\int_{-\infty}^{\infty} \lambda(|x|) \, dF(x) < \infty, \tag{2.4}$$

define

$$J(F) = \int_{-\infty}^{\infty} \{F(x) + F(-x) - 1\}^2 d\lambda(x). \tag{2.5}$$

$J(F)$ may be regarded as a measure of the deviation from symmetry about 0. Condition (2.4) means that (2.5) can be rewritten in the form

$$J(F) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w(x_1, x_2) \, dF(x_1) \, dF(x_2)$$

where $w(x_1, x_2) = 2 \operatorname{sgn}(x_1 x_2) \lambda(\min(|x_1|, |x_2|))$. Now consider the family of all probability measures P absolutely continuous with respect to Lebesgue measure, whose distribution functions $F = F_P$ satisfy the conditions (2.4) and

$$J(F) = \alpha; \tag{2.6}$$

(2.6) is equivalent to (2.2) with $u(x_1, x_2) = w(x_1, x_2) - \alpha$. We distinguish two cases:

(i) λ *bounded.* Then (2.4) is no restriction. If $\alpha = 0$, (2.6) defines the class of all continuous distribution functions symmetric about 0. It is clear in this case that a symmetric unbiased estimator of zero need not have the form (2.3), since every bounded, measurable function $f(x)$ such that $f(-x) = -f(x)$ is an unbiased estimator of zero. This particular example is

discussed using a different approach, in Hoeffding (1977b), where further discussion is given. If $\alpha > 0$ (and not too large), the function $u(x_1, x_2)$ may be shown to satisfy one of the conditions sufficient for a symmetric unbiased estimator of zero to have the form (2.3). The relevant condition is C4 in Section 3, namely that there exist points $x_1, x_2 \in \mathbf{X}$ such that $u(x_1, x_1) < 0$ and $u(x_2, x_2) > 0$.
(ii) λ *unbounded*, e.g. $\lambda(x) = x$. For $\alpha > 0$, $u(x_1, x_2)$ is unbounded and the family is boundedly complete.

EXAMPLE 2. Consider the functional $J_0(F) = \int \{F(x) - F_0(x)\}^2 \, dF_0(x)$, where F_0 is a fixed distribution function, continuous and strictly increasing. $J_0(F)$ is the Cramer-von Mises goodness-of-fit statistic. The condition $J_0(F) = \alpha$ is of similar type to (2.6). It can be reexpressed as $\int \int w_0(x_1, x_2) \, dF(x_1) \, dF(x_2) = \alpha$, where

$$w_0(x_1, x_2) = 1/3 + 1/2\{F_0^2(x_1) + F_0^2(x_2)\} - \max\{F_0(x_1), F_0(x_2)\}.$$

With $\alpha = 0$, the condition implies $F_0(x) \equiv F(x)$. With α positive but not too large, the results are similar to those in Example 1.

The key to representing functionals like $J_0(F)$ in the form $\int \int w_0(x_1, x_2) \, dF(x_1) \, dF(x_2)$ is the identity $F(x) = \int I[y \leq x] \, dF(y)$. This does not even require that F be continuous. The functional $\int \{F(x) - F_0(x)\}^2 dF(x)$ can in general be rewritten in the general form $\int \cdots \int W(x_1, \cdots, x_k) \, dF(x_1) \cdots dF(x_k)$ with $k = 3$, but it will reduce to the form with $k = 2$ if both F and F_0 are continuous (in which case $J_0(F)$ and $\int \{F(x) - F_0(x)\}^2 dF(x)$ are identically equal, since $\int (F - F_0)^2 d(F - F_0) = 0$). Another example, $\int F(x)\{1 - F(x)\} \, dx = 1/2 \int \int |x_1 - x_2| \, dF(x_1) \, dF(x_2)$, is considered in Example 5 below.

EXAMPLE 3. The condition $\Pr\{(X_1 + X_2) > 0\} = 1/2$ (or $= \alpha$) is related to the Wilcoxon one-sample signed-rank test for symmetry about 0. The function $u(x_1, x_2) = I[x_1 + x_2 > 0] - \alpha$ satisfies the condition on u given in Example 1, for the family of distributions considered therein.

EXAMPLE 4. Consider a random sample of vectors $\{X_i = (X_i^{(1)}, X_i^{(2)}), 1 \leq i \leq n\}$ with the condition

$$P\{(X_1^{(1)} - X_2^{(1)})(X_1^{(2)} - X_2^{(2)}) > 0\} = 1/2 \text{ (or } = \alpha);$$

the left hand side is the probability that X_1 and X_2 are concordant. This is related to the rank correlation coefficient known as Kendall's tau. The remarks in Example 3 are also pertinent here.

Either of the conditions mentioned in Examples 3 and 4 imposes a severe restriction on the class of parameters $\theta(P)$ admitting UMVUE's. The characterization theorem for UMVUE's implies that $\theta(X_1, \cdots, X_n)$ can only be a UMVUE of its expectation if $\int \int u(x_1, x_2) \, \theta(x_1, \cdots, x_n) \, dP(x_1) \, dP(x_2) = 0$ for all P in the relevant family $\mathbf{P}$, and examples of such statistics are not easy to find.

EXAMPLE 5. Let $u(x, y) = \delta(|x - y|) - \alpha$, where $\delta(0) = 0$, and $\delta(z)$ is an increasing function of $z (> 0)$. Then u is of a form which defines a class of measures of dispersion; for example, with $\delta(z) \equiv z$, condition (2.2) fixes the mean deviation of P at a particular value, and with $\delta(z) \equiv z^2$, (2.2) fixes the variance of P at some other prescribed value. For this general form of u, $\alpha = 0$ implies that the family of distributions comprises only degenerate distributions, whereas if $\alpha > 0$, the following condition on u (see C5 in Section 2) is both necessary and sufficient for the validity of the representation (2.3) for symmetric unbiased estimators of zero: namely, that $u(x, x) \geq 0, \not\equiv 0$, all x (resp. $\leq 0, \not\equiv 0$, all x) and that there exist $x_1, x_2 \in \mathbf{X}$ such that

$$u(x_1, x_2) < -\{u(x_1, x_1)u(x_2, x_2)\}^{1/2} \text{(resp. } u(x_1, x_2) > \{u(x_1, x_1)u(x_2, x_2)\}^{1/2}).$$

3. Statement of Results. Let $X_1, \cdots, X_n$ be a random sample from $(\mathbf{X}, \mathbf{S}_x, P)$ where $\mathbf{S}_x$ is the collection of all subsets of an arbitrary set $\mathbf{X}$, and $P \in \mathbf{P}$, the family of all proability measures (pm's) concentrated on finite subsets of $\mathbf{X}$. The induced family of pm's of the order statistic $T \equiv \{X_1, \cdots, X_n\}$ will be denoted by $\mathbf{P}^T$.

Let $\mathbf{P}_k$ be the sub-family of $\mathbf{P}$ comprising those P in $\mathbf{P}$ for which (2.1) holds, where $u_1, \cdots, u_k$ are $\mathbf{S}_x$-measurable functions.

THEOREM 1. (Hoeffding, 1977a). (i) *Let $\mathbf{C}_k$ be a convex family of probability measures on* $(\mathbf{X}, \mathbf{S}_x)$ *which satisfy* (2.1), *and let $\mathbf{C}_k \supset \mathbf{P}_k$. If g is a symmetric $\mathbf{S}_x^{(n)}$-measurable function such that $\int g dP^n = 0$ for all $P \in \mathbf{C}_k$, then there exist k symmetric $\mathbf{S}_x^{(n-1)}$-measurable functions $h_1, \cdots, h_k$ which are P^{n-1}-integrable for each $P \in \mathbf{C}_k$, such that*

$$g(x_1, \cdots, x_n) = \textstyle\sum_{i=1}^k \sum_{j=1}^n u_i(x_j) h_i(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n) \tag{3.1}$$

for all $(x_1, \cdots, x_n) \in \mathbf{X}^n$.

(ii) *If, in addition, g is bounded and every non-trivial linear combination of $u_1, \cdots, u_k$ is unbounded, then $g(x_1, \cdots, x_n) = 0$ for all $(x_1, \cdots, x_n) \in \mathbf{X}^n$.*

In the same paper, Hoeffding derives corresponding results for dominated families of pm's, and Hoeffding (1977b) provides extensions of these results to families of pm's symmetric about zero, and to two-sample families.

Denote by $\mathbf{P}^{(k)}$ the sub-family of pm's $P \in \mathbf{P}$ for which (1.3) holds (with c assumed zero), where u is a symmetric $\mathbf{S}_x^{(k)}$-measurable function. (Henceforth, we assume that $k = 2$ except where specified.) Theorem 2, the analogue of Theorem 1 for the family $\mathbf{P}^{(2)}$, has been established under certain conditions on $u(x_1, x_2)$. Each of the following conditions C1 through C6 is sufficient for the validity of Theorem 2.

C1: $u(x, x) \equiv 0$.

C2: $u(x_1, x_2) = v_1(x_1)v_1(x_2) - v_2(x_1)v_2(x_2) (\equiv v_1^*(x_1)v_2^*(x_2) + v_2^*(x_1)v_1^*(x_2))$ with $v_1(x) \not\equiv$ constant $\times\ v_2(x)$.

C3: for every $x_1, \cdots, x_n \in \mathbf{X}$ there exist $N > n$, $x_{n+1}, \cdots, x_n \in \mathbf{X}$, $p_1 > 0, \cdots, p_{N-1} > 0$ such that (i) $Q > 0$; (ii) $u(x_N, x_N) \neq 0$ and $u(x_N, x_N)(Q^{1/2} - L) > 0$; (iii) for $x_1, \cdots, x_N$ fixed, $Q^{1/2}$ is not rational, considered as a function of $p_1, \cdots, p_{N-1}$ where

$$L = \textstyle\sum_{i=1}^{N-1} u(x_i, x_N)p_i, \quad Q = \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} \{u(x_i, x_N)u(x_j, x_N) - u(x_i, x_j)u(x_N, x_N)\}p_i p_j.$$

C3 is implied by any of the following conditions.

C4: there exist $x_1, x_2 \in \mathbf{X}$ such that $u(x_1, x_1)$ and $u(x_2, x_2)$ have opposite signs.

C5: $u(x, x) \geq 0, \not\equiv 0$, all x (resp. $\leq 0, \not\equiv 0$, all x) and there exist $x_1, x_2 \in \mathbf{X}$ such that

$$u(x_1, x_2) < -\{u(x_1, x_1)u(x_2, x_2)\}^{1/2} \quad (\text{resp. } u(x_1, x_2) > \{u(x_1, x_1)u(x_2, x_2)\}^{1/2}).$$

C6: there exist $x_1, x_2 \in \mathbf{X}$ such that $u(x_1, x_1) = 0$, $u(x_1, x_2) < 0$, and $u(x, x) \geq 0 (\not\equiv 0)$; provided that u does not satisfy C5, and that

$$\textstyle\sum_{j=2}^N u(x_1, x_j)p_j \nmid \sum_{i=2}^N \sum_{j=2}^N u(x_i, x_j)p_i p_j$$

for at least one pair of sets $\{x_1, \cdots, x_N \in \mathbf{X}\}$, $\{p_1 > 0, \cdots, p_N > 0\}$ satisfying

$$\textstyle\sum_{i=1}^N \sum_{j=1}^N u(x_i, x_j)p_i p_j = 0.$$

(The notation $\nmid$ signifies "does not divide".)

To obtain some idea of the stringency of these conditions, suppose that u is a two-valued function. Without loss of generality, the two possible values can be taken to be $1 - \alpha$ and $-\alpha$, $0 < \alpha < 1$. Then it is easily seen that u satisfies either C4 or C5 unless

$$u(x, x) \equiv 1 - \alpha \quad \text{and} \quad \alpha \geq \tfrac{1}{2}, \tag{3.2}$$

or equivalently, $u(x, x) \equiv -\alpha$ and $\alpha \geq \frac{1}{2}$. The particular case of (3.2) with $\alpha = \frac{1}{2}$ gives $u(x_1, x_2) = v(x_1)v(x_2)$ where $v(x) = \pm 1/\sqrt{2}$, which is covered by Theorem 1. Other cases of (3.2) can be investigated easily by choosing $N = 3$ in C4. For example, if $u(x_1, x_2) \equiv -\alpha$ when $x_1 \neq x_2$, then u satisfies C4 for $\frac{1}{2} > \alpha > \frac{1}{3}$, and C2 for $\alpha = \frac{1}{3}$.

Thus, the class of two-valued functions to which Theorem 2 applies is quite extensive.

THEOREM 2. *Let $\mathbf{P}^{(2)}$ be the sub-family of probability measures P in $\mathbf{P}$ which satisfy* (2.2), *where u is a symmetric $\mathbf{S}_x^{(2)}$-measurable function satisfying one of the conditions* C1 *through* C6. *If $g(x_1, \cdots, x_n)$ is a symmetric function such that $\int g\, dP^n = 0$ for all $P \in \mathbf{P}^{(2)}$, then there exists a symmetric function h which is P^{n-2}-integrable for each $P \in \mathbf{P}^{(2)}$, such that*

$$g(x_1, \cdots, x_n) = \textstyle\sum_{\sigma_{2,n-2}^{1,\cdots,n}} u(x_{i_1}, x_{i_2}) h(x_{i_3}, \cdots, x_{i_n}) \tag{3.3}$$

for all $(x_1, \cdots, x_n) \in \mathbf{X}^n$.

(ii) *If, in addition, g is bounded and u is unbounded, then $g(x_1, \cdots, x_n) = 0$ for all $(x_1, \cdots, x_n) \in \mathbf{X}^n$. That is, $\mathbf{P}^{(2)T}$ is boundedly complete.*

REMARK 1. In contrast to Theorem 1, the results in Theorem 2 do not extend to a convex family $\mathbf{C}^{(2)}$ of pm's P on $(\mathbf{X}, \mathbf{S}_x)$ which satisfy (2.2) and such that $\mathbf{C}^{(2)} \supset \mathbf{P}^{(2)}$, since the assumption on convexity of $\mathbf{C}^{(2)}$, together with (2.2), would imply that

$$\int\int u(x_1, x_2)\, dP_1(x_1)\, dP_2(x_2) = 0, \quad \text{all } P_1, P_2 \in \mathbf{C}^{(2)}.$$

REMARK 2. Theorem 2 does not apply to functions $u(x_1, x_2)$ of the type $u_1(x_1)u_1(x_2) + \cdots + u_k(x_1)u_k(x_2)$, because in this case the condition (2.2) is equivalent to (2.1), the condition under which Theorem 1 implies another specific form (3.1) for $g(x_1, \cdots, x_n)$. Only in special cases will this form also be of the form (3.3).

Now let $\mathbf{S}$ be a σ-field of subsets of $\mathbf{X}$, μ a σ-finite measure on $(\mathbf{X}, \mathbf{S})$, and $\mathbf{P}(\mu)$ the family of all pm's on $(\mathbf{X}, \mathbf{S})$ dominated by μ, so that $\mathbf{P}(\mu)$ contains all pm's P for which $dP/d\mu$ is a simple function of sets in $\mathbf{S}$. Let $\mathbf{A}$ be a class of sets A in $\mathbf{S}$ such that

$$\mu^2(A_1 \times A_2) + \int_{A_1 \times A_2} |u|\, d\mu^2 < \infty$$

for all pairs A_1, A_2 in $\mathbf{A}$, and define

$$U(A_1, A_2) = \int_{A_1 \times A_2} u\, d\mu^2,$$

all $A_1, A_2 \in \mathbf{A}$. Further, let $\mathrm{C1}(\mu), \cdots, \mathrm{C6}(\mu)$ denote the analogues in terms of U of the conditions C1, $\cdots$, C6.

THEOREM 3. (i) *Let $\mathbf{P}^{(2)}(\mu)$ be the sub-family of probability measures P in $\mathbf{P}(\mu)$ which satisfy* (2.2), *where u is a symmetric $\mathbf{S}^{(2)}$-measurable function which satisfies one of the conditions* $\mathrm{C1}(\mu)$ *through* $\mathrm{C6}(\mu)$. *If $g(x_1, \cdots, x_n)$ is a symmetric $\mathbf{S}^{(n)}$-measurable function such that $\int g\, dP^n = 0$, all $P \in \mathbf{P}^{(2)}(\mu)$, then there exists a symmetric $\mathbf{S}^{(n-2)}$-measurable function h which is P^{n-2}-integrable for all $P \in \mathbf{P}^{(2)}(\mu)$, such that* (3.3) *is satisfied* a.e. $[\mathbf{P}^{(2)}(\mu)]$.

(ii) *If, in addition, u is $\mathbf{P}(\mu)$-unbounded and g is $\mathbf{P}(\mu)$-bounded, then $g(x_1, \cdots, x_n) =$* 0 a.e. $[\mathbf{P}^{(2)}(\mu)]$. (An *$\mathbf{S}^{(2)}$-measurable function $u(x_1, x_2)$ is $\mathbf{P}(\mu)$-unbounded if, given $a \in R^1$, there exists $P \in \mathbf{P}(\mu)$ such that $P(|u(x_1, x_2)| > a) > 0$.) In general, h will not be independent of μ.*

For the family $\mathbf{C}_k$ defined in Theorem 1, and with a square-integrability condition on $u_1, \cdots, u_k$, the following result characterizes the class of functions $\theta(P)$ which admit UMVUE's.

THEOREM 4. *Suppose that the functions $u_1, \cdots, u_k$ defining the family $\mathbf{P}_k$ are $\mathbf{C}_k$-square integrable. In order that a symmetric $\mathbf{S}_x^{(n)}$-measurable function $\hat{\theta}(X_1, \cdots, X_n)$ be the unique uniform minimum variance unbiased estimator of its expectation, relative to $\mathbf{P}_k$, it is necessary and sufficient that, for all $x_2, \cdots, x_n \in \mathbf{X}$,*

$$\int u_i(x_1)\hat{\theta}(x_1, \cdots, x_n)\, dP(x_1) = 0, \quad 1 \leq i \leq k, \quad \text{all } P \in \mathbf{C}_k. \tag{3.4}$$

It will be clear from the proof of this theorem that analogous results hold for the family $\mathbf{P}^{(2)}$ considered in Theorem 2 and for the corresponding dominated families.

Condition (3.4) is quite stringent; however, non-trivial examples can be found in which it can be applied. Thus, if $\mathbf{P}_1$ is the particular subfamily of $\mathbf{P}$ satisfying $\int x\, I[|x| > a]\, dP(x) = 0$, all $P \in \mathbf{P}_1$, for some $a \in R^1$, then any symmetric statistic $\hat{\theta}(X_1, \cdots, X_n)$ which is zero outside $\{|x_1| \leq a, \cdots, |x_n| \leq a\}$ will be the UMVUE of its expectation.

4. Some sufficient conditions on u for the validity of Theorem 2(i). The nature of the proof of Theorem 2(i) will depend on which of the sufficient conditions C1, C2, or C3 u satisfies. Apart from the condition C2, in which u has a particular functional form, the other conditions arise from the assumption in the proofs given in Sections 5.3 and 5.4 that, given n arbitrary points $x_1, \cdots, x_n \in \mathbf{X}$, there exist points $x_{n+1}, \cdots, x_N \in \mathbf{X}$ and real numbers $p_1 > 0, \cdots, p_N > 0$ such that

$$\textstyle\sum_{i=1}^{N} \sum_{j=1}^{N} u(x_i, x_j) p_i p_j = 0. \tag{4.1}$$

Condition C1 asserts that if $u(x, x) \equiv 0$, then this can always be achieved. Condition C3 is the complementary assertion which would have to be checked for the given function u, whilst C4, C5 and C6 are particular cases in which C3 is satisfied. In this section, the condition C1 will be justified, and some limited discussion of C4, C5, C6 given.

To avoid trivialities, assume that u takes both positive and negative values. For $u(x, x) \equiv 0$, (4.1) reduces to

$$\textstyle\sum\sum_{1 \leq i < j \leq N} u_{ij} p_i p_j = 0 \tag{4.2}$$

whence, assuming for convenience that $u(x_{N-1}, x_N) \neq 0$,

$$p_N = -\left(\textstyle\sum\sum_{1 \leq i < j \leq N-1} u_{ij} p_i p_j\right) / \textstyle\sum_{1 \leq i \leq N-1} u_{iN} p_i.$$

There are two distinct cases to consider, depending on whether or not

$$\textstyle\sum_{i=1}^{N-1} u_{iN} p_i \mid \textstyle\sum\sum_{1 \leq i < j \leq N-1} u_{ij} p_i p_j$$

for all N and all $x_1, \cdots, x_N \in \mathbf{X}$, as a function of $p_1, \cdots, p_{N-1}$.

I. *The case* $\sum_{i=1}^{N-1} u_{iN} p_i \mid \sum_{1 \leq i < j \leq N-1} u_{ij} p_i p_j$.

If there exist three points $x_{N-2}, x_{N-1}, x_N \in \mathbf{X}$ such that $u(x_{N-2}, x_{N-1}) < 0$, $u(x_{N-1}, x_N) > 0$ then it is easily seen that (4.2) can be satisfied for some $p_1 > 0, \cdots, p_N > 0$. Otherwise, it must be the case that, for any $x_0 \in \mathbf{X}$, either $u(x_0, x) \geq 0$, all $x \in \mathbf{X}$, or $u(x_0, x) \leq 0$, all $x \in \mathbf{X}$. Let $x_{N-3}, x_{N-2}, x_{N-1}, x_N$ be four points in $\mathbf{X}$ for which $u(x_{N-3}, x_{N-2}) < 0$, $u(x_{N-1}, x_N) > 0$; then (4.2) can again be satisfied for $p_1 > 0, \cdots, p_N > 0$.

II. *The case* $\sum_{i=1}^{N-1} u_{iN} p_i \nmid \sum\sum_{1 \leq i < j \leq N-1} u_{ij} p_i p_j$, *all* N, *all* $x_1, \cdots, x_N \in \mathbf{X}$.

PROPOSITION. *Let $u(x_1, x_2)$ be a function symmetric in its arguments such that $u(x,$*

$x) \equiv 0$. A necessary and sufficient condition that u has the form $v_1(x_1)v_1(x_2) - v_2(x_1)v_2(x_2)$ is that for any $N = 3, 4, \cdots$, and $x_1, \cdots, x_N \in \mathbf{X}$, the (non-trivial) linear form $u(x_1, x_N)p_1 + \cdots u(x_{N-1}, x_N)p_{N-1}$ divides the quadratic form

$$\sum\sum_{1 \le i < j \le N-1} u(x_i, x_j)p_i p_j.$$

PROOF. Suppose

$$\sum_{i=1}^{N-1} u(x_i, x_N)p_i \mid \sum\sum_{1 \le i \le j < N-1} u(x_1, x_j)p_i p_j$$

for all $N = 3, 4, \cdots$, and $x_1, \cdots, x_N \in \mathbf{X}$ (at least one $u(x_1, x_N) \neq 0$). Then it is easy to show that (i) for every $x_1, x_2, x_3 \in \mathbf{X}$, either $u_{12} = u_{13} = u_{23} = 0$ or precisely one of these is zero, (ii) for every x_1 there exist $x_2, x_3 \in \mathbf{X}$ such that $u_{12} = 0$, $u_{13} \neq 0$, and (iii) for every $x_1, x_2, x_3, x_4 \in \mathbf{X}$ such that $u_{12} \neq 0$, $u_{34} \neq 0$, either $u_{12}u_{34} = u_{13}u_{24}$ or $u_{12}u_{34} = u_{14}u_{23}$, depending on whether $u_{14} = u_{23} = 0$ or $u_{13} = u_{24} = 0$.

Now let $x_0 \in \mathbf{X}$ and define $\mathbf{X}_1 = \{x \in \mathbf{X} : u(x, x_0) = 0\}$, $\mathbf{X}_2 = \{x \in \mathbf{X} : u(x, x_0) \neq 0\}$. Then (i), (ii) and (iii) may be used to prove that $\{X_1, X_2\}$ is a partition of $\mathbf{X}$ independent of the choice of x_0 and with the following properties:

P1: $x_1 \in \mathbf{X}_1, x_2 \in \mathbf{X}_2 \Rightarrow u(x_1, x_2) \neq 0$;
P2: $x, y \in \mathbf{X}_i \Rightarrow u(x, y) = 0, \quad i = 1, 2$;
P3: $x_1, y_1 \in \mathbf{X}_1, x_2, y_2 \in \mathbf{X}_2 \Rightarrow u(x_1, x_2)u(y_1, y_2) = u(x_1, y_2)u(x_2, y_1)$.

Choose $y_1, y_2 \in \mathbf{X}$ such that $u(y_1, y_2) > 0$. Then for any $x_1 \in \mathbf{X}_1, x_2 \in \mathbf{X}_2$, P3 $\Rightarrow u(x_1, x_2) = w_1(x_1)w_2(x_2)$ say, where $w_1(x_1) = u(x_1, y_2)/\{u(y_1, y_1)\}^{1/2}$, $w_2(x_2) = u(x_2, y_1)/\{u(y_1, y_2)\}^{1/2}$. The definition of w_1 and w_2 may be extended to all $x \in \mathbf{X}$ by defining $w_1(x) \equiv 0$, $x \in \mathbf{X}_2$, $w_2(x) \equiv 0$, $x \in \mathbf{X}_1$. Then

$$u(x_1, x_2) = w_1(x_1)w_2(x_2) + w_1(x_2)w_2(x_1) \equiv v_1(x_1)v_1(x_2) - v_2(x_1)v_2(x_2),$$

where $2v_1 = w_1 + w_2$, $2v_2 = w_1 - w_2$.

Conversely, suppose $u(x_1, x_2) \equiv v_1(x_1)v_1(x_2) - v_2(x_1)v_2(x_2)$. Then $u(x, x) \equiv 0 \Rightarrow v_1^2(x) \equiv v_2^2(x)$. Define $\mathbf{X}_1 = \{x \in \mathbf{X} : v_1(x) = v_2(x)\}$, $\mathbf{X}_2 = \{x \in \mathbf{X} : v_1(x) = -v_2(x)\}$. It is easily seen that $\{\mathbf{X}_1, \mathbf{X}_2\}$ is a partition of $\mathbf{X}$ satisfying P1, P2 and P3, and hence that

$$\sum_{i=1}^{N-1} u(x_i, x_N)p_i \mid \sum\sum_{1 \le i < j \le N-1} u(x_i, x_j)p_i p_j$$

for all $N = 3, 4, \cdots$, and $x_1, \cdots, x_N \in \mathbf{X}$ (at least one $u(x_1, x_2) \equiv 0$). The proof is complete.

Thus in this case u has the special form given by condition C2; the proof of Theorem 2(i) for this situation is given in Section 5.2.

When $u(x, x) \not\equiv 0$, suppose $u(x_N, x_N) \neq 0$ and solve (3.1) for p_N to yield

$$\begin{aligned} u_{NN}p_N &= -\sum_{i=1}^{N-1} u_{iN}p_i \pm \{\sum\sum_{1 \le i < j \le N-1} (u_{iN}u_{jN} - u_{ij}u_{NN})p_i p_j\}^{1/2} \\ &\equiv -L(p_1, \cdots, p_{N-1}) \pm Q^{1/2}(p_1, \cdots, p_{N-1}). \end{aligned} \tag{4.3}$$

The proof of Theorem 2(i) under condition C3 relies on the assumption that $Q^{1/2}$ is not a rational function. The exceptional cases when this is not so are (i) $Q \equiv 0$, and (ii) $Q \equiv$ a perfect square. It is easily seen that (i) $\Rightarrow u(x_1, x_2) \equiv v(x_1)v(x_2)$, in which case Theorem 1 is applicable, and (ii) $\Rightarrow u(x_1, x_2) \equiv v_1(x_1)v_1(x_2) - v_2(x_1)v_2(x_2)$, which is condition C2.

These cases aside, the problem is to find necessary and sufficient conditions on u such that one form of (4.3) is satisfied for $p_1 > 0, \cdots, p_N > 0$. Such a solution will not be possible if, for example, Q is non-positive definite; a particular instance of this occurs when $u(x_1, x_2) = v_1(x_1)v_1(x_2) + \cdots + v_k(x_1)v_k(x_2)$. Conditions C4, C5 and C6 are sufficient conditions which may be derived from (4.3) by elementary manipulation. None of these conditions is, in general, necessary, even under the assumptions of the conditions, although there are two interesting special cases in which C5 is also necessary (c.f. the general result quoted in Example 5 in Section 2).

(a) $\mathbf{X}$ = interval of R^1; $u(x_1, x_2) = (x_1 - x_2)^2 - 2c^2$, which restricts the family of pm's on

$(\mathbf{X}, \mathbf{S}_x)$ to those with variance c^2. $u(x, x) \equiv -2c^2$, so C5 requires $u(x_1, x_2) > 4c^2$, that is, $|x_1 - x_2| > 2c$, for at least one pair $x_1, x_2 \in \mathbf{X}$. Thus C5 is also necessary if the family is to be of interest, for any distribution whose support is the interval $[a - c, a + c]$ must have variance less than or equal to c^2, with equality if and only if the distribution assigns mass ½ to each end point.

(b) $\mathbf{X}$ = interval of R^1; $u(x_1, x_2) = |x_1 - x_2| - c^2$. Arguing as in (a), C5 requires the existence of $x_1, x_2 \in \mathbf{X}$ such that $|x_1 - x_2| > 2c^2$. This condition is also necessary, since X_1, X_2 are independent with common distribution P concentrated on $[a - c^2, a + c^2]$, then $E_{P^2}|X_1 - X_2| \le 2c^2$, with equality if and only if $P\{a - c^2\} = P\{a + c^2\} = ½$.

5. Proof of Theorems 2(i) and 3(i).

5.1. *Preliminary results.*

LEMMA 1. *Let $R(z)$ be a rational function of the complex variable z, analytic in a domain D containing the open interval* (a, b) *of the real line, and let $I(z)$ be a non-rational function of z, analytic in a domain D' which also contains* (a, b). *If $R(z) \equiv I(z)$, $z \in$* (a, b) *then R and I must be identically zero throughout the complex plane C. (For the purpose of this statement, the function which is identically zero in C may be considered either rational or non-rational.)*

LEMMA 2. *A homogeneous polynomial on R^N which is zero on a non-degenerate N-dimensional interval is identically zero on R^N.*

The next two lemmas are immediate corollaries of Theorem 2(i).

LEMMA 3. *Let $x_1, \cdots, x_n, y$ be any $n + 1$ points in $\mathbf{X}$, with $u(y, y) \neq 0$. If $g(x_1, \cdots, x_n)$ admits a representation of the form* (3.3), *then*

$$h(x_1, \cdots, x_{n-2}) = \sum_{r=2}^{n} \sum_{s=1}^{[r/2]} \sum_{S_{r,s}} \{D(r, s; n) g(x_{i_{r-1}}, \cdots, x_{i_{n-2}}, y, \cdots, y)$$
$$\times [\textstyle\prod_{t=1}^{s-1} u(x_{i_{2t-1}}, x_{i_{2t}})][\prod_{t=2s-1}^{r-2} u(x_{i_t}, y)]\}/[u(y, y)]^{r-s}, \tag{5.1.1}$$

where $S_{r,s} = \{(i_1, \cdots, i_{n-2}) \in \sigma_{n-r,2s-2,r-2s}^{1,\cdots,n-2}\}$ and the quantities $D(r, s; n)$ do not depend on $x_1, \cdots, x_{n-2}$ or y.

LEMMA 4. *Suppose that $u(x, x) \equiv 0$, and let $x_1, \cdots, x_n, y_1, y_2$ be any $n + 2$ points in $\mathbf{X}$, with $u(y_1, y_2) \neq 0$. If $g(x_1, \cdots, x_n)$ admits a representation of the form* (3.3), *then*

$$h(x_1, \cdots, x_{n-2}) = \sum_{m=2}^{n} \sum_{\ell_1,\ell_2=1,2;\ell_1\neq\ell_2} \sum_{r=1}^{[m/2]} \sum_{s=1}^{r} \sum_{S_{m,r,s}} \{D(r, m-r, s; n)$$
$$\times g(\underbrace{x_{i_{m-1}}, \cdots, x_{i_{n-2}}}_{i}, \underbrace{y_{\ell_1}, \cdots, y_{\ell_1}}_{r}, \underbrace{y_{\ell_2}, \cdots, y_{\ell_2}}_{m-r}) \tag{5.1.2}$$
$$\times [\textstyle\prod_{t=1}^{s-1} u(x_{i_{2t-1}}, x_{i_{2t}})][\prod_{t=2s-1}^{r+s-2} u(x_{i_t}, y_{\ell_1})]$$
$$\times [\textstyle\prod_{t=r+s-1}^{m-2} u(x_{i_t}, y_{\ell_2})]\}/[u(y_1, y_2)]^{m-s}$$

where $S_{m,r,s} = \{(i_1, \cdots, i_{n-2}) \in \sigma_{n-m,2s-2,r-s-1}^{1,\cdots,n-2}\}$ and the quantities $D(r, m - r, s; n)$ do not depend on $x_1, \cdots, x_{n-2}, y_1$ or y_2.

5.2. *Proof of Theorem* 2(i) *under Condition* C2. It is given that $u(x_1, x_2) \equiv v_1(x_1)v_1(x_2) - v_2(x_1)v_2(x_2)$. Then

$$E_{P^2}u = 0 \Leftrightarrow (E_P v_1)^2 - (E_P v_2)^2 = 0 \Leftrightarrow E_P(v_1 + v_2) \times E_P(v_1 - v_2) = 0.$$

Define $w_1 = v_1 + v_2$, $w_2 \equiv v_1 - v_2$, and $\mathbf{Q}_1 = \{P \in \mathbf{P}^{(2)} : E_P w_1 = 0\}$, $\mathbf{Q}_2 = \{P \in \mathbf{P}^{(2)} : E_P w_2 = 0\}$, so that $\mathbf{Q}_1 \cup \mathbf{Q}_2 = \mathbf{P}^{(2)}$. It follows from the assumption that $v_1(x) \not\equiv \text{constant} \times v_2(x)$, that both $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are proper subsets of $P^{(2)}$. By Theorem 1, there exist symmetric $\mathbf{S}_x^{(n-1)}$-measurable functions h_1 and h_2 such that

$$g(x_1, \cdots, x_n) = \begin{cases} \sum_{i=1}^n w_1(x_i) h_1(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n), \text{ using } \mathbf{Q}_1 \\ \sum_{i=1}^n w_1(x_i) h_2(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n), \text{ using } \mathbf{Q}_2 \end{cases} \tag{5.2.1}$$

for all $x_1, \cdots, x_n \in \mathbf{X}$. To prove the theorem, it is sufficient to establish that there exists a symmetric function h such that

$$g(x_1, \cdots, x_n) = \sum_{\sigma_{1,1,n-2}^{1,\cdots,n}} w_1(x_{i_1}) w_2(x_{i_2}) h(x_{i_3}, \cdots, x_{i_n}), \tag{5.2.2}$$

for this implies that

$$\begin{aligned} g(x_1, \cdots, x_n) &= \tfrac{1}{2} \sum \{w_1(x_{i_1}) w_2(x_{i_2}) + w_1(x_{i_2}) w_2(x_{i_1})\} \times h(x_{i_3}, \cdots, x_{i_n}) \\ &= \sum u(x_{i_1}, x_{i_2}) h(x_{i_3}, \cdots, x_{i_n}). \end{aligned}$$

Equation (5.2.2) can be verified by proving that $h_2(x_1, \cdots, x_{n-1})$ has the form

$$\sum_{j=1}^{n-1} w_1(x_j) h(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_{n-1}) \tag{5.2.3}$$

for some symmetric function h. To this end, let y, z be points in $\mathbf{X}$ such that $V \equiv w_1(y)w_2(z) - w_1(z)w_2(y) \neq 0$, and assume that $w_1(y)w_2(z) \neq 0$, $w_1(z)w_2(y) \neq 0$. From (5.2.1),

$$\begin{aligned} &w_1(y) h_1(x_1, \cdots, x_{n-1}) - w_2(y) h_2(x_1, \cdots, x_{n-1}) \\ &\quad = \sum_{\sigma_{1,n-2}^{1,\cdots,n-1}} \{w_2(x_{i_1}) h_2(x_{i_2}, \cdots, x_{i_{n-1}}, y) - w_1(x_{i_1}) h_1(x_{i_2}, \cdots, x_{i_{n-1}}, y)\}. \end{aligned} \tag{5.2.4}$$

Appropriate manipulations involving (5.2.4) lead to (5.2.3) and hence to the result. There are trivial modifications if $V \equiv 0$.

5.3. *Proof of Theorem* 2(i) *under Condition* C3 *or* C1. Let $x_1, \cdots, x_N$ and $p_1, \cdots, p_N$ satisfy (i), (ii) and (iii) of C3. Then using (2.2) and the assumptions of Theorem 2,

$$\sum_{i=1}^N \sum_{j=1}^N u_{ij} p_i p_j = 0, \quad \text{and} \tag{5.3.1}$$

$$\sum_{\Delta(n,N)} (n; r_1, \cdots, r_N) g^*_{r_1 \cdots r_N} p_1^{r_1} \cdots p_N^{r_N} = 0. \tag{5.3.2}$$

The structure of the proof is as follows.

I. Solve (5.3.1) for p_N and substitute into (5.3.2), yielding an equation of the form $P_L(p_1, \cdots, p_{N-1}) = -Q^{1/2}(p_1, \cdots, p_{N-1})\, P_R(p_1, \cdots, p_{N-1})$ where P_L, Q and P_R are homogeneous polynomials of respective degrees n, 2, and $n-1$, in the variables $p_1, \cdots, p_{N-1}$. Lemma 1 and Lemma 2 imply that P_L and P_R are identically zero, hence the coefficients of distinct individual power products of $p_1, \cdots, p_{N-1}$ in each are zero. The equations obtained by setting these coefficients to zero will yield a representation of the form (3.3) for $g(x_1, \cdots, x_n)$.

II. The function h determined implicitly in I will contain a dummy variable x_N; it can be shown that h is independent of the choice of x_N (given the condition $u(x_N, x_N) \neq 0$).

Similar methods are employed to prove the theorem if C1 is assumed instead.

5.4. *Proof of Theorem* 3(i) *under Conditions* C1 (μ), C2 (μ), C3 (μ). As in the proof of Theorem 1B in Hoeffding (1977a), the proof of this theorem may be accomplished by structuring the assumptions in such a way that the methods in Section 5.2 and 5.3 apply. Again, according to the various conditions on u (C1(μ), C2(μ), C3(μ)), different forms of proof are required; the proof for conditions C3(μ) will be sketched as an example. Let A_1,

$\cdots, A_n \in \mathbf{A}$, where

$$\int_{A_N \times A_N} u \, d\mu^2 > 0$$

and choose $a_1, \cdots, a_N$ positive so that $\Sigma \, a_i \mu(A_i) = 1$ and $\sum \sum a_i a_j U_{ij} = 0$, where $U_{ij} \equiv U(A_i, A_j)$. Then $p(x) = \sum a_i I[x \in A_i]$ is a probability density with respect to μ of a distribution in $\mathbf{P}^{(2)}(\mu)$: hence

$$\textstyle\sum_{\Delta(n,N)} (n; r_1, \cdots, r_N) G^*_{r_1 \cdots r_N} a_1^{r_1} \cdots a_N^{r_N} = 0$$

where

$$G_{1 \cdots n} \equiv G(A_1, \cdots, A_n) = \int_{A_1 \times \cdots \times A_n} g \, d\mu^n$$

and $G^*_{i_1 \cdots i_N}$ is the analogue of $g^*_{i_1 \cdots i_N}$. The form of proof in Section 5.3, with p_j, $\mathbf{X}$, x_j, u_{ij} and $g_{i_1 \cdots i_N}$ replaced by a_j, $\mathbf{A}$, A_j, U_{ij} and $G_{i_1 \cdots i_N}$ respectively, may be used to conclude that there exists a symmetric real-valued function H on $\mathbf{A}^{n-2}$ such that, for any $(A_1, \cdots, A_n) \in \mathbf{A}^n$,

$$G(A_1, \cdots, A_n) = \textstyle\sum_{\sigma^{1, \cdots, n}_{2, n-2}} U(A_{i_1}, A_{i_2}) H(A_{i_3}, \cdots, A_{i_n}).$$

Lemma 3 implies a representation for $H(A_1, \cdots, A_{n-2})$ in terms of G and U. Further, H can be written as

$$\int_{A_1} \cdots \int_{A_{n-2}} h(x_1, \cdots, x_{n-2}) \, d\mu(x_{n-2}) \cdots d\mu(x_1)$$

where h is given by (5.1.2) in Lemma 3 with the following alterations: $g^*_{1 \cdots 10 \cdots 0k}$ is replaced by

$$\int_{A_N} \cdots \int_{A_N} g(x_1, \cdots, x_{n-k}, y_1, \cdots, y_k) \, d\mu(y_k) \cdots d\mu(y_1)$$

and $u(x_i, y)$ by

$$\int_{A_N} u(x_i, y) \, d\mu(y).$$

Now write

$$w(x_1, \cdots, x_n) = g(x_1, \cdots, x_n) - \textstyle\sum_{\sigma^{1, \cdots, n}_{2, n-2}} u(x_{i_1}, x_{i_2}) h(x_{i_3}, \cdots, x_{i_n}).$$

So far, it has been shown that $F(C) \equiv \int_C w \, dx^n = 0$ for all sets $C = A_1 \times \cdots \times A_n \in \mathbf{A}^n$. Following the proof of Theorem 1B in Hoeffding (1977a), let $B \in \mathbf{A}$. Then $F(E \cap B^n) = 0$ for all E in $\mathbf{S}^{(n)}$, hence $w(x_1, \cdots, x_n) = 0$ a.e. $[\mu^n]$ on B^n.

Finally, let $P \in \mathbf{P}^{(2)}(\mu)$, let P be a version of $dP/d\mu$ (hence, a simple function of the form $a_1 I[x \in A_1] + \cdots + a_k I[x \in A_k]$ for some integer k, positive numbers $a_1, \cdots, a_k$, and disjoint sets $A_1, \cdots, A_k \in \mathbf{A}$), let $\mathbf{B}_\varepsilon = \{x : p(x) > \varepsilon\}$, and let $A \in \mathbf{A}$. It is necessary to show that $B_\varepsilon \in \mathbf{A}$, that is

$$\int_{B_\varepsilon} \int_A (1 + |u(x, y)|) \, d\mu(x) \, d\mu(y) < \infty \tag{5.4.1}$$

for all $A \in \mathbf{A}$. Since B_ε is just the union of those sets A_j among $A_1, \cdots, A_k$ for which $a_j > \varepsilon$, the LHS of (5.4.1) is just the sum of integrals over sets $A_j \times A$, all of which are finite by definition. Thus $B_\varepsilon \in \mathbf{A}$ for all $\varepsilon > 0$. Therefore $w(x_1, \cdots, x_n) = 0$ a.e. $[\mu^n]$ on $\cup_{m=1}^\infty B^n_{1/m}$, a set of P^n-measure 1. It follows that $g(x_1, \cdots, x_n) = \sum u(x_{i_1}, x_{i_2}) h(x_{i_3}, \cdots, x_{i_n})$ a.e. $[P^n]$ for all $P \in \mathbf{P}^{(2)}(\mu)$. h may be shown to be independent of the arbitrary set A_N.

6. Proof of Theorems 2(ii) and 3(ii). Depending on whether or not $u(x, x) \equiv 0$, Lemma 3 or Lemma 4 can be used to prove, sequentially, that $h(x_1, \cdots, x_1) = 0$, $h(x_1, \cdots, x_1, x_2) = 0, \cdots, h(x_1, \cdots, x_n) = 0$ for all $x_1, \cdots, x_n \in \mathbf{X}$.

7. Proof of Theorem 4. Let $\hat{\theta}(X_1, \cdots, X_n)$ be a symmetric statistic with finite variance for all $P \in \mathbf{C}_k$ and let $g(X_1, \cdots, X_n)$ be an unbiased estimator of zero with finite variance, for all $P \in \mathbf{C}_k$. Then g has the representation (3.1), where $h_1, \cdots, h_k$ are each $\mathbf{C}_k^{(n-1)}$-square integrable; c.f. Lemma 2 of Hoeffding (1977a). By a well-known result (Lehmann and Scheffé, 1950, Theorem 5.3), $\hat{\theta}$ is the UMVUE of its expectation if and only if $\hat{\theta}$ is uncorrelated with all such g, for all $P \in \mathbf{P}_k$, i.e. if and only if

$$\sum_{i=1}^k \sum_{j=1}^n \int \cdots \int \hat{\theta}(x_1, \cdots, x_n) u_i(x_j) h_i(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n) \times dP(x_1) \cdots dP(x_n) = 0 \tag{7.1}$$

for all $P \in \mathbf{C}_k$ and all symmetric $\mathbf{C}_k^{(n-1)}$-square integrable $h_1, \cdots, h_k$. By symmetry, (7.1) reduces to

$$\sum_{j=1}^k \int \cdots \int \hat{\theta}(x_1, \cdots, x_n) u_i(x_1) h_i(x_2, \cdots, x_n)\, dP(x_1) \cdots dP(x_n) = 0 \tag{7.2}$$

for all $P \in \mathbf{C}_k$, all such $h_1, \cdots, h_k$, and (3.2) is sufficient for (7.2) to hold.

Conversely, if (7.2) is true for all $h_1, \cdots, h_k$, it is true in particular when $h_2 = \cdots = h_k \equiv 0$, leaving

$$\int \cdots \int \hat{\theta}(x_1, \cdots, x_n) u_1(x_1) h_1(x_2, \cdots, x_n)\, dP(x_1) \cdots dP(x_n) = 0, \tag{7.3}$$

all $P \in \mathbf{C}_k$. Let A_{n-1} be an arbitrary set in $\mathbf{S}_x^{(n-1)}$. Then

$$\begin{aligned}
&\int_{A_{n-1}} \cdots \int \left\{ \int \hat{\theta}(x_1, \cdots, x_n) u_1(x_1)\, dP(x_1) \right\} h_1(x_2, \cdots, x_n)\, dP(x_2) \cdots dP(x_n) \\
&= \{(n-1)!\}^{-1} \sum_{\sigma_{n-1}^{i_2, \cdots, i_n}} \int_{A_{n-1}} \cdots \int \left\{ \int \hat{\theta}(x_1, x_{i_2}, \cdots, x_{i_n}) u_1(x_1)\, dP(x_1) \right\} \\
&\qquad \times h_1(x_{i_2}, \cdots, x_{i_n})\, dP(x_{i_2}) \cdots dP(x_{i_n}) \\
&= \{(n-1)!\}^{-1} \sum_{j=1}^m \int_{B_j} \cdots \int \left\{ \int \hat{\theta}(x_1, \cdots, x_n) u_1(x_1)\, dP(x_1) \right\} \\
&\qquad \times h_1(x_2, \cdots, x_n)\, dP(x_2) \cdots dP(x_n)
\end{aligned} \tag{7.4}$$

for some integer $m \geq 1$, and some $\mathbf{S}_x^{(n-1)}$-measurable disjoint sets $B_1, \cdots, B_m$ which are invariant under permutations of their coordinates. By choosing successively $h_1(x_2, \cdots, x_n) = I[(x_2, \cdots, x_n) \in B_i]$, for $i = 1, 2, \cdots, m$, it follows that each integral in (7.4) is zero, whence

$$\int_{A_{n-1}} \cdots \int \{\hat{\theta}(x_1, \cdots, x_n) u_1(x_1)\, dP(x_1)\}\, dP(x_2) \cdots dP(x_n) = 0$$

for all $A_{n-1} \in \mathbf{S}_x^{(n-1)}$ and all $P \in \mathbf{C}_k$. Thus $\int u_1(x_1)\hat{\theta}(x_1, \cdots, x_n)\, dP(x_1) = 0$, all $P \in \mathbf{C}_k$, and similarly for $u_2, \cdots, u_k$. The proof is thus complete.

The proof of the corresponding result for the family $\mathbf{P}^{(2)}$ in Theorem 2 is quite similar to this; the proofs for the corresponding dominated families are modifications of the above proofs along the lines of the proof of Theorem 3.

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