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H_0-CONTIGUITY IN NONPARAMETRIC TESTING PROBLEMS AND SAMPLE PITMAN EFFICIENCY

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The concept of H_0-contiguity is studied for certain nonparametric testing problems. Furthermore, it is shown that a meaningful sample definition of Pitman efficiency is possible under H_0-contiguity, and this definition turns out to coincide with the usual efficiency concept based on comparing asymptotic distributions.

1. Introduction. The aim of the present note is twofold. On the one hand, we introduce the notion of "H_0-contiguity" of a sequence of alternatives to a compound null-hypothesis H_0. This concept is essentially the same as the notion of "contiguity to H_0" which was used by Hájek and Šidák (1967). We show that a consequent application of the H_0-contiguity concept simplifies the derivation of the limiting distribution of linear and other rank statistics under (not necessarily translation) alternatives.

On the other hand, we show that a meaningful sample definition of the asymptotic relative Pitman efficiency of certain rank tests is possible under (nearly) arbitrary H_0-contiguous alternatives. For the sake of concreteness, we deal mainly with linear rank tests for the two-sample problems of "randomness" versus "positive stochastic deviation of the first sample." In the last section some extensions are given to other tests and other testing problems.

2. H_0-contiguity and linear rank statistics. Let $X_1, \cdots, X_m$, respectively $Y_1, \cdots, Y_n$, be i.i.d. real random variables having continuous distribution functions (df's) F_m and G_n, and denote by $R_{11}, \cdots, R_{1m}$ and $R_{21}, \cdots, R_{2n}$ their ranks in the pooled sample consisting of $N = m + n$ observations. We want to consider the null-hypothesis of randomness $H_0: F_m = G_n$ versus the alternative, K, that the first sample is stochastically larger than the second sample, i.e. $K: F_m \le G_n$, $F_m \ne G_n$.

Let $b_{N1} \le \cdots \le b_{NN}$ be given scores such that the step functions $b_N: (0, 1) \to \mathbb{R}$, defined by $b_N(u) = b_{Ni}$ for $(i - 1)/N \le u < i/N$, $i = 1, \cdots, N$ converge in L_2 space of Lebesgue (λ)-square integrable functions on (0, 1) to some nondecreasing function $b: (0, 1) \to \mathbb{R}$ with $\langle \mathbf{1}, b\rangle = 0$ and $\|b\| = 1$:

$$\lim_{N\to\infty} \|b_N - b\| = 0, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in L_2 and $\|\cdot\|$ the corresponding norm, $\mathbf{1}$ being the function with constant value 1.

It is a well known fact that under H_0 the two-sample linear rank statistics

$$S_{mn} = (mn/N)^{1/2} \cdot \left\{\frac{1}{m}\sum_{i=1}^m b_N\left(\frac{R_{1i}}{N+1}\right) - \frac{1}{n}\sum_{j=1}^n b_N\left(\frac{R_{2j}}{N+1}\right)\right\} \tag{2.2}$$

converge in distribution to the standard normal distribution $\mathcal{N}(0, 1)$.

$$S_{mn} \to_{\mathscr{D}} \mathcal{N}(0, 1) \quad \text{for} \quad \min(m, n) \to \infty, \tag{2.3}$$

see e.g. Hájek and Šidák (1967). The crucial step in the proof of (2.3) is the result that

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under $H_0: F_m = G_n = H_{m,n}$, say, the statistics S_{mn} can be approximated in probability by

$$S_{mn}^* = (mn/N)^{1/2}\left\{\frac{1}{m}\sum_{i=1}^m b(U_i) - \frac{1}{n}\sum_{j=1}^n b(U_{m+j})\right\}, \tag{2.4}$$

$U_i = H_{m,n}(X_i)$, $i = 1, \cdots, m$, $U_{m+j} = H_{m,n}(Y_j)$, $j = 1, \cdots, n$; i.e.

$$S_{mn} - S_{mn}^* \to_{\text{Pr}} 0 \quad \text{under} \quad H_0 \quad \text{for} \quad \min(m, n) \to \infty. \tag{2.5}$$

From (2.5) and Lindeberg's Theorem applied to S_{mn}^*, (2.3) follows at once.

In order to judge the asymptotic power of the test with critical region $S_{mn} \geq u_\alpha$, u_α being the α-fractile of $\mathcal{N}(0, 1)$, one often considers alternatives (F_m, G_n) which are contiguous to the null hypothesis H_0 in the sense of Hájek and Šidák (1967, Chap. VI). Here we want to consider a slightly different concept, called H_0-contiguity.

Before giving the exact definition of the notion "H_0-contiguity", let us restrict the sequence of sample numbers (m, n), by assuming that the total sample number $N = m + n$ determines m and n, i.e. $m = m(N)$, $n = n(N)$ and assume that the relative sample number converges:

$$\lim_{N\to\infty} m(N)/N = \eta, \quad \text{say,} \quad \text{for} \quad \eta \in (0, 1). \tag{2.6}$$

Henceforth all limits are for $N \to \infty$. The reason for such a restriction is that later on we want to judge the efficiency of the test by the total sample number needed to raise the power of the test above some given value, rather than to consider the amount of asymptotic translation of S_{mn} under H_0 as in Hájek and Šidák (1967). To make such a sample efficiency concept meaningful, the above restrictions are very natural, though they are not really needed in the following definition.

DEFINITION 2.1. A sequence $\{(F_N, G_N)\}$, $N \geq 1$, is called H_0-contiguous, abbreviated $\{(F_N, G_N)\} \lhd H_0$, if there exists a sequence $\{(\tilde{H}_N, \tilde{H}_N)\}$, $N \geq 1$, in H_0 such that

$$\{\otimes_1^{m(N)} F_N \otimes \otimes_1^{n(N)} G_N\} \lhd \{\otimes_1^N \tilde{H}_N\}$$

in the usual one-sided contiguity sense, see e.g. Hájek and Šidák (1967) or Oosterhoff and van Zwet (1979). We want to give some characterizations of H_0-contiguity. Note first that $H_N = \eta_N F_N + (1 - \eta_N)G_N$, $\eta_N = m(N)/N$, dominates F_N and G_N. Then define a function d_N on (0, 1) by

$$d_N = \{mn/N\}^{1/2}(f_N - g_N)\circ H_N^{-1}, \tag{2.7}$$

with $f_N = dF_N/dH_N$, $g_N = dG_N/dH_N$ and H_N^{-1} the left continuous pseudoinverse of H_N. Since $\eta_N f_N + (1 - \eta_N)g_N = 1[H_N]$, f_N and g_N are bounded:

$$0 \leq f_N \leq 1/\eta_N, \quad 0 \leq g_N \leq 1/(1 - \eta_N)[H_N].$$

The following lemma characterizes H_0-contiguity.

LEMMA 2.2. *Under condition* (2.6), *the following four statements are equivalent.*

$$\{(F_N, G_N)\} \lhd H_0, \tag{2.8}$$

$$\limsup_N \mathscr{H}^2(F_N, G_N) < \infty, \tag{2.9}$$

$$\limsup \| d_N \| < \infty, \tag{2.10}$$

$$\{\otimes_1^{m(N)} F_N \otimes \otimes_1^{n(N)} G_N\} \lhd \{\otimes_1^N H_N\}. \tag{2.11}$$

In (2.9) $\mathscr{H}$ denotes the Hellinger distance defned by

$$\mathscr{H}(F, G) = \left\{\int (f^{1/2} - g^{1/2})^2 \, d\mu\right\}^{1/2}$$

for two probability measures (or df's) F and G, with μ-densities f and g, where μ is any σ-finite measure dominating F and G; for $\rho(F, G) = \int (fg)^{1/2}\, d\mu$ one has $\mathscr{H}^2(F, G) = 2\{1 - \rho(F, G)\}$.

The proof of the Lemma makes use of a result of Oosterhoff and van Zwet (1979) which we cite in advance for easier reference: Let P_{Ni}, Q_{Ni} be probability measures on arbitrary σ-finite measure spaces $(\mathscr{X}_{Ni}, \mathscr{A}_{Ni}, \mu_{Ni})$ such that μ_{Ni} dominates $P_{Ni} + Q_{Ni}$ with $p_{Ni} = dP_{Ni}/d\mu_{Ni}$ and $q_{Ni} = dQ_{Ni}/d\mu_{Ni}$ $(i = 1, \cdots, N, N = 1, 2, \cdots)$. Then according to Oosterhoff and van Zwet (1979), Theorem 1, the conditions

(A) $\limsup \sum_{i=1}^{N} \mathscr{H}^2(P_{Ni}, Q_{Ni}) < \infty$,

and

(B) $\lim \sum_{i=1}^{N} Q_{Ni}\{q_{Ni}/p_{Ni} \geq c_N\} = 0,\ c_N \to \infty$,

are jointly equivalent to $\{\otimes_{i=1}^{N} Q_{Ni}\} \lhd \{\otimes_{i=1}^{N} P_{Ni}\}$.

Proof of Lemma: From $\mathscr{H}(F_N, G_N) \leq \mathscr{H}(F_N, \tilde{H}_N) + \mathscr{H}(\tilde{H}_N, G_N)$, (2.6) and condition (A), it follows that (2.8) implies (2.9). The equivalence of (2.9) and (2.10) follows from

$$\|d_N\|^2 \leq \{\eta_N(1 - \eta_N)\}^{-1} N \mathscr{H}^2(F_N, G_N) \leq \{\eta_N(1 - \eta_N)\}^{-2} \|d_N\|^2, \quad \forall N \geq 1. \tag{2.12}$$

The first inequality in (2.12) is a consequence of

$$\begin{aligned} \left| \int h\circ H_N (f_N - g_N)\, dH_N \right| &= \left| \int h\circ H_N (f_N^{1/2} + g_N^{1/2})(f_N^{1/2} - g_N^{1/2})\, dH_N \right| \\ &\leq \{\eta_N(1 - \eta_N)\}^{-1} \int |h\circ H_N|\, |f_N^{1/2} - g_N^{1/2}|\, dH_N \\ &\leq \{\eta_N(1 - \eta_N)\}^{-1} \|h\| \mathscr{H}(F_N, G_N), \quad h \in L_2; \end{aligned} \tag{2.13}$$

for the first inequality in (2.13) note that $(f_N^{1/2} + g_N^{1/2}) \leq \{\eta_N(1 - \eta_N)\}^{-1}$, which follows from the concavity of the square root and from $\eta_N f_N + (1 - \eta_N) g_N = 1$. Putting $h = (f_N - g_N)\circ H_N^{-1}$ in (2.13) yields the first half of (2.12), while the second one is entailed by $|f_N^{1/2} - g_N^{1/2}| \leq |f_N - g_N|$. This is true since $\eta_N f_N + (1 - \eta_N) g_N = 1$. Since (2.11) $\Rightarrow$ (2.8) is trivially fufilled, it remains to show that (2.9) implies (2.11). Omitting the index N, one gets from the concavity of the square root function

$$\begin{aligned} \rho(H, F) &= \int \{\eta f + (1 - \eta) g\}^{1/2} f^{1/2}\, dH \geq \int \{\eta f^{1/2} + (1 - \eta) g^{1/2}\} f^{1/2}\, dH \\ &= \eta \int f\, dH + (1 - \eta) \int g^{1/2} f^{1/2}\, dH = \eta + (1 - \eta)\rho(F, G), \end{aligned}$$

and, similarly, $\rho(H, G) \geq (1 - \eta) + \eta\rho(F, G)$. Hence

$$\mathscr{H}^2(H, F) = 2\{1 - \rho(H, F)\} \leq 2\{1 - \eta - (1 - \eta)\rho(F, G)\} = (1 - \eta)\mathscr{H}^2(F, G)$$

and $\mathscr{H}^2(H, G) \leq \eta \mathscr{H}^2(F, G)$. Combining the last two inequalities yields

$$\begin{aligned} &\limsup\{m(N)\mathscr{H}^2(F_N, H_N) + n(N)\mathscr{H}^2(G_N, H_N)\} \\ &\qquad \leq 2\cdot \limsup \eta_N(1 - \eta_N) N \mathscr{H}^2(F_N, G_N) < \infty. \end{aligned}$$

Therefore condition (A) is met for the sequences in (2.11). Since condition (B) is automatically fulfilled for uniformly bounded densities, (2.11) follows. □

The interesting feature of Lemma 2.2 is that under (2.6) H_0-contiguity is expressible completely by means of the Hellinger distance $\mathscr{H}(F_N, G_N)$.

Now assume that $\{(F_N, G_N)\}$ is H_0-contiguous. Choosing $H_{m,n} = H_N$ in (2.5) and (2.6) and writing $S_N^{(*)} = S_{m(N),n(N)}^{(*)}$, we obtain

$$S_N - S_N^* \to_{\Pr} 0 \quad \text{under} \quad \{(F_N, G_N)\}. \tag{2.14}$$

Then

$$E_{F_N,G_N} S_N^* = (mn/N)^{1/2} \int b \circ H_N (f_N - g_N)\, dH_N = < b, d_N > \tag{2.15}$$

with d_N as above. Since f_N and g_N are bounded, the expectation in (2.15) is always finite. This is a technical advantage of H_0-contiguity compared with the usual approach using fixed elements (H, H) in H_0, because in the latter case the corresponding expectations are in general finite only after a truncation of the score function b, see e.g. Behnen (1972). Moreover, (2.12) gives an explicit bound for $\{< b, d_N >\}$, $N \geq 1$:

$$\limsup < b, d_N >^2 \leq \{\eta(1-\eta)\}^{-1} \limsup N\mathscr{H}^2(F_N, G_N) < \infty. \tag{2.16}$$

Using (2.13) with $h = b$ yields, after simple calculations, $\lim \mathrm{Var}_{(F_N,G_N)} S_N^* = 1$, implying that $\{S_N^* - < b, d_N >\}$ obeys the Lindeberg condition. Therefore, using (2.14) one finally gets

$$S_N - < b, d_N > \to_{\mathscr{D}} \mathscr{N}(0, 1) \quad \text{under} \quad \{(F_N, G_N)\} \lhd H_0, \tag{2.17}$$

which is the basis for judging the asymptotic power of the rank test with critical region $S_N \geq u_\alpha$.

For our efficiency considerations in the next section, it will be useful to extend (2.17) a little bit. Let $\{k(N)\}$, $N \geq 1$, be a subsequence of the sequence of total sample numbers $\{N\}$ with

$$\lim k(N) = \infty \quad \text{and} \quad \liminf N/k(N) > 0. \tag{2.18}$$

Then

$$S_{k(N)} - \{k(N)/N\}^{1/2} < b, d_N > \to_{\mathscr{D}} \mathscr{N}(0, 1) \quad \text{under} \quad \{(F_N, G_N)\} \lhd H_0. \tag{2.19}$$

PROOF OF (2.19). Without loss of generality, assume $k(N) < k(N+1) \forall N \geq 1$; otherwise one may take a suitable subsequence. Define $\tilde{F}_{k(N)} \equiv F_N$, $\tilde{G}_{k(N)} \equiv G_N \forall N \geq 1$ and let $\tilde{F}_k = \tilde{G}_k$ be arbitrary df's for $k \notin \{k(N): N \geq 1\}$. Then (2.18) and (2.9) imply $\{(\tilde{F}_k, \tilde{G}_k)\} \lhd H_0$. Using (2.17) for $\{(\tilde{F}_k, \tilde{G}_k)\}$ instead of $\{(F_N, G_N)\}$ and taking the subsequence $\{k(N)\}$, one gets (2.19).

REMARK 2.3. It is easy to show that

$$F_m \underset{\neq}{\leq} G_n \Leftrightarrow \int_0^t d_N\, d\lambda \leq 0 \ \forall t \in (0, 1) \quad \text{and} \quad \text{"<"} \quad \text{for some} \quad t \in (0, 1). \tag{2.20}$$

Therefore, d_N characterizes K. Using the fact that b is assumed to be nondecreasing, one gets $< b, d_N > \geq 0$ for $(F_m, G_n) \in K$ with strict inequality for strictly increasing b, implying that the translation in (2.19) is positive in that case.

REMARK 2.4. Every fixed pair (F, G) of alternatives generates a corresponding family of alternatives (F_Δ, G_Δ), $0 \leq \Delta \leq 1$, by

$$f_\Delta = dF_\Delta/dH = 1 + \Delta(1 - \eta)\, d(F - G)/dH \tag{2.21}$$

and

$$g_\Delta = dG_\Delta/dH = 1 - \Delta\eta\, d(F - G)/dH, \quad 0 \leq \Delta \leq 1, \tag{2.22}$$

with $\eta = \lim m(N)/N$, $H = \eta F + (1 - \eta)G$. Clearly, $F_1 = F$, $G_1 = G$, $F_0 = G_0 = H$ and $\eta F_\Delta + (1 - \eta)G_\Delta = H \forall \Delta$. Under condition (2.6) for $\Delta = \Delta_N$, the following equivalence holds true:

(2.23) $$\{(F_{\Delta_N}, G_{\Delta_N})\} \lhd H_0 \Leftrightarrow \Delta_N = O(N^{-1/2}).$$

PROOF OF (2.23). Using

$$|f_\Delta^{1/2} - g_\Delta^{1/2}| \le |f_\Delta - g_\Delta| = \Delta\,|d(F-G)/dH| \le \Delta\{\eta(1/\eta)\}^{-1}$$

one gets

$$N\mathscr{H}^2(F_{\Delta N}, G_{\Delta N}) \le N\Delta_N^2\{\eta(1-\eta)\}^{-2},$$

yielding together with Lemma 2.2 that $\Delta_N = O(N^{-1/2})$ is sufficient for H_0-contiguity. On the other hand, using the well known inequality $\|F_\Delta - G_\Delta\|_1 \le \mathscr{H}(F_\Delta, G_\Delta)$ where $\|\cdot\|_1$ denotes the total variation distance, one gets

$$N^{1/2}\Delta_N\|F-G\|_1 = N^{1/2}\|F_{\Delta N} - G_{\Delta N}\|_1 \le N^{1/2}\mathscr{H}(F_{\Delta N}, G_{\Delta N}).$$

An application of Lemma 2.2 to the last inequality yields (2.23). □

The quantity $d(F-G)/dH$ represents in some sense the deviation of (F, G) from H_0. Multiplying $d(F-G)/dH$ by a factor $\Delta_N \to 0$ maintains the character of deviation while $\{(F_{\Delta_N}, G_{\Delta_N})\}$ approaches H_0.

Therefore, the strictly nonparametric family (F_Δ, G_Δ), $0 \le \Delta \le 1$, seems to be a natural one for applying local asymptotic power results to finite N. Moreover, technical advantages also exist for generating a H_0-contiguous sequence by starting from elements of the alternative in contrast to the usual way of generating contiguous sequences by starting from elements of H_0 (translation alternatives, Lehmann alternatives). E.g., it is very easy to prove consistency of the linear rank tests by noting that $F_\Delta \le F$, $G_\Delta \ge G \forall \Delta \in [0, 1]$ and that $N^{1/2}\Delta_N$ may be chosen to converge to arbitrary large values; see (3.2).

3. A sample definition of Pitman efficiency under H_0-contiguity. Let $\{\varphi_N\}$ and $\{\psi_N\}$ be two sequences of level α tests for the two-sample testing problem of Section 2 and $\{(F_N, G_N)\}$ be any sequence of alternatives. Denote by $\underset{\sim}{k}(N)$ the smallest $k \ge 1$ with

$$E_{F_N,G_N}\varphi_k \ge E_{F_N,G_N}\psi_N = \beta_N,$$

say, and $\underset{\sim}{k}(N) = \infty$ otherwise and define the asymptotic relative efficiency (ARE) of $\{\varphi_N\}$ with respect to $\{\psi_N\}$ on $\{(F_N, G_N)\}$ by

(3.1) $$\mathrm{ARE}(\varphi:\psi) = \liminf N/\underset{\sim}{k}(N).$$

We want to compute the ARE of linear (and other) rank tests for the two sample testing problem of Section 2 under H_0-contiguous alternatives $\{(F_N, G_N)\}$. Therefore, let φ_N be the one-sided level α test rejecting for large values of S_N, where S_N is taken from Section 2.

Then under H_0-contiguous alternatives $\{(F_N, G_N)\}$, (2.17) yields for subsequences $\{k(N)\}$ of $\{N\}$ with (2.18) the following asymptotic formula for the power of φ_N:

(3.2) $$E_{F_N,G_N}\varphi_{k(N)} = M\left(\left\{\frac{k(N)}{N}\right\}^{1/2} T(d_N)\right) + o(1)$$

with $M(z) = 1 - \Phi(u_\alpha - z)$, Φ denoting the df of $\mathscr{N}(0, 1)$, and $T(x) = \langle b, x \rangle$, for $x \in L_2$.

THEOREM 3.1. *Let $\{\varphi_N\}$ resp. $\{\psi_N\}$ be level α tests fulfilling* (3.2.) *resp.*

(3.3) $$0 < \alpha < \liminf \beta_N \le \limsup \beta_N < 1, \quad \beta_N \equiv E_{F_N,G_N}\psi_N.$$

Then

(3.4) $\liminf N/\underset{\sim}{k}(N) = 0$ iff 0 *is an accumulation point of* $\{T(d_N): N \ge 1\}$, *and*

(3.5) $\liminf N/\underline{k}(N) > 0$ *implies* $N/\underline{k}(N) = \{T(d_N)/M^{-1}(\beta_N)\}^2 + o(1)$,

and, consequently,

(3.6) $$\text{ARE}(\varphi:\psi) = \liminf\{T(d_N)/M^{-1}(\beta_N)\}^2.$$

PROOF. We show the "only if" part of (3.4): Fix a number $\rho > 0$ and define $k(N)$ to be the integer part of $N \cdot \rho^2$. Then $k(N) \to \infty$ and $k(N)/N \to \rho^2$. Now, let $\{N'\}$ be a subsequence of $\{N\}$ with $N'/\underline{k}(N') \to 0$ and $T(d_{N'}) \to a(\geq 0)$, the last assumption being possible because of the boundedness of $\{T(d_N)\}$. Clearly, $k(N') < \underline{k}(N') \forall N' \geq N_0$ for some N_0. Hence from the definition of $\{\underline{k}(N)\}$ and (3.2)

$$\beta_{N'} \geq E_{F_{N'},G_{N'}}\varphi_{k(N')} = M(\{k(N')/N'\}^{1/2}T(d_{N'})) + o(1)$$
$$= M(\rho a) + o(1) \forall N' \geq N_0,$$

implying $\limsup \beta_N \geq M(\rho a) \forall \rho > 0$. Because of (3.3), this is only possible for $a = 0$, i.e. 0 is an accumulation point of $\{T(d_N)\}$.

In order to show the "if" part of (3.4), note first that H_0-contiguity of $\{(F_N, G_N)\}$ implies $\int |f_N - g_N| \, dH_N \to 0$ and therefore $\lim_{N\to\infty} E_{F_N,G_N}\varphi_r = E_{H_0}\varphi_r = \alpha$. Hence $\alpha < \liminf \beta_N$ forces $\{\underline{k}(N)\}$ to converge to ∞. Now assume that $\liminf N/\underline{k}(N) > 0$. Then $\{\underline{k}(N)\}$ fulfills (2.18), especially $\underline{k}(N) < \infty$ for $N \geq N_0$, say. Therefore,

(3.7) $$\beta_N < E_{F_N,G_N}\varphi_{\underline{k}(N)} = M(\{\underline{k}(N)/N\}^{1/2}T(d_N)) + o(1).$$

The last inequality shows that 0 cannot be an accumulation point of $\{T(d_N)\}$, since $T(d_{N'}) \to 0$ would imply $\{\underline{k}(N')/N'\}^{1/2}T(d_{N'}) \to 0$ and consequently $\beta_{N'} \to \alpha$, contradicting (3.3); (3.4) follows. Moreover, for $\liminf N/\underline{k}(N) > 0$

(3.8) $$\beta_N \geq E_{F_N,G_N}\varphi_{\underline{k}(N)-1} = M(\{(\underline{k}(N) - 1)/N\}^{1/2}T(d_N)) + o(1).$$

(3.7) and (3.8) and the boundedness of $\{T(d_N)\}$ entail

(3.9) $$\beta_N = M(\{\underline{k}(N)/N\}^{1/2}T(d_N)) + o(1).$$

Since M^{-1} is continuous on $[\alpha, 1)$ and $\{\underline{k}(N)/N\}^{1/2}T(d_N)$, $N \geq 1$, is bounded, (3.9) may be inverted yielding

(3.10) $$M^{-1}(\beta_N) = \{\underline{k}(N)/N\}^{1/2}T(d_N) + o(1).$$

From $\alpha < \liminf \beta_N$ it follows that $\{M^{-1}(\beta_N)\}$, $N \geq 1$, is bounded away from 0, consequently, $1/M^{-1}(\beta_N) = \{N/\underline{k}(N)\}^{1/2}/T(d_N) + o(1)$, implying (3.5), since $\{T(d_N)\}$, $N \geq 1$, is bounded. □

EXAMPLE 3.2. Let us compare two linear rank tests $\{\varphi_N\}$ and $\{\psi_N\}$ with asymptotic score functions b_1 and b_2, respectively. Then

(3.11) $$\text{ARE}(\varphi:\psi) = \liminf < b_1, d_N >^2 / < b_2, d_N >^2,$$

if $\liminf < b_2, d_N >^2 > 0$.

PROOF. (3.2) applied to $\{\psi_N\}$ with $k(N) = N$ yields $\beta_N = E_{F_N,G_N}\psi_N = M(< b_2, d_N >) + o(1)$. Using (3.6) with $T(x) = < b_1, x >$ yields (3.10). □

The right side coincides (apart from the explicit form of the asymptotic translations $< b_i, d_N >$, $i = 1, 2$) with the definition of asymptotic relative Pitman efficiency based on asymptotic translations of the H_0-distribution, see e.g. Behnen (1972), thus yielding a sample efficiency interpretation of all results based on that ARE definition.

4. Complements. Similar results as in Theorem 3.1 are also possible for non-linear rank tests. Let us consider, for example, the one-sided two-sample Kolmogorov-Smirnov (KS) test based on the KS statistic

$$K_N = \sup_{x\in\mathbb{R}}(mn/N)^{1/2}\{\hat{G}_n(x) - \hat{F}_m(x)\}, \tag{4.1}$$

where $\hat{F}_m$ resp. $\hat{G}_n$ denotes the empirical df based on the X's resp. Y's. It is well known that the KS test with rejection region $K_N \geq (-\tfrac{1}{2}\log\alpha)^{1/2}$ has the asymptotic level α. Moreover, one can show that under H_0-contiguous alternatives $\{(F_N, G_N)\}$, formula (3.2) holds with

$$T: L_2 \to C[0, 1], \qquad T(x)(t) = -\int_0^t x\, d\lambda, \qquad 0 \leq t \leq 1, \tag{4.2}$$

and

$$M: C[0, 1] \to \mathbb{R}, \qquad M(z) = \Pr[\sup_{0\leq t\leq 1}\{W_0(t) + z(t)\} > (-\tfrac{1}{2}\log\alpha)^{1/2}], \tag{4.3}$$

where W_0 denotes the Brownian Bridge on [0, 1].

For fixed $z \in C[0, 1]$ with $z(t) \geq 0\forall t$, $z \neq 0$, the mapping $M_z:[0, \infty) \to [\alpha, 1)$ defined by $M_z(\rho) = M(\rho z)$ is continuous, strictly increasing and bijective.

Note that for $(F_N, G_N) \in K$, (2.20) implies $T(d_N)(t) \geq 0\forall t$, $T(d_N) \neq 0$, and that the boundedness of $\{\| d_N \|\}$ implies relative compactness of $\{T(d_N)\}$ in $C[0, 1]$. Using these facts, one can show that Theorem 3.1 holds in the KS setting with (3.5) replaced by

$$\liminf N/\underline{k}(N) > 0 \quad \text{implies} \quad N/\underline{k}(N) = \{M^{-1}_{T(d_N)}(\beta_N)\}^2 + o(1) \tag{3.5$'$}$$

and (3.6) replaced by

$$\mathrm{ARE}(\varphi:\psi) = \liminf\{M^{-1}_{T(d_N)}(\beta_N)\}^2, \tag{3.6$'$}$$

where φ represents the KS test.

Formula (3.6′) can serve as a basis for efficiency comparisons of linear rank tests with the two sample KS test similarly, as was done by Yu (1971) for translation alternatives.

As mentioned in the introduction, the concept of H_0-contiguity also makes sense for other testing problems. Especially simple is the 1-dimensional one-sample problem of testing "symmetry" versus "positive asymmetry": For a sequence $X_1, \cdots, X_N$ of i.i.d. real random variables having continuous df's F_N, one considers the null-hypothesis of symmetry $H_0: F_N(x) = 1 - F_N(-x)\forall x \in \mathbb{R}$ versus the alternative of positive asymmetry $K: F_N(x) \leq 1 - F_N(-x) \equiv F_N^-(x)$, $\forall x \in \mathbb{R}$, and "<" for some x. Defining H_0-contiguity of $\{F_N\}$ in the same manner as in Definition 2.1, and noting that the symmetrization $F_{N*} = (F_N + F_N^-)/2$ of F_N dominates F_N, one gets with $d_N = N^{1/2}(dF_N/dF_{N*} - 1)\circ F_{N*}^{-1}$, an analogue of the characterization of H_0-contiguity in Lemma 2.2, namely the following.

LEMMA 2.2′. *The following four statements are equivalent*

$$\{F_N\} \lhd H_0, \tag{2.8$'$}$$

$$\limsup N\mathscr{H}^2(F_N, F_{N*}) < \infty, \tag{2.9$'$}$$

$$\limsup \| d_N \| < \infty, \tag{2.10$'$}$$

$$\{\otimes_1^N F_N\} \lhd \{\otimes_1^N F_{N*}\}. \tag{2.11$'$}$$

Notice that conditions like (2.6) are not needed here. All other results of Section 2 and 3 can be carried over to the present testing problem without any difficulty.

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