The Annals of Statistics
1982, Vol. 10, No. 2, 394–400

SEQUENTIAL TESTS CONSTRUCTED FROM IMAGES[1]

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A method is described for constructing sequential test boundaries for which the sample size distribution is easily computed. Assuming a diffusion approximation to be adequate, a few suitably placed images with negative weights are added to a basic unit source at the origin to create absorbing boundaries with intuitively desirable properties. A simple family of closed boundaries generated in this way is compared with Ferebee's (1980) reversed parabolic boundaries and Anderson's (1960) trapezoidal boundaries with regard to minimax expected sample size. The use of a delayed image to generate two-lobed closed boundaries is also briefly described.

1. Introduction and summary. The Wald sequential probability ratio test has the attractive property that its characteristics are, at least approximately, easy to evaluate. But in general, sequential tests based on some optimum or intuitively reasonable ad hoc principle often present considerable technical difficulties in determining power, expected sample size and so on. The idea of the present paper is to reverse the usual procedure. The proposed tests are chosen so that these calculations are straightforward, provided a diffusion approximation is acceptable—a matter which would require further investigation. Simple solutions $f(x, t)$ of the diffusion equation $\frac{1}{2}\partial^2 f/\partial x^2 = \partial f/\partial t$ are constructed by adding a few image sources with negative weights to a basic unit source $1/\sqrt{2\pi t}\,\exp(-x^2/2t)$. The density $f(x, t)$ vanishes on a boundary $x = \xi(t)$, and by suitably siting the images, the absorbing boundary $\xi(t)$ can form the basis of an approximate sequential test for the mean θ with intuitively desirable properties.

For simplicity, the scales of x and t are assumed to be chosen so that t is the actual sample size. The first exit density for paths starting at (0, 0) and crossing at $\xi(t)$ is $h(t) = -\frac{1}{2}\partial f/\partial \xi$. When there is a drift θ, the corresponding density for the same boundary is $h(t \mid \theta) = h(t)\exp(\theta\xi - \frac{1}{2}t^2)$. The distribution of sample size t is then readily computable for any value of θ, and the power function, expected sample size etc. can be calculated from it.

The main example discussed here is a family of closed sequential tests generated by two symmetrically placed images. It was motivated by the recent work of B. Ferebee (1980) on a family of tests with reversed parabolic boundaries $\xi(t) = \lambda(t_1 - t)^{1/2}$ for testing $\theta > 0$ against $\theta \le 0$. By varying the parameters subject to a fixed acceptance probability at a given value of θ, he determined the boundary with minimum expected sample size in the worst case $\theta = 0$. His results differed little from those for trapezoidal boundaries computed in the pioneering paper by T. W. Anderson (1960). This insensitivity to the form of the boundary is confirmed with the present family.

The distribution of sample size when $\theta = 0$ for the present minimax region is compared with the corresponding distribution for Anderson's triangular region which has almost the minimax expected sample size, with a view to examining the proportion of excessively large sample sizes in the latter case. Finally, it is shown how a delayed image introduced at $(0, \tau)$ can produce two-lobed regions which may be of use for sequential tests.

2. Some simple boundaries. The simplest example of the present approach is the

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Received October 1981; revised December 1981.

[1] Presented at the Symposium on Statistical Inference and Applications in honour of George Barnard's 65th birthday, University of Waterloo, August 5–8, 1981.

AMS 1980 *subject classifications.* Primary 62L10; secondary 60J70.

Key words and phrases. Boundary crossing, diffusion, images, minimax expected sample size.

familiar use of an image source of negative weight $-\kappa$ (i.e. a sink) at $(a, 0)$ to produce a straight line boundary. The density is

$$f(x, t) = \frac{e^{-x^2/2t}}{\sqrt{2\pi t}} - \kappa \frac{e^{-(x-a)^2/2t}}{\sqrt{2\pi t}} \tag{2.1}$$

$$= \frac{e^{-x^2/2t}}{\sqrt{2\pi t}} \{1 - \kappa e^{-a(x-a/2)/t}\} \tag{2.2}$$

which vanishes on $\xi(t) = \frac{1}{2}a + \beta t$ where $\beta = a^{-1}\log(1/\kappa)$.

The effect of introducing a second image is to produce some curvature in the boundary. Images at $(a_1, 0)$, $(a_2, 0)$ of weights $-\kappa_1$, $-\kappa_2$, respectively, give the density

$$f(x, t) = \frac{e^{-x^2/2t}}{\sqrt{2\pi t}} - \kappa_1 \frac{e^{-(x-a_1)^2/2t}}{\sqrt{2\pi t}} - \kappa_2 \frac{e^{-(x-a_2)^2/2t}}{\sqrt{2\pi t}} \tag{2.3}$$

which vanishes on the boundary $\xi(t)$ satisfying

$$\kappa_1 \exp\left(-\frac{a_1^2}{2t} + a_1 \frac{\xi}{t}\right) + \kappa_2 \exp\left(-\frac{a_2^2}{2t} + a_2 \frac{\xi}{t}\right) = 1. \tag{2.4}$$

In general there is no explicit formula for $\xi(t)$. If $0 < a_1 < a_2$, then when t is small, $\xi \sim \frac{1}{2}a_1 + \beta_1 t$ as before, with $\kappa_1 = e^{-a\beta_1}$. When t is large, $\xi(t)$ behaves like $\xi \sim \beta_2 t$ where

$$\kappa_1 e^{a_1\beta_2} + \kappa_2 e^{a_2\beta_2} = 1. \tag{2.5}$$

Evidently $\beta_2 \gtrless 0$ according as $\kappa_1 + \kappa_2 \lesseqgtr 1$, and the probability of eventually crossing the boundary is $\min(\kappa_1 + \kappa_2, 1)$. In the special case $a_1 = a$, $a_2 = 2a$, there is an explicit formula

$$\xi(t) = \frac{1}{2}a - \frac{t}{a}\log\left\{\frac{1}{2}\kappa_1 + \left(\frac{1}{4}\kappa_1^2 + \kappa_2 e^{-a^2/t}\right)^{1/2}\right\}. \tag{2.6}$$

This boundary was suggested as an approximating model for a certain diffusion problem in Daniels (1965) and was used by Durbin (1971) as a convenient exact result against which to check his numerical calculation of the crossing probability. As it stands, it is not particularly useful for a sequential test, but we shortly discuss another special case of (2.4) which is useful.

3. Some basic results. We need some known results concerning first exit densities. The first, which is essential to the present approach, is that for a probability density $f(x, t)$ satisfying $\frac{1}{2}\partial^2 f/\partial x^2 = \partial f/\partial t$ and vanishing on $x = \xi(t)$, the first exit density at $\xi(t)$ is

$$h(t) = -\frac{1}{2}\frac{\partial f(\xi, t)}{\partial \xi}. \tag{3.1}$$

A proof may be constructed on the following lines. In a small time interval $(t, t + \tau)$, $\xi(t + \tau) \sim \xi(t) + \tau\xi'(t)$. The probability that a path starting at $\xi - z\tau^{1/2}$ at time t crosses the boundary in $(t, t + \tau)$ is, from (2.2),

$$P(z) \sim 1 - \Phi(z + \xi'\tau^{1/2}) + (1 - 2z\xi'\tau^{1/2})\Phi(-z + \xi'\tau^{1/2}) \sim 2\{1 - \Phi(z)\}(1 - 2z\xi'\tau^{1/2}),$$

where z is taken to be $O(1)$. Since $f(\xi - z\tau^{1/2}, t) \sim -z\tau^{1/2}\partial f/\partial\xi$, the probability of crossing the boundary in $(t, t + \tau)$ is

$$\tau h(t) \sim -2\tau \frac{\partial f}{\partial \xi}\int_0^\infty z\{1 - \Phi(z)\}\, dz + O(\tau^{3/2}),$$

whence $h(t) = -\frac{1}{2}\partial f/\partial\xi$.

Next, suppose there is a drift θ. Then the first exit density $h(t \mid \theta)$ is obtained by multiplying $h(t)$ by the factor $\exp\{\theta\xi(t) - \frac{1}{2}\theta^2 t\}$. This result is usually established by a martingale argument but the following derivation is more in the spirit of the present approach.

If $f(x, t)$ is any density satisfying $\frac{1}{2}\partial^2 f/\partial x^2 = \partial f/\partial t$ and vanishing on the boundary $\xi(t)$, then

$$f(x, t \mid \theta) = e^{\theta x - (1/2)\theta^2 t} f(x, t)$$

satisfies $\frac{1}{2}\partial^2 f/\partial x^2 + \theta \partial f/\partial x = \partial f/\partial t$ and vanishes on the *same* boundary. It is obviously true of a single source (where there is no boundary) and can be verified in general. We therefore have a family of densities $f(x, t \mid \theta)$ for Brownian paths starting at (0, 0) with mean drift θ in the presence of the same absorbing boundary. The corresponding first exit density is

$$h(t \mid \theta) = -\frac{1}{2}\frac{\partial f(x, t \mid \theta)}{\partial x}\bigg|_{x=\xi(t)} = -\frac{1}{2}\frac{\partial}{\partial x}\{e^{\theta x - (1/2)\theta^2 t} f(x, t)\}\bigg|_{x=\xi(t)} = e^{\theta \xi(t) - (1/2)\theta^2 t} h(t),$$

since $f(\xi(t), t) = 0$.

4. A family of boundaries. A simple image system leading to an explicit formula for a family of boundaries with useful properties is now described. Starting with a unit source at (0, 0) as before, images of weight $-\frac{1}{2}\kappa$, $\kappa > 0$, are placed at $(\pm a, 0)$. The density $f(x, t)$ is a special case of (2.3) with $a_1 = a$, $a_2 = -a$, $\kappa_1 = \kappa_2 = \frac{1}{2}\kappa$. It vanishes on the boundary

$$\xi(t) = \pm \frac{t}{a} \cosh^{-1}\left(\frac{1}{\kappa} e^{a^2/2t}\right). \tag{4.1}$$

We distinguish the positive and negative branches of $\xi(t)$ by an appropriate suffix. Alternative forms for $\xi_+(t)$ are

$$\xi_+(t) = \frac{t}{a} \log\left\{\frac{1}{\kappa} e^{a^2/2t} + \left(\frac{1}{\kappa^2} e^{a^2/t} - 1\right)^{1/2}\right\} \tag{4.2}$$

$$= \frac{1}{2} a + \beta t + \frac{t}{a} \log\left\{\frac{1}{2} + \left(\frac{1}{4} - e^{-a^2/t} - 2a\beta\right)^{1/2}\right\} \tag{4.3}$$

where $\beta = \frac{1}{a} \log(2/\kappa)$.

The following properties of $\xi(t)$ are easily established.

(i) When $\kappa > 1$, only values of t less than $t_1 = a^2/(2 \log \kappa)$ give real values of $\xi(t)$ and the boundary is closed. When $\kappa \leq 1$, t is unrestricted and the boundary is open.

(ii) Initially, from (4.3), $\xi_+(t) \sim \frac{1}{2}a + \beta t$. The slope β is positive or negative according as $\kappa \lessgtr 2$. So for example, if $1 < \kappa < 2$, the boundary is closed but the initial slope of $\xi_+(t)$ is positive.

(iii) When $\xi(t)$ is closed ($\kappa > 1$) it behaves near t_1 like

$$\xi(t) \sim \pm(t_1 - t)^{1/2}.$$

(iv) When $\kappa < 1$ and t is large, $\xi(t) \sim \pm\gamma t + O(1)$ where $\gamma = 1/a \cosh^{-1}(1/\kappa) < \beta$. The probability of eventually crossing one of the branches is $\kappa < 1$.

(v) In the transitional case $\kappa = 1$, $\xi(t)$ behaves for large t like

$$\xi(t) \sim \pm\left\{t + \frac{1}{6} a^2\right\}^{1/2} + O(t^{-1}).$$

This boundary may be of interest as an approximation to a sequential test of the repeated significance test type.

The first exit density is most easily calculated as

$$h(t \mid 0) = \frac{1}{2t}\left\{\frac{\xi}{\sqrt{t}} Z\left(\frac{\xi}{\sqrt{t}}\right) - \frac{1}{2}\kappa \frac{(\xi - a)}{\sqrt{t}} Z\left(\frac{\xi - a}{\sqrt{t}}\right) - \frac{1}{2}\kappa \frac{(\xi + a)}{\sqrt{t}} Z\left(\frac{\xi + a}{\sqrt{t}}\right)\right\}$$

and $h(t \mid \theta) = h(t \mid 0)\exp(\theta\xi - \frac{1}{2}\theta^2 t)$, where $Z(\cdot)$ is the standard normal density.

5. A test for the mean drift. The closed boundaries with $\kappa > 1$ provide test for $\theta > 0$ against $\theta \leq 0$ which can be compared with those of Anderson (1960) and Ferebee (1980). Our boundaries include the fixed sample size test as a limiting case, but unlike the others do not include the Wald test.

Some boundaries for the acceptance probability $P = 0.95$ are shown in Figure 1 which may be compared with Ferebee's Figure 1a. The corresponding values of t_1 and $E(t \mid 0)$ are given in Table 1.

For comparison with Anderson's and Ferebee's results, calculations have been made for $P = 0.95$ and 0.99. Table 2 is an extension of Ferebee's Table 4 to include the results for the present tests. It will be seen that the minimax expected sample size is almost the same for all three types of boundary. (Notice that our minimax region is nearly parabolic).

Ferebee also found that for a given acceptance probability P at $\theta = 0.1$ the power function was remarkably similar for all members of his family of parabolas (see his Tables

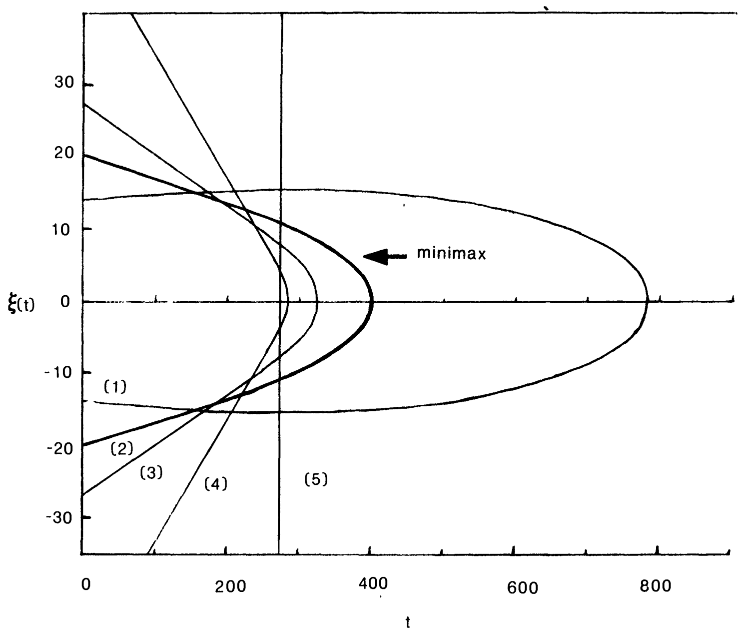

FIG. 1 *Image boundaries* $\xi(t) = \pm(t/a)\cosh^{-1}(\kappa^{-1}e^{a^2/2t})$ *for cases* (1)–(5) *in Table* 1. $P = 0.95$, $\theta = 0.1$.

TABLE 1

	(1)	(2)*	(3)	(4)	(5)
t_1	784	401	329	286	271
E(t | 0)	216	192	198	220	271

* Minimax.

TABLE 2
Minimax region

	P = 0.95			P = 0.99		
	E(t | 0)	E(t | 0.1)	t_1	E(t | 0)	E(t | 0.1)	t_1
Trapezoid	192	139	529	402	249	783
Parabola	193	139	417	405	245	691
Images	192	140	401	403	252	699

TABLE 3

Q(θ) for minimax region

$\theta =$	.05	.10	.12	.14	.16	.18	.20
$Q(\theta)$	.201	.050	.026	.012	.006	.003	.001
	.119	.010	.006	.002	.001	—	—

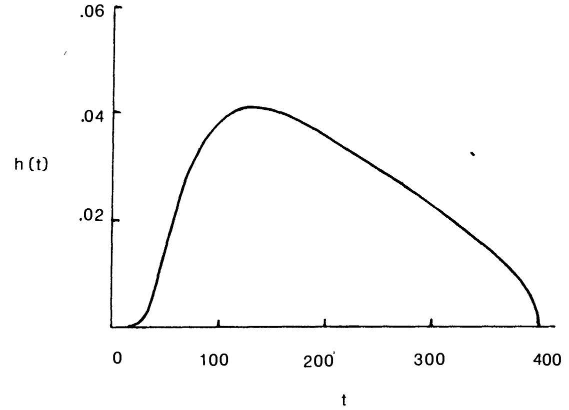

FIG. 2 *First exit density for minimax image boundary.* $P = 0.95$, $\theta = 0.1$.

2a, 2b). The same phenomenon was observed with the present family of regions, the power function being almost the same as Ferebee's.

Table 3 gives the rejection probabilities $Q(\theta) = 1 - \int_0^{t_1} h_+(t \mid \theta)\, dt$ for the minimax region when $Q(.1) = .05$ and .01.

Fig. 2 shows the distribution of sample size t when $P = 0.95$ and $\theta = 0$.

The calculations were most conveniently carried out in terms of $u = t/t_1$, $c = at_1^{-1/2}$. To facilitate numerical integration, the transformation $v = \{1 - (1 - u)^{1/2}\}^\alpha$, $0 < \alpha < 1$, was used. It has the effect of making ξ_+ and h_+ behave linearly in v near $u = 1$, and of stretching the scale near $u = 0$ to counteract the extreme skewness of h for some values of c.

6. Another approach. An alternative way of closing the boundary of the Wald test is to alter the weights of its appropriate image system. The density for a Wald test with parallel boundaries at $\xi(t) = \pm\frac{1}{2}a$ is

$$f(x, t) = \sum_{r=-\infty}^{\infty} (-1)^r \frac{e^{-(x-ra)^2/2t}}{\sqrt{2\pi t}} \tag{6.1}$$

or, in its dual form,

$$f(x, t) = \frac{2}{\pi} \sum_{r=0}^{\infty} e^{-(2r+1)^2\pi^2 t/2a^2} \cos\{(2r + 1)\pi x/a\}. \tag{6.2}$$

The simplest modification is to increase the weight of images from $(-1)^r$ to $(-1)^r \exp(\beta r^2)$, $\beta > 0$, the new density being

$$f(x, t) = \sum_{r=-\infty}^{\infty} (-1)^r \frac{e^{\beta r^2 - (x-ra)^2/2t}}{\sqrt{2\pi t}} \tag{6.3}$$

which converges for $0 \le t < t_1 = a^2/2\beta$. But on rewriting (6.3) as

$$(6.4) \quad f(x, t) = \exp\left\{\frac{x^2}{2(t_1 - t)}\right\} \sum_{r=-\infty}^{\infty} (-1)^r \exp\left[-\frac{1}{2t(1 - t/t_1)}\{x - ra(1 - t/t_1)\}^2\right]$$

it is seen from (6.2) to have the dual form

$$(6.5) \quad f(x, t) = \exp\left\{\frac{x^2}{2(t_1 - t)}\right\} \frac{2}{a}\left(1 - \frac{t}{t_1}\right)^{1/2} \sum_{r=0}^{\infty} \exp\left\{-\frac{(2r+1)^2\pi^2 t}{2a^2(1 - t/t_1)}\right\} \cos\left\{\frac{(2r+1)\pi x}{a(1 - t/t_1)}\right\},$$

which vanishes on $\xi(t) = \pm\frac{1}{2}a(1 - t/t_1)$. We are therefore back to Anderson's trapezoidal boundaries. These formulae can, of course, be derived by applying the transformation $t = t_1 T/(1 + T)$, $y(T) = (1 + T)x(t)$ to the parallel boundary process $x(t)$; Doob (1949).

A remarkable feature of Anderson's Tables 1 and 2 is that his $E(t \mid 0)$ has a very flat minimum. Practically the same $E(t \mid 0)$ is found when the converging boundary lines actually meet on the t axis, in which case t_1 takes the considerably larger values 600 and 870 for $P = 0.95$ and 0.99 respectively. Lai (1973) has proved that this triangular region is asymptotically optimal (in the present sense) among *all* sequential tests when the rejection probability is small. But large sample sizes might arguably be considered more disadvantageous than is allowed for in the average. It is perhaps worth examining the proportion of sample sizes for such a triangular region which exceed the maximum attainable in the other optimal regions.

The two forms of the first exit density for ξ_+ are

$$(6.6) \quad h_+(t \mid 0) = -\frac{1}{2}\frac{\partial f}{\partial \xi_+}$$

$$= \frac{a t_1^{1/2}}{2\sqrt{2\pi} t^{3/2}} e^{a^2(t_1 - t)/8} \sum_{r=0}^{\infty} (-)^r (2r+1) \exp\{-(2r+1)^2 a^2 (t_1 - t)/8 t_1 t\}$$

$$(6.7) \quad = \frac{\pi t_1^{1/2}}{a^2 (t_1 - t)^{3/2}} e^{a^2(t_1 - t)/8} \sum_{r=0}^{\infty} (-)^r (2r+1) \exp\left\{-\frac{(2r+1)^2 \pi^2 t_1 t}{2a^2 (t_1 - t)}\right\}$$

from which the density $h(t \mid 0) = 2h_+(t \mid 0)$ was calculated for the triangular region when $P = 0.95$ (Figure 3). Comparison with Figure 2 shows that about 2% of samples will exceed the maximum of 401 attained by our optimal region, and the average amount by which it is exceeded is about 20 observations.

7. Delayed image. Finally, we briefly consider an extension of the technique which can produce regions of possible use in sequential analysis. This is to introduce a delayed

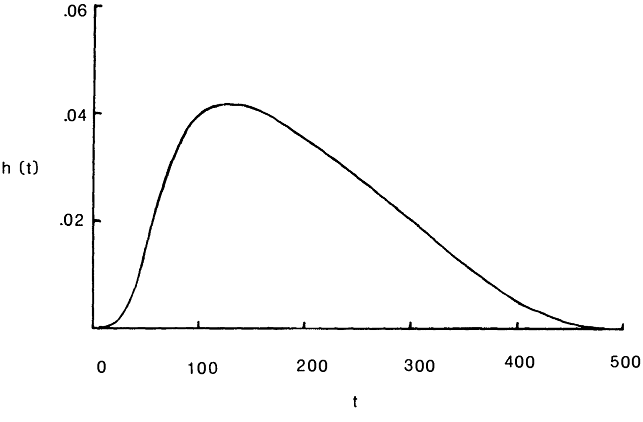

FIG. 3 *First exit density for triangular boundary.* $P = 0.95$, $\theta = 0.1$.

image at $(0, \tau)$. Consider first what happens if we start with a unit source at (0, 0) and introduce a source of strength $-\lambda$ at $(0, \tau)$. During $0 \le t < \tau$ there is no boundary, but after τ a boundary appears on which the density

$$f(x, t) = \frac{e^{-x^2/2t}}{\sqrt{2\pi t}} - \frac{\lambda e^{-x^2/2(t-\tau)}}{\sqrt{2\pi(t-\tau)}} \tag{7.1}$$

vanishes. It has the explicit formula

$$\xi(t) = \pm \left\{\frac{t(t-\tau)}{\tau} \log\left(\frac{\lambda^2 t}{t-\tau}\right)\right\}^{1/2} \tag{7.2}$$

which defines a closed oval on $\tau \le t \le \tau/(1-\lambda^2)$ if $\lambda < 1$, or defines two open branches approaching the asymptotes $\xi \sim \pm(2 \log \lambda/\tau)^{1/2} t$ if $\lambda > 1$, with a transitional form when $\lambda = 1$ which is asymptotically parabolic. But such regions are not of much interest in the present context.

A more promising situation arises if the delayed source at $(0, \tau)$ is added to the image system of Section 4, particularly if we start with an open region having $\kappa < 1$. When t exceeds τ an inner boundary begins to form and the region $\xi(t)$ develops two symmetrical diverging bands. The boundary is defined implicitly by

$$1 - \kappa e^{-a^2/2t} \cosh \frac{a\xi}{t} = \lambda \exp\left\{-\frac{\xi^2 \tau}{2t(t-\tau)}\right\} \tag{7.3}$$

and if λ exceeds a certain critical value, the bands close after a finite time, forming two symmetrical lobes. Such a region may be useful as the basis of a closed test of the Sobel-Wald type. The values of $\xi(t)$ will have to be calculated from (7.3) by iteration, but if a Newton-Raphson routine is used it will automatically generate $\partial f/\partial \xi$ and so produce the first exit density $h(t)$ at the same time.

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