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SUFFICIENCY AND INVARIANCE IN CONFIDENCE SET ESTIMATION

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This paper describes how sufficiency and invariance considerations can be applied in problems of confidence set estimation to reduce the class of set estimators under investigation. Let X be a random variable taking values in $\mathscr{X}$ with distribution P_θ, $\theta \in \Theta$, and suppose a confidence set is desired for $\gamma = \gamma(\theta)$, where γ takes values in Γ. The main tools used are (i) the representation of randomized set estimators as functions $\varphi: \mathscr{X} \times \Gamma \rightarrow [0,1]$, and (ii) the definition of sufficiency in terms of a certain family of distributions on $\mathscr{X} \times \Gamma$. Sufficiency and invariance reductions applied in tandem to $\mathscr{X} \times \Gamma$ yield a class of set estimators that is essentially complete among all invariant set estimators, provided the risk function depends only on $E_\theta \varphi(X, \gamma)$, $(\theta, \gamma) \in \Theta \times \Gamma$. Several illustrations are given.

1. Introduction and summary. Little exists in the literature concerning the principle of sufficiency as it applies in confidence set estimation. The results of Bahadur (1954) apply in problems of confidence interval estimation, but they do not extend to the general situation. The difficulty lies in the non-Euclidean nature of the action space (a set of subsets) and in describing the class of randomized procedures. The primary purpose of this paper is to introduce an appropriate statistical model such that sufficiency considerations can be used to reduce the class of set estimators under investigation. Sufficiency reductions combined with invariance reductions are of particular interest.

This work was motivated by the general MANOVA problem, presented in Example 4.3. Sufficiency and invariance reductions have been carried out for the associated testing problem in Kariya (1978). The results of the present paper make it possible to carry out the corresponding reductions in set estimation problems.

The discussion will be carried out in the following framework. Let $\mathscr{X}$ be a measurable space with σ-field $\mathscr{F}$. Let X be a random variable taking values in $\mathscr{X}$ and having distribution P_θ, $\theta \in \Theta$. Suppose inference is desired about some function of θ, say $\gamma = \gamma(\theta)$, taking values in Γ. The symbol γ will be used to denote both the function $\gamma: \Theta \rightarrow \Gamma$ and a point in Γ, depending on the context.

In Section 2 randomized set estimators are described as functions φ mapping $\mathscr{X} \times \Gamma$ into [0, 1]. In Section 3 the principles of sufficiency and invariance are both shown to lead to reductions of $\mathscr{X} \times \Gamma$. In Section 4 the results are applied in three examples.

2. Randomized set estimators. In general, to justify restricting one's attention to procedures based on a sufficient statistic, it is necessary to allow for the use of randomized procedures. While it is true that one usually wants to end up using a nonrandomized procedure and while it often turns out that only nonrandomized procedures are admissible, still randomization is needed for general theoretical results. Thus we shall need to adopt one of several possible definitions of randomized set estimators.

First, by a nonrandomized set estimator we shall mean a subset C of $\mathscr{X} \times \Gamma$ with the property that, for each fixed γ, the cross section

$$C(\cdot, \gamma) \equiv \{x : (x, \gamma) \in C\} \in \mathscr{F}.$$

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If desired, stronger measurability restrictions may be imposed; see the discussion at (3.1). When X takes the value x, one chooses as a confidence set the cross section

$$C(x, \cdot) = \{\gamma : (x, \gamma) \in C\}. \tag{2.1}$$

Joshi (1969, page 1044) discusses randomized set estimators. Every such estimator determines a function $\varphi : \mathscr{X} \times \Gamma \to [0, 1]$ defined by: $\varphi(x, \gamma)$ is the conditional probability that γ is included in the confidence set given that X takes the value x. It follows that, for fixed γ, $\varphi(\cdot, \gamma)$ is $\mathscr{F}$-measurable. Conversely every function $\varphi : \mathscr{X} \times \Gamma \to [0, 1]$ with $\varphi(\cdot, \gamma)\mathscr{F}$-measurable for each γ corresponds to at least one randomized set estimator; see the discussion at (2.2). However, the function φ does not uniquely specify a randomized set estimator. As a trivial example let $\{\Gamma_1, \Gamma_2\}$ be a partition of Γ and consider the procedure: pick Γ_i with probability ½, $i = 1, 2$. Then $\varphi(x, \gamma) \equiv ½$ obtains no matter which partition is used. This lack of uniqueness is unimportant for describing set estimators because set estimators with identical φ functions can be regarded as equivalent. Most of the usual criteria for comparing set estimators depend only on $E_\theta \varphi(X, \gamma)$, specifically

(i) the probability of covering the true value of $\gamma : E_\theta\varphi(X, \gamma)$ for $\gamma = \gamma(\theta)$;
(ii) the probability of covering false values of $\gamma : E_\theta\varphi(X, \gamma)$ for $\gamma \neq \gamma(\theta)$;
(iii) the expected "size" of the confidence set: $\int E_\theta\varphi(X, \gamma)m(d\gamma, \theta)$ where, for each θ, $m(d\gamma, \theta)$ is a measure on Γ.

For (iii), Γ must be a measurable space and it is convenient to take φ to be jointly measurable. Joshi follows this convention. See the discussion at (3.1). Usually either (i) and (ii) or (i) and (iii) are used. See Cohen and Strawderman (1973) for a fuller discussion of these criteria. We assume below that all criteria are based on $E_\theta\varphi(X, \gamma)$, $(\theta, \gamma) \in \Theta \times \Gamma$ with γ not necessarily equal to $\gamma(\theta)$.

Thus, in characterizing randomized set estimators, one need consider only equivalence classes of estimators, each class represented by a φ function. Using this representation it is still necessary to define an explicit procedure for each φ; i.e., one has to describe, for each x, how one uses $\varphi(x, \cdot)$ to obtain a subset of Γ. We will use the following convention. Let U be uniformly distributed on $[0, 1]$ independently of X. Then for a given φ we have in mind the set estimator C_φ based on (X, U) given as

$$C_\varphi = \{(x, u, \gamma) \in \mathscr{X} \times [0, 1] \times \Gamma : u \leq \varphi(x, \gamma)\}. \tag{2.2}$$

For a particular value (x, u) of (X, U) one chooses the cross section $C_\varphi(x, u, \cdot)$ as in (2.1).

As an aside we note that the above representation makes it possible to impose restrictions on the shape of confidence sets determined by φ by imposing these restrictions on the level sets of $\varphi(x, \cdot)$. Also the definition allows one to invert randomized tests to obtain set estimators without explicitly writing down acceptance regions in terms of the randomization device. If, for each $\gamma^* \in \Gamma$, $\varphi_{\gamma^*}(x)$ is a level α test for $H : \gamma = \gamma^*$ then $\varphi(x, \gamma) = 1 - \varphi_\gamma(x)$ is a level $1 - \alpha$ randomized set estimator for γ. The converse obviously holds as well.

3. Invariance and sufficiency. Invariance reductions are considered first. Suppose G is an invariance group that also acts on Γ; for definitions see Wijsman (1980, Section 3). The actions will be denoted $x \to gx$, $\theta \to g\theta$, $\gamma \to g\gamma$. A nonrandomized set estimator C is said to be equivariant under G provided $C(gx, \cdot) = gC(x, \cdot)$ for all $x \in \mathscr{X}$, $g \in G$; see Lehmann (1959, page 243). The natural extension of this definition to randomized set estimators φ is to require φ to be invariant; i.e., $\varphi(gx, g\gamma) = \varphi(x, \gamma)$ for all g, x, γ. It is easily checked, using (2.2), that $C_\varphi(x, u, \cdot)$ is equivariant for all u if and only if φ is invariant. So the principle of invariance allows us to restrict our attention to set estimators φ which are invariant under G; or equivalently, to consider only set estimators defined in terms of a maximal invariant on $\mathscr{X} \times \Gamma$. Actually this last statement requires some mild measurability assumptions; see Remark 2.

The main device that makes the principle of sufficiency useful in set estimation is the introduction of an appropriate σ-field $\mathscr{A}$ of subsets of $\mathscr{X} \times \Gamma$ and a certain family

$\{P_{\theta,\gamma}: (\theta, \gamma) \in \Theta \times \Gamma\}$ of distributions on $\mathscr{A}$. Let $\mathscr{A}_0$ be the σ-field defined by

$$\mathscr{A}_0 = \{C \subseteq \mathscr{X} \times \Gamma : C(\cdot, \gamma) \in \mathscr{F} \text{ for each } \gamma \in \Gamma\}. \tag{3.1}$$

Then take $\mathscr{A}$ to be any sub-σ-field of $\mathscr{A}_0$. Define $P_{\theta,\gamma}$ on $\mathscr{A}$ by

$$P_{\theta,\gamma}(C) = P_\theta\{C(\cdot, \gamma)\}. \tag{3.2}$$

Taking $\mathscr{A} = \mathscr{A}_0$ corresponds to the least restrictive definition of set estimators; i.e. φ is $\mathscr{A}_0$-measurable if and only if $\varphi(\cdot, \gamma)$ is $\mathscr{F}$-measurable for each γ. In certain situations it may be desirable to take $\mathscr{A}$ strictly coarser than $\mathscr{A}_0$; e.g., if Γ is a measurable space and criteria (iii) (expected size) is used, it is convenient to take $\mathscr{A}$ to be the product σ-field.

Suppose $\mathscr{T}$ is a measurable space and $T: \mathscr{X} \times \Gamma \to \mathscr{T}$ is measurable. (Here and below, "measurable" means $\mathscr{A}$-measurable.) Let $\mathscr{P}^T = \{P^T_{\theta,\gamma}: (\theta, \gamma) \in \Theta \times \Gamma\}$ denote the family of distributions induced by $\mathscr{P}$ on $\mathscr{T}$; i.e., for measurable $A \subseteq \mathscr{T}$,

$$P^T_{\theta,\gamma}(A) = P_\theta\{T(\cdot, \gamma)^{-1}A\}.$$

Suppose $\mathscr{S}$ is a measurable space and that $S: \mathscr{T} \to \mathscr{S}$ is sufficient for $\mathscr{P}^T$. Then if φ_1 is a set estimator defined in terms of T, $\varphi_1(x, \gamma) = F_1\{T(x, \gamma)\}$, there exists a set estimator φ_2 defined in terms of S, $\varphi_2(x, \gamma) = F_2[S\{T(x, \gamma)\}]$, such that $E_\theta\varphi_1(X, \gamma) = E_\theta\varphi_2(X, \gamma)$ for all $(\theta, \gamma) \in \Theta \times \Gamma$; here γ is not necessarily equal to $\gamma(\theta)$. This follows by taking $F_2(S) = E\{F_1(T) \mid S\}$; the assumption of sufficiency guarantees existence of a version of this conditional expectation which is free of (θ, γ). Therefore we have the following result.

LEMMA 1. *If S is sufficient for $\mathscr{P}^T$, then the class of randomized set estimators based on S is essentially complete among those based on T, provided the risk depends only on $E_\theta\varphi(X, \gamma)$, $(\theta, \gamma) \in \Theta \times \Gamma$.*

The following theorem describes two important situations where sufficiency relative to $\mathscr{P}^T$ is equivalent to the usual, weaker notion of sufficiency relative to $\{P^T_{\theta,\gamma}: \theta \in \Theta\}$ for γ fixed.

THEOREM 1. (i) *Sufficiency reduction of X alone: Suppose $\mathscr{T} = \mathscr{X} \times \Gamma$, $T =$ the identity function, and that $Y: \mathscr{X} \to \mathscr{Y}$ is sufficient for $\{P_\theta: \theta \in \Theta\}$. Then $S(X, \gamma) = (Y(X), \gamma)$ is sufficient for $\mathscr{P}$.* (ii) *Sufficiency reduction after an invariance reduction: Suppose G is an invariance group acting transitively on Γ. Let $T: \mathscr{X} \times \Gamma \to \mathscr{T}$ be invariant under G. Suppose, for some particular $\gamma^* \in \Gamma$, that $S(T(X, \gamma^*))$ is sufficient for $\{P^T_{\theta,\gamma^*}, \theta \in \Theta\}$. Then S is sufficient for $\mathscr{P}^T$.*

PROOF. (i) is immediate. For (ii), fix $\gamma \in \Gamma$ and let $g \in G$ be such that $g\gamma = \gamma^*$. Then, for measurable $A \subseteq \mathscr{T}$, the following holds almost surely:

$$\begin{aligned} P^T_{\theta,\gamma}(A \mid S) &= P_\theta\{T(X, \gamma) \in A \mid S(T(X, \gamma))\} = P_{g\theta}\{T(g^{-1}X, \gamma) \in A \mid S(T(g^{-1}X, \gamma))\} \\ &= P_{g\theta}\{T(X, g\gamma) \in A \mid S(T(X, g\gamma))\} = P\{T(X, \gamma^*) \in A \mid S(T(X, \gamma^*))\}. \end{aligned}$$

The second equality follows because G is an invariance group, the third because T is invariant, and the fourth because S is sufficient when $\gamma = \gamma^*$. ☐

In applying sufficiency and invariance reductions one is usually interested in reducing first by invariance, then by sufficiency, obtaining an invariantly sufficient statistic; see Hall et al (1965, page 579). However, in practice it is often easier to perform a sufficiency reduction first, obtaining a sufficient statistic U upon which the group acts, and then find a maximal invariant Y on the range space of U. Hall et al (1965) give conditions under which the two routes yield the same result. The following provides a useful tool for verifying these regularity conditions.

THEOREM 2. *Let $T: \mathscr{X} \times \Gamma \to \mathscr{T}$ be measurable. Suppose G is an invariance group* *acting transitively on* Γ *and that* G *acts also on* $\mathscr{T}$ *so that* $T(gx, g\gamma) = gT(x, \gamma)$. *Let* V *be a* G*-invariant function defined on* $\mathscr{T}$ *and suppose that, for some particular* γ^*, V *is invariantly sufficient for* $\{P^T_{\theta,\gamma^*} : \theta \in \Theta\}$ *relative to the group* $G_{\gamma^*} \equiv \{g \in G : g\gamma^* = \gamma^*\}$. *Then* V *is invariantly sufficient for* $\mathscr{P}^T$ *relative to* G.

PROOF. Let A be a G-invariant measurable subset of $\mathscr{T}$. It sufficies to establish the existence of a version of $P^T_{\theta,\gamma}(A \mid V)$ which is free of (θ, γ). Fix $\gamma \in \Gamma$ and let $g \in G$ be such that $g\gamma = \gamma^*$. As described in Hall et al (1965, page 598), the transformation g produces an isomorphism between $(\mathscr{T}, P^T_{\theta,\gamma})$ and $(\mathscr{T}, P^T_{g\theta,g\gamma})$ so that, defining $gV(gt) = V(t)$,

$$P^T_{\theta,\gamma}(A \mid V) = P^T_{g\theta,g\gamma}(gA \mid gV) \quad \text{a.s.}$$

But by assumption, $gV = V$ and $gA = A$, so

$$P^T_{\theta,\gamma}(A \mid V) = P^T_{g\theta,\gamma^*}(A \mid V) \quad \text{a.s.}$$

Since A G-invariant implies A G_{γ^*}-invariant, the right side can be taken as free of θ. □

REMARK 1. In Examples 4.1 and 4.3 an invariantly sufficient function for $\mathscr{P}^T$ is obtained by reducing first to a sufficient statistic U and then to a maximal invariant Y on the range space of U. The resulting function $Y \circ U$ is shown to be invariantly sufficient by verifying Assumption A of Hall et al (1965, page 600). This verification is carried out by applying Theorem 4 of Lehmann (1959, page 225). Lehmann's theorem requires the existence of a σ-finite measure on G possessing a certain invariance property. This condition is satisfied in the examples. To apply the theorem it is also necessary that the family of distributions of U be dominated by a σ-finite measure. This is often not true of the family $\{P^U_{\theta,\gamma} : (\theta, \gamma) \in \Theta \times \Gamma\}$. However, in many parametric problems (including the examples), the family $\{P^U_{\theta,\gamma^*} : \theta \in \Theta\}$ is dominated for each $\gamma^* \in \Gamma$. Now U sufficient for $\mathscr{P}^T$ implies U sufficient for $\{P^T_{\theta,\gamma^*} : \theta \in \Theta\}$. Also in the examples it is easy to check that the restriction of Y to the range of $U(T(\cdot, \gamma^*))$ is a maximal invariant for G_{γ^*}. This shows (via Assumption A) that Y is invariantly sufficient for $\{P^T_{\theta,\gamma^*} : \theta \in \Theta\}$ relative to G_{γ^*}. Then the desired result follows from Theorem 2.

REMARK 2. Let $T : (\mathscr{X} \times \Gamma, \mathscr{A}) \rightarrow (\mathscr{T}, \mathscr{B})$ be a maximal invariant under G and let $\mathscr{A}_I$ be the sub-σ-field of all invariant sets in $\mathscr{A}$. By Lemma 1 of Lehmann (1959, page 37), all $\mathscr{A}$-measurable invariant functions φ on $\mathscr{X} \times \Gamma$ are of the form $f \circ T$ for $\mathscr{B}$-measurable f if and only if $\mathscr{A}_I = T^{-1}\mathscr{B}$. Convenient sufficient conditions for this follow from Blackwell (1956, Theorem 3 and Corollary 2). These conditions require, among other things, separability of $\mathscr{A}$, which does not hold if $\mathscr{A} = \mathscr{A}_0$ and Γ is uncountable. Thus it would be useful only to have to apply Blackwell's conditions to $\mathscr{X}$. The following theorem makes this possible. The conditions apply in particular when both $\mathscr{X}$ and $\mathscr{T}$ are Euclidean.

THEOREM 3. *Suppose* G *is an invariance group acting transitively on* Γ, $T : (\mathscr{X} \times \Gamma, \mathscr{A}) \rightarrow (\mathscr{T}, \mathscr{B})$ *is a maximal invariant under* G, $(\mathscr{X}, \mathscr{F})$ *is a Lusin space, and* $\mathscr{B}$ *is separable and contains all singletons. Then, given an* $\mathscr{A}$*-measurable invariant* $\varphi : \mathscr{X} \times \Gamma \rightarrow [0, 1]$, *there exists a* $\mathscr{B}$*-measurable* $f : \mathscr{T} \rightarrow [0, 1]$ *such that* $\varphi = f \circ T$.

PROOF. Since $\mathscr{A}$-measurability implies $\mathscr{A}_0$-measurability, we may assume without loss of generality that $\mathscr{A} = \mathscr{A}_0$. Fix $\gamma^* \in \Gamma$. Let $G_{\gamma^*} = \{g \in G : g\gamma^* = \gamma^*\}$ and $\mathscr{A}_{\gamma^*} = \{C(\cdot, \gamma^*) : C \in \mathscr{A}_I\}$, where $\mathscr{A}_I$ consists of the invariant sets in $\mathscr{A}_0$. Then $\mathscr{A}_{\gamma^*}$ is the σ-field consisting of all G_{γ^*}-invariant $A \in \mathscr{F}$. (Suppose $A \in \mathscr{F}$ is G_{γ^*}-invariant. Set $C = \cup_{g \in G}(gA \times \{g\gamma^*\})$. Then C is G-invariant. Since $g_1\gamma^* = g_2\gamma^* \Leftrightarrow g_2^{-1}g_1 \in G_{\gamma^*} \Rightarrow g_1A = g_2A$, we have, for arbitrary $\gamma \in \Gamma$, $C(\cdot, \gamma) = gA$ for any $g \in G$ such that $g\gamma^* = \gamma$. So $C \in \mathscr{A}_0$, since $gA \in \mathscr{F}$, and $C(\cdot, \gamma^*) = A$. The converse is obvious.) Also, $\varphi(\cdot, \gamma^*)$ and $T(\cdot, \gamma^*)$ are $\mathscr{A}_{\gamma^*}$-measurable, hence G_{γ^*}-invariant, and $T(\cdot, \gamma^*)$ distinguishes orbits of G_{γ^*}. (T is a maximal invariant under G, so $T(x_1, \gamma^*) = T(x_2, \gamma^*)$ implies $(x_2, \gamma^*) = (gx_1, g\gamma^*)$ for some $g \in G$. But then $g \in G_{\gamma^*}$.)

Thus, by the results of Blackwell mentioned above, $\mathscr{A}_{\gamma^*} = T(\cdot, \gamma^*)^{-1}\mathscr{B}$, and so there exists a $\mathscr{B}$-measurable f such that $\varphi(\cdot, \gamma^*) = f \circ T(\cdot, \gamma^*)$. Now fix $\gamma \in \Gamma$ and let $g \in G$ be such that $g\gamma = \gamma^*$. Then, by the G-invariance of φ and T, $\varphi(x, \gamma) = \varphi(gx, \gamma^*) = f(T(gx, \gamma^*)) = f(T(x, \gamma))$, so $\varphi = f \circ T$.

Remark 3. Suppose G acts transitively on Θ; this is the case in each of our examples below. If $T = T(X, \gamma)$ is invariant under G, then T is a pivotal quantity; see Wijsman (1980, Lemma 3.1). Thus invariantly sufficient functions are pivotal quantities. In Hooper (1981b) a method is given for constructing confidence sets having smallest expected measure among all invariant level $1 - \alpha$ confidence sets. The family of distributions $P^T_{\theta,\gamma}$ of an invariantly sufficient function T plays a key role in this construction. The results on the general MANOVA problem in Section 4.3 below are used in Hooper (1981a) as a starting point for studying problems of simultaneous estimation.

4. Applications.

Example 4.1. Let $X_1, \cdots, X_n$ be independent uniform $(\mu - \sigma, \mu + \sigma)$, $n \geq 2$. Suppose a confidence set is desired for $\gamma(\mu, \sigma) = \mu$. Let G be the group generated by the actions $X_i \to X_i + a$, $\mu \to \mu + a$ for $a \in \mathbb{R}$, and $X_i \to cX_i$, $\mu \to c\mu$, $\sigma \to c\sigma$ for $c > 0$. By Theorem 1(i) a sufficient function is (U, V, μ) where $U = \min\{X_i\}$, $V = \max\{X_i\}$. A maximal invariant under the action of G on (U, V, μ) is $S = (V - \mu)/(U - \mu)$. Remark 1 (with $\gamma^* = 0$) shows that S is invariantly sufficient for the family of distributions of $(X_1, \cdots, X_n, \mu)$. An example of a confidence interval for μ based on S is

$$[\tfrac{1}{2}(U + V) - \tfrac{1}{2}t(V - U), \tfrac{1}{2}(U + V) + \tfrac{1}{2}t(V - U)], \quad 0 \leq t < \infty,$$

which covers μ with probability $1 - (1 + t)^{-(n-1)}$.

If a confidence set is desired for $\gamma(\mu, \sigma) = \sigma$, a similar argument shows that $(V - U)/\sigma$ is invariantly sufficient. If one is interested in both parameters, $((V - \mu)/\sigma, (U - \mu)/\sigma)$ is invariantly sufficient.

Example 4.2. Let $X_1, \cdots, X_n$ be independent, identically distributed, continuous random variables. The distribution is otherwise unspecified except that the median Δ is assumed to be unique. The problem of obtaining a confidence set for Δ is invariant under the group of strictly increasing continuous functions f mapping $\mathbb{R}$ onto $\mathbb{R}$, the actions being $X_i \to f(X_i)$, $\Delta \to f(\Delta)$. A maximal invariant on $(X_1, \cdots, X_n, \Delta)$ is the set of corresponding ranks $(R_1, \cdots, R_n, R_\Delta)$ among the $n + 1$ different numbers (ties occurring with probability zero). Conditional on a given value of R_Δ, the $n!$ different possibilities for $(R_1, \cdots, R_n)$ are equally likely. Thus R_Δ is an invariantly sufficient function. Each nonrandomized confidence set based on R_Δ is the union of a subcollection of the $n + 1$ intervals determined by the ordered values of $(X_1, \cdots, X_n)$.

Example 4.3. The general MANOVA problem. Our notation will follow mainly that of Kariya (1978). The model, in its canonical form, is as follows:

$$Z: (m + n) \times p \sim N(\tilde{\Theta}, I_{m+n} \otimes \Sigma), \quad n \geq p;$$

i.e., the $m + n$ rows of Z are independent p-dimensional multivariate normal with common covariance matrix Σ. It is assumed that Σ is positive definite and that $\tilde{\Theta}$ has the form

$$\tilde{\Theta} = \begin{array}{c} \\ \\ \\ \\ \end{array}\!\!\begin{array}{cccc} p_1 & p_2 & p_3 & \\ \left[\begin{matrix} \Theta_{11} \\ \Theta_{21} \\ 0 \end{matrix}\right. & \begin{matrix} \Theta_{12} \\ \Theta_{22} \\ 0 \end{matrix} & \left.\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right] & \begin{matrix} m_1 \\ m_2 \\ n \end{matrix} \end{array} \qquad \begin{matrix} m_1 + m_2 = m \\ p_1 + p_2 + p_3 = p. \end{matrix}$$

The parameters Σ and $\tilde{\Theta}$ are otherwise unknown. We consider the problem of obtaining confidence sets for $\gamma(\tilde{\Theta}, \Sigma) = \Theta_{12}$.

Let $\mathscr{G}$ be the group $\tilde{\mathscr{O}} \times \mathscr{A} \times \mathscr{F}$ where (see Kariya, 1978, page 202): $\tilde{\mathscr{O}}$ consists of all block diagonal orthogonal $m \times m$ matrices P, the action being multiplication on the left of the first m rows of Z and $\tilde{\Theta}$; $\mathscr{A}$ consists of all nonsingular lower block-triangular matrices A, the action being $Z \to ZA$, $\tilde{\Theta} \to \tilde{\Theta} A$, and $\Sigma \to A'\Sigma A$; $\mathscr{F}$ consists of all $m \times (p_1 + p_2)$ matrices, the action being translation of the (1, 1), (1, 2), (2, 1), and (2, 2) blocks of Z and $\tilde{\Theta}$. (This last differs from the testing problem where $\mathscr{F}$ consists of translations only of the (1, 1), (2, 1), and (2, 2) blocks.) Then $\mathscr{G}$ is an invariance group acting on Θ_{12}.

The following diagram illustrates the order in which reductions are performed. Horizontal and vertical lines represent respectively sufficiency and invariance reductions. Dotted lines represent reductions not explicitly carried out.

$$\begin{tikzcd}
Z, \Theta_{12} \arrow[r] \arrow[dd, dashed] & \tilde{Z}, V, \Theta_{12} \arrow[d] & \\
& T_1(\Theta_{12}), T_2, T_3, T_4 \arrow[r] \arrow[d, dashed] & T_1(\Theta_{12}), T_2 \arrow[d, dashed] \\
{} \arrow[r, dashed] & B \arrow[r, dashed] & W
\end{tikzcd}$$

We begin by reducing (Z, Θ_{12}) by sufficiency. Applying Theorem 1 (i), a sufficient function is $(\tilde{Z}, V, \Theta_{12})$, where $\tilde{Z}$ is the first m rows of Z and $V = [Z_{31}Z_{32}Z_{33}]'[Z_{31}Z_{32}Z_{33}]$. Note that Kariya (1978) takes $(\tilde{Z}, V)$ as his starting point in specifying the general MANOVA model. The action of $\mathscr{G}$ on V is $V \to A'VA$. Next, apply an invariance reduction under the subgroup of $\mathscr{F}$ consisting of translations of the (1, 2) block. A maximal invariant is $(Z(\Theta_{12}), V)$ where $Z(\Theta_{12})$ is $\tilde{Z}$ with Z_{12} replaced by $Z_{12} - \Theta_{12}$. Now reduce under $\mathscr{A}$ and the remainder of $\mathscr{F}$. Following Kariya (1978, page 203) a maximal invariant is $(T_1(\Theta_{12}), T_2, T_3, T_4)$, where

$$T_1(\Theta_{12}) = X(\Theta_{12})V_{22.3}^{-1}X(\Theta_{12})', \quad V_{22.3} = V_{22} - V_{23}V_{33}^{-1}V_{32},$$

$$X(\Theta_{12}) = (I + T_2)^{-1/2}(Z_{12} - \Theta_{12} - Z_{13}V_{33}^{-1}V_{32}), \quad T_2 = Z_{13}V_{33}^{-1}Z_{13}',$$

$$T_3 = Z_{23}V_{33}^{-1}Z_{23}', \quad \text{and} \quad T_4 = Z_{13}V_{33}^{-1}Z_{23}'.$$

The action under $\tilde{\mathscr{O}}$ is

$$(T_1(\Theta_{12}), T_2, T_3, T_4) \to (P_1T_1(\Theta_{12})P_1', P_1T_2P_1', P_2T_3P_2', P_1T_4P_2').$$

Using Theorem 1 (ii) with $\gamma^* = 0$ and Kariya (1978, Lemma 3.2) it is seen that $(T_1(\Theta_{12}), T_2)$ is sufficient for the family of distributions of $(T_1(\Theta_{12}), T_2, T_3, T_4)$.

Finally one can reduce $(T_1(\Theta_{12}), T_2)$ under $\tilde{\mathscr{O}}$. Let W be a maximal invariant. By Remark 1 (with $\gamma^* = 0$, $G_{\gamma^*} = \tilde{\mathscr{O}}$) W is invariantly sufficient for the family of distributions of $(T_1(\Theta_{12}), T_2, T_3, T_4)$. Let B be a maximal invariant under $\tilde{\mathscr{O}}$ *of* $(T_1(\Theta_{12}), T_2, T_3, T_4)$. Then, again by Remark 1 (with $\gamma^* = 0$ and $G_{\gamma^*} = \mathscr{G}$ without translations of the (1, 2) block) B is invariantly sufficient for the family of distributions of (Z, Θ_{12}). These two facts imply that W is invariantly sufficient for the family of distributions of (Z, Θ_{12}); see the diagram.

Since it is difficult to find a tractable form for W, we use the definition of invariance directly. A set estimator φ based on $(T_1(\Theta_{12}), T_2)$ only, $\varphi(Z, \Theta_{12}) = \varphi_0(T_1(\Theta_{12}), T_2)$, is $\mathscr{G}$-invariant if $\varphi_0(P_1T_1(\Theta_{12})P_1', P_1T_2P_1') = \varphi_0(T_1(\Theta_{12}), T_2)$ for all $m_1 \times m_1$ orthogonal P_1. We have shown that the class of $\mathscr{G}$-invariant set estimators based on $(T_1(\Theta_{12}), T_2)$ only is essentially complete among all $\mathscr{G}$-invariant set estimators.

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