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QUALITATIVE ROBUSTNESS OF RANK TESTS

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An asymptotic notion of robust tests is studied which is based on the requirement of equicontinuous error probabilities. If the test statistics are consistent, their robustness in Hampel's sense and robustness of the associated tests turn out to be equivalent. Uniform extensions are considered. Moreover, test breakdown points are defined. The main applications are on rank statistics: they are generally robust, under a slight condition even uniformly so; their points of final breakdown coincide with the breakdown points of the corresponding $R-$ estimators.

1. Introduction. When one compares e.g. the one-sample normal scores rank test with its local parametric competitor, based on the mean, the following aspects would suggest superior robustness behavior of the rank test:

(a) the scores that incoming outliers successively occupy in the worst case are strictly decreasing–as opposed to constant maximum influence of each outlier on the mean;
(b) the absolute value of observations can be increased without the rank statistic changing its value; i.e. outliers are automatically brought in.

As has been pointed out in Rieder (1981a), cf. Remark (2) following Proposition 2.2, the effect of (b) will disappear if the fraction of gross errors tends to zero. As for the effect of (a), apparently, one has to be afraid of a similar disappearance. For, if the percentage of outliers in the sample becomes less and less, as the sample size increases, the minimum of the scores they occupy in the worst case will still tend to infinity. (Only if outliers are very scarce will it matter that the maximum score at each sample size is finite even though it tends to infinity.) Actually, this is how the corresponding theoretical results of Rieder (1981a) must finally be interpreted. Therefore, the exclusive use of infinitesimal neighborhoods cannot be said to do full justice to the intuitive robustness properties of rank tests.

In this paper, robustness of test sequences is defined by the requirement of equicontinuous error probabilities (with respect to Prokhorov or Kolmogorov metrics). Thus the notion is still asymptotic, however fixed-size neighborhoods are employed; the idea behind equicontinuity is the same as in Hampel's (1971) qualitative definition of robust estimators. For tests which are of one-sided form and are based on consistent statistics, an equivalence is established between robustness of the statistics in Hampel's sense and robustness of the tests (Theorem 2.2). Thus continuous sequences of statistics define robust and consistent tests (Corollary 2.4). Uniform extensions, which seem relevant in case of composite hypotheses, are pointed out.

Section 3 introduces asymptotic breakdown points of sequences of tests and test statistics; e.g. the "point of final breakdown" of a sequence of statistics denotes the critical amount δ^* of contamination that renders the statistics unable to asymptotically distinguish any two δ^*-contaminated probabilities.

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Section 4 gives the applications on rank test. By Theorem 4.4, they now turn out to be generally robust (even if the scores may be unbounded) due to the continuity of an associated functional (Lemma 4.1) and an approximation result (Lemma 4.2), which also yields a law of large numbers (Proposition 4.3). Under a slight condition, covering e.g. the sign-Wilcoxon-, and normal scores rank tests, even uniform robustness and consistency can be established. The "points of final breakdown" of rank statistics coincide with the breakdown points (Hampel, 1971) of the corresponding rank estimators.

Only the one-sample case is considered in this paper; the complete two-sample analogue and relevant proofs can be found in the author's research reports (Rieder, 1979, 1981c). As for the correlation case, see Rieder (1981b).

2. Qualitatively robust sequences of tests and test statistics. Let $(\Omega, \mathscr{B})$ be a measurable space, $\mathscr{M}$ the set of probability measures (pm's) on $\mathscr{B}$, for every positive integer n let $(\Omega^n, \mathscr{B}^n)$ denote the n-fold product space of $(\Omega, \mathscr{B})$, $\otimes_{i=1}^n G_i$ the product measure of the pm's $G_1, \cdots, G_n (G^n$ if $G_1 = \cdots = G_n = G)$, for a subset $\mathscr{P} \subset \mathscr{M}$ let $\mathscr{P}^{(n)} = \{\otimes_{i=1}^n G_i \mid G_i \in \mathscr{P}, i = 1, \cdots, n\}$. Sequences of tests (ϕ_n) and test statistics (T_n) are understood to be sequences of measurable mappings $\phi_n : \Omega^n \to [0, 1]$, $T_n : \Omega^n \to \mathbb{R}$ (the reals), with the possible exception of some initial segments $n \le n_0$. The law of T_n under $W_n \in \mathscr{P}^{(n)}$ is denoted by $W_n \circ T_n^{-1}$. We imagine a sequence of test statistics (T_n) to be accompanied by a collection of associated test sequences $(\psi_{\gamma_n, k_n}(T_n))$, where $\psi_{\gamma_n, k_n}(T_n) = (1 - \gamma_n) I(T_n > k_n) + \gamma_n I(T_n \ge k_n)$, and $(\gamma_n) \subset [0, 1]$ ranges over all sequences of randomization constants, $(k_n) \subset \mathbb{R}$ over all sequences of critical values.

First, Ω is assumed to be complete, separable, metric with $\mathscr{B}$ its Borel σ-field and $\mathscr{M}$ being metrized by Prokhorov distance d_P; the Prokhorov ball of radius $\delta \in [0, 1]$ and center $F \in \mathscr{M}$ is denoted by $\mathscr{P}_P(F, \delta)$.

DEFINITION 2.1. (a) A test sequence (ϕ_n) is q_P-robust at a pm F iff

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \exists n_0 \quad \forall n > n_0 : W_n \in \mathscr{P}_P(F, \delta)^{(n)} \Rightarrow \left| \int \phi_n \, dW_n - \int \phi_n \, dF^n \right| < \varepsilon.$$

(b) A sequence of test statistics (T_n) is q_P-robust at a pm F iff

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \exists n_0 \quad \forall n > n_0 : W_n \in \mathscr{P}_P(F, \delta)^{(n)} \Rightarrow d_P(W_n \circ T_n^{-1}, F^n \circ T_n^{-1}) < \varepsilon.$$

REMARK (1). Obviously, (ϕ_n) is q_P-robust at F iff $(1 - \phi_n)$ is q_P-robust at F, and (T_n) is q_P-robust at F iff $(-T_n)$ is q_P-robust at F. Therefore, we need not distinguish between null hypothesis and alternative, and, as $\psi_{\gamma_n, k_n}(-T_n) = 1 - \psi_{1-\gamma_n, -k_n}(T_n)$, we may restrict attention to tests which reject for large values of T_n.

REMARK (2). The definition essentially restates Hampel's (1971) definition of qualitatively robust sequences of estimators. This is apparent in case (b). As for (a) note that, after randomization, the ϕ_n's take their values in the discrete space $\{0, 1\}$, where the Prokhorov distance between any two pm's F, G coincides with their total variation distance $|G(\{1\}) - F(\{1\})|$.

Qualitative robustness of sequences of log-transformed p-values has been studied by Lambert (1977). □

Under the assumption of consistency, the following basic equivalence holds.

THEOREM 2.2. *Assume that $T_n \to T_\infty(F)$ in F^n-probability as $n \to \infty$, for some $T_\infty(F) \in \mathbb{R}$.*

(a) *If (T_n) is q_P-robust at F and $k \ne T_\infty(F)$ then $(\psi_{\gamma, k}(T_n))$ is q_P-robust at F.*

(b) *If for every $\varepsilon > 0$ there are $k' \in (T_\infty(F) - \varepsilon, T_\infty(F))$ and $k'' \in (T_\infty(F), T_\infty(F) + \varepsilon)$ such that $(\psi_{0, k'}(T_n))$ and $(\psi_{0, k''}(T_n))$ are q_P-robust at F, then (T_n) is q_P-robust at F.*

Qualitative robustness of (T_n) is usually verified by means of Hampel's (1971) continuity criterion. It presumes statistics T_n that do not depend on the order of the observations and hence can be written as functions of the empirical distribution function $\hat{F}_n$. We denote the set of all possible values of $\hat{F}_n$, i.e. the set of all pm's whose atoms carry probabilities mn^{-1}, $m = 1, \cdots, n$, by $\mathscr{M}_n$. Only statistics of the previously mentioned kind shall be considered in the context of this continuity criterion.

DEFINITION 2.3. (Hampel, 1971) A sequence of test statistics (T_n) is d_P-continuous at a pm F iff

$$\forall\varepsilon > 0 \quad \exists\delta > 0 \quad \exists n_0 \quad \forall m, n > n_0 \quad \forall F_m \in \mathscr{M}_m \quad \forall F_n \in \mathscr{M}_n:$$
$$d_P(F_m, F) \vee d_P(F_n, F) < \delta \Rightarrow | T_m(F_m) - T_n(F_n) | < \varepsilon.$$

If (T_n) is d_P-continuous at F, then (T_n) is (strongly) consistent under F for some $T_\infty(F) \in \mathbb{R}$ (Hampel, 1971, Lemma 2); and, moreover, (T_n) is q_P-robust at F, which is essentially Hampel's (1971) Theorem 1.

Thus, on one hand, essential portions of Hampel's theory can equivalently be cast in testing form. On the other hand, results about q_P-robust and consistent tests, like the following, can be deduced.

COROLLARY 2.4. *If (T_n) is d_P-continuous at F and $k > T_\infty(F)$, then*

$$\forall\varepsilon > 0 \quad \exists\delta > 0 \quad \exists n_0 \quad \forall n > n_0 : W_n \in \mathscr{P}_P(F, \delta)^{(n)} \Rightarrow \int \psi_{\gamma,k}(T_n)\, dW_n < \varepsilon.$$

In case of composite hypotheses $\mathscr{P} \subset \mathscr{M}$, it is natural to consider uniform q_P-robustness of (ϕ_n), (T_n) on $\mathscr{P}$, in the sense that the equicontinuity conditions of Definition 2.1 are required to hold uniformly in $F \in \mathscr{P}$.

If one now wants to infer uniform robustness on $\mathscr{P}$ of (T_n) from a uniform version of the continuity condition (Definition 2.3), along the proof of Hampel's (1971) Theorem 1, one encounters the obstacle that d_P-convergence of the empirical $\hat{F}_n$ towards F, in F^n-probability, does in general not hold uniformly in $F \in \mathscr{P}$ (see an example by R. M. Dudley in Rieder, 1979). In case $\Omega = [-\infty, \infty]$, however, this convergence does hold uniformly, even a.s. F^∞, $F \in \mathscr{M}$, if it is measured by Kolmogorov distance d_K. So, in the context of uniform qualitative robustness, we assume that $\Omega = [-\infty, \infty]$ and measure uniform continuity of (T_n) by d_K (or Lévy-metric d_L).

Another ingredient of the proof would be the implication that, if two pm's G, F are close to each other, so are the empiricals $\hat{G}_n$, $\hat{F}_n$ with high probability. Hampel's proof of this (Hampel, 1971, Lemma 1), which employs Prokhorov distance and uses a result due to V. Strassen, carries over to the present situation (non i.i.d., uniform), if (T_n) is uniformly d_L-continuous on $\mathscr{P}$. If, as in later examples, (T_n) is only uniformly d_K-continuous, the proof still remains applicable provided Prokhorov balls $\mathscr{P}_P(F, \delta)^{(n)}$ are replaced by the smaller total variation balls $\mathscr{P}_V(F, \delta)^{(n)}$. Actually, by virtue of L. LeCam's generalization of the Kiefer and Wolfowitz (1958) result to independent, not necessarily identically distributed observations, even Kolmogorov balls $\mathscr{P}_K(F, \delta)^{(n)}$ can be used.

Thus, if $\Omega = [-\infty, \infty]$ and if (T_n) is uniformly d_K-continuous on $\mathscr{P} \subset \mathscr{M}$, we obtain: (T_n) is uniformly (strongly) consistent on $\mathscr{P}$, (T_n) is uniformly q_K-robust on $\mathscr{P}$, i.e.,

$$\forall\varepsilon > 0 \quad \exists\delta > 0 \quad \exists n_0 \quad \forall n > n_0 \quad \forall F \in \mathscr{P}: \ W_n \in \mathscr{P}_K(F, \delta)^{(n)} \Rightarrow d_P(W_n \circ T_n^{-1}, F^n \circ T_n^{-1}) < \varepsilon;$$

hence, for example, if $k > \sup\{T_\infty(F) \mid F \in \mathscr{P}\}$, the test sequence $(\psi_{\gamma,k}(T_n))$ satisfies the conclusion of Corollary 2.4 uniformly in $F \in \mathscr{P}$.

EXAMPLE. Let IC: Ω (polish) $\rightarrow \mathbb{R}$ be measurable, and consider the sequence of statistics $T_n = \int IC\, d\hat{F}_n$.

(a) If IC is unbounded, then (T_n) is not q_V-robust at any $F \in \mathscr{M}$, $\int |IC| \, dF < \infty$.
(b) If IC is bounded and continuous a.e. F, then (T_n) is d_P-continuous at F.
(c) If $\Omega = [-\infty, \infty]$, and if IC is of bounded variation, then (T_n) is uniformly d_K-continuous on $\mathscr{M}$.
(d) If $\Omega = [-\infty, \infty]$, and if IC is of bounded variation, absolutely continuous, and $\lim_{\varepsilon \to 0} \int | \dot{IC}(t+\varepsilon) - \dot{IC}(t) | \, dt = 0$, then (T_n) is uniformly d_L-continuous on $\mathscr{M}$ (e.g., $IC(t) = c' \vee t \wedge c''$, $t \in \Omega$, for $c', c'' \in \mathbb{R}$, $c' < c''$).

Statistics of this special kind are considered here for reasons of simplicity only. If the setup were local, many other statistics would suitably be approximated by such statistics, and IC could then be interpreted as influence curve of the sequence of statistics; cf. Hampel (1974), Rousseeuw and Ronchetti (1979). However, in view of nonshrinking balls, local approximations are in general not valid in the present framework. □

3. Test breakdown points. The breakdown point of a test sequence shall denote the maximum distance from the ideal pm F, up to which the tests still decide for either zero or one with positive probability, if they have done so at F. Analogously, the breakdown point of a sequence of test statistics shall denote the critical radius, or fraction of gross errors, beyond which the associated tests are absolutely unable to asymptotically distinguish any two pm's out of the underlying set of ideal pm's, when these are blown up to balls.

To demonstrate (in subsequent examples) the independence of breakdown points of the type of balls, we define them with respect to contamination balls ($\mathscr{P}_C(F, \delta) = \{(1-\delta)F + \delta E \,|\, E \in \mathscr{M}\}$), total variation ($V$), Prokhorov ($P$), Lévy ($L$) and Kolmogorov ($K$) balls. Accordingly, $(\Omega, \mathscr{B})$ may be general measurable, or polish, or $\Omega = [-\infty, \infty]$.

Let $F \in \mathscr{M}$, $\mathscr{P} \subset \mathscr{M}$, and $H = C, V, P, L, K$.

DEFINITION 3.1. (a) The H-breakdown point δ^* of a test sequence (ϕ_n) at F is defined as follows,

$$\begin{aligned}\delta^* &= \delta^*_H((\phi_n), F)\\ &= \sup\left\{\delta \in [0,1] \,\middle|\, \liminf_n \int \phi_n \, dF^n > 0 \Rightarrow \liminf_n \alpha_n(\delta) > 0,\right.\\ &\qquad \left.\limsup_n \int \phi_n \, dF^n < 1 \Rightarrow \limsup_n \beta_n(\delta) < 1\right\},\end{aligned}$$

where $\alpha_n(\delta)$ is the infimum and $\beta_n(\delta)$ is the supremum of $\int \phi_n \, dW_n$ when W_n ranges through $\mathscr{P}_H(F, \delta)^{(n)}$.

(b) The H-breakdown point δ^* of a sequence of test statistics (T_n) on $\mathscr{P}$ is defined as follows,

$$\begin{aligned}\delta^* &= \delta^*_H((T_n), \mathscr{P})\\ &= \sup\left\{\delta \in [0,1] \,\middle|\, \exists(\gamma_n) \subset [0,1] \exists (k_n) \subset \mathbb{R} \exists F_0, F_1 \in \mathscr{P}:\right.\\ &\limsup_n \int \phi_n \, dF_0^n \vee \int (1-\phi_n) \, dF_1^n < 1, \left.\delta^*_H((\phi_n), F_0) \wedge \delta^*_H((\phi_n), F_1) > \delta\right\},\end{aligned}$$

where $\phi_n = \psi_{\gamma_n, k_n}(T_n)$.

REMARK (1). Obviously,

$$\delta^*_H((\phi_n), F) = \delta^*_H((1-\phi_n), F), \; \delta^*_H((T_n), \mathscr{P}) = \delta^*_H((-T_n), \mathscr{P}).$$

The breakdown points equal 1 if the ϕ_n, T_n each are identically constant.

REMARK (2). The definitions are certainly in the same spirit as Hampel's (1971) definition of estimator breakdown, although formal relations seems to be tedious. A finite sample size version of test breakdown point, named test resistance, has been defined by Ylvisaker (1977), who also points out connections with Hampel's notion.

REMARK (3). The breakdown points usually do not depend on the type of balls. If $\Omega = [-\infty, \infty]$, this is clear for $H = V, P, L, K$ if the ϕ_n, T_n are monotone with respect to stochastic ordering, since the stochastically extreme elements of $\mathscr{P}_V(F, \delta)$ and $\mathscr{P}_K(F, \delta)$ are the same, as well as those of $\mathscr{P}_P(F, \delta)$ and $\mathscr{P}_L(F, \delta)$. Moreover, those of $\mathscr{P}_P(F, \delta)$ coincide with those of $\mathscr{P}_V(\tilde{F}, \delta)$, where $\tilde{F}$ denotes F shifted by δ or $-\delta$, respectively. □

EXAMPLE. Let $IC:\Omega \to \mathbb{R}$ be measurable, and consider the sequence of test statistics $T_n = \int IC\, d\hat{F}_n$.

(a) If IC is unbounded, then $\delta_H^*((T_n), \mathscr{P}) = 0$ for $\mathscr{P} = \{F \in \mathscr{M} \mid \int |IC|\, dF < \infty\}$ and all H.

(b) If IC is bounded, nonconstant, then $\delta_H^*((T_n), \mathscr{M}) = 1/2$ for $H = C, V$. In case $\Omega = [-\infty, \infty]$, and if IC is also monotone, the same is true for $H = P, L, K$, by the argument given in the preceding remark.

(c) Let the pm's F_θ, $\theta > 0$, approach a pm F_0 in such a way that $F_\theta \ll F_0$ and, for some function $\Lambda \in L^1(F_0)$, $\int | dF_\theta - dF_0 - \theta \cdot \Lambda\, dF_0 | \to 0$ as $\theta \to 0$. Then, if IC is bounded, nonconstant, if follows that

$$\delta_C^*((T_n), \{F_0, F_\theta\}) = \frac{\left| \int IC \cdot \Lambda\, dF_0 \right|}{\sup IC - \inf IC} \cdot \theta + o(\theta) \quad \text{as} \quad \theta \to 0.$$

Thus, the locally evaluated breakdown point is related to the local asymptotic unbiasedness criterion of Rieder (1978), Theorem 5.1. Incidentally, $\delta_C^*((T_n), \{F_0, F_\theta\})$ has maximum slope 1/2 when Λ ranges over the set $\{\Lambda \in L^1(F_0) \mid \int \Lambda\, dF_0 = 0, \int |\Lambda|\, dF_0 = 1\}$ (neglecting the difference between pointwise and F_0-essential extrema of IC). □

4. Implications for rank statistics. The foregoing theory shall be applied in this section to one-sample rank statistics and thus, implicitly, also to one-sample rank tests.

For every sample size n let scores $a_n(1), \cdots, a_n(n) \in \mathbb{R}$ be given. Let $\Omega = [-\infty, \infty]$. The absolute ranks $r_1^+, \cdots, r_n^+$ of the observations $x_1, \cdots, x_n \in \Omega$ are given by $r_i^+ = \sum_{j=1}^n I\{|x_j| \le |x_i|\}$, $i = 1, \cdots, n$. Rank statistics R_n of the following form (simple, linear) are considered,

(1) $$R_n = n^{-1} \textstyle\sum_{i=1}^n \operatorname{sign}(x_i) a_n(r_i^+),$$

where it has been assumed that the observations fulfill the condition

(2) $$x_i \neq 0, \qquad |x_j| \neq |x_k| \quad \text{for all} \quad i, j, k = 1, \cdots, n, \qquad j \neq k.$$

For such observations, R_n can indeed be written as a function of the empirical $\hat{F}_n$, namely

$$\begin{aligned} R_n &= R_n(\hat{F}_n) \\ &= 2 \int_{(0,\infty)} a_n(n(\hat{F}_n(t) - \hat{F}_n(-t - 0)))\hat{F}_n(dt) - n^{-1} \textstyle\sum_{i=1}^n a_n(i). \end{aligned}$$

The subset of $\mathscr{M}_n$ that corresponds to observations which fulfill condition (2) shall be denoted by $\mathscr{M}'_n$. If, furthermore, the set of all pm's F with the property $F(\{t\}) = 0$, $t \in \Omega$, is denoted by $\mathscr{M}_c$ then $F^\infty(\{\forall n, \hat{F}_n \in \mathscr{M}'_n\}) = 1$, $F \in \mathscr{M}_c$.

In general, however, ties occur, which have to be treated by extra methods; cf. Rieder

(1981a). We only notice the following property of the generally defined R_n: If the scores are nonnegative, increasing, i.e.

$$0 \leq a_n(1) \leq \cdots \leq a_n(n) \tag{3}$$

then the stochastically extreme laws of R_n under $\mathscr{P}^{(n)}$, $\mathscr{P} = \mathscr{P}_H(F, \delta)$, $F \in \mathscr{M}_c$, $H = C, V, P, L, K$, can already be computed under the assumption of continuously and identically distributed observations (Rieder, 1981a, Propositions 2.1, 2.2).

The scores are connected over different sample sizes by the requirement that there exist a function $a \in L^1(\lambda)$, with λ Lebesgue measure on (0, 1), such that

$$\lim_n \textstyle\sum_{i=1}^n \left| \displaystyle\int_{[(i-1)/n, i/n]} (a_n(i) - a(s))\, \lambda(ds) \right| = 0. \tag{4}$$

For every $a \in L^1(\lambda)$ let the functional $R_a: \mathscr{M}_c \to \mathbb{R}$ be defined by

$$R_a(F) = 2 \int_{(0,\infty)} a(F(t) - F(-t))F(dt) - \int a \, d\lambda.$$

The steps towards qualitative robustness and "point of final breakdown" of such sequences of rank statistics are now as follows.

LEMMA 4.1. *For $a \in L^1(\lambda)$ the functional R_a is d_P-continuous on $\mathscr{M}_c$.*

LEMMA 4.2. *Let R_n be of form* (1), *with scores $a_n(i) \in \mathbb{R}$, and let $a \in L^1(\lambda)$. Then, for every $F_n \in \mathscr{M}'_n$,*

$$\inf\{ |R_a(F) - R_n(F_n)| \mid F \in \mathscr{M}_c, d_K(F, F_n) \leq n^{-1} \} \leq 3 \textstyle\sum_{i=1}^n \left| \displaystyle\int_{[(i-1)/n, i/n]} (a_n(i) - a(s))\lambda(ds) \right|.$$

Thus, under assumption (4), the continuity of R_a carries over to (R_n) and, as a first result, yields consistency under weaker assumptions than e.g. Theorem 1 of Sen (1970).

PROPOSITION 4.3. *Let R_n be of form* (1), *with the scores satisfying* (4) *for some $a \in L^1(\lambda)$. Then, for every $F \in \mathscr{M}_c$,*

$$R_n(\hat{F}_n) \to R_a(F) \quad \text{as} \quad n \to \infty, \qquad \text{a.s.} \quad F^\infty.$$

Secondly, general qualitative robustness is obtained:

THEOREM 4.4. *Let R_n be of form* (1), *with the scores satisfying conditions* (3) *and* (4). *Then (R_n) is q_P-robust at every $F \in \mathscr{M}_c$.*

By means of the consistency result, the "point of final breakdown," $\delta_H^*((R_n), \mathscr{M}_c)$, can be computed as follows.

THEOREM 4.5. *Let R_n be of form* (1), *with the scores satisfying conditions* (3), *and* (4) *for some monotone increasing, nonnegative $a \in L^1(\lambda)$. Then $\delta_H^*((R_n), \mathscr{M}_c)$, $H = C, V, P, L, K$, coincides with the maximum solution $\delta^* \in [0, 1]$ of the equation*

$$\int_{(0, 1-\delta^*)} a \, d\lambda = \int_{(1-\delta^*, 1)} a \, d\lambda.$$

EXAMPLE. For a $\equiv 1$ (sign-test) one gets $\delta^* = 0.5$, for $a = id_{(0,1)}$ (Wilcoxon) $\delta^* = 1 - 2^{-1/2} \approx 0.293$, and for $a(s) = \Phi^{-1}(1/2 + s/2)$, $0 < s < 1$ (normal scores), $\delta^* = 2\Phi(-(2 \log 2)^{1/2}) \approx 0.239$. □

Uniform extensions of the consistency and robustness results are based on the requirement that R_a be uniformly d_K-continuous on $\mathscr{M}_c$. This holds if

(5) a is of bounded variation on each compact inside (0, 1);

as e.g. in the case of the sign-, Wilcoxon- and the normal scores rank tests.

Thus, the following uniform consistency and robustness results are finally obtained.

PROPOSITION 4.6. *Let R_n be of form* (1), *with the scores satisfying* (4) *for some* $a \in L^1(\lambda)$ *such that* (5) *holds. Then, for every* $\varepsilon > 0$,

$$\lim_n \inf\{F^\infty(\{\forall m > n : |R_m(F_m) - R_a(F)| < \varepsilon\}) \mid F \in \mathscr{M}_c\} = 1.$$

THEOREM 4.7. *Let R_n be of form* (1), *with the scores satisfying* (3), *and* (4) *for some* $a \in L^1(\lambda)$ *such that* (5) *holds. Then* (R_n) *is uniformly* q_K*-robust on* $\mathscr{M}_c$.

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