The Annals of Statistics
1981, Vol. 9, No. 6, 1267–1275

ROBUSTNESS OF MULTIVARIATE TESTS

BY TAKEAKI KARIYA[1]

University of Pittsburgh and Hitotsubashi University

This paper gives necessary and sufficient conditions for the null distribution of a test statistic to remain the same in the class of left $\mathscr{O}(n)$-invariant distributions and in the class of elliptically symmetric distributions. Secondly, it is shown that in certain special cases, the usual MANOVA tests are still uniformly most powerful invariant in a class of left $\mathscr{O}(n)$-invariant distributions including elliptically symmetric distributions.

1. Introduction. Dempster (1969), Dawid (1977) and Eaton (1979) considered the invariance of distributions of some matrix variates under orthogonal transformations. Applications of these results to certain multivariate tests yield the invariance or uniqueness of the null distributions of the tests among certain classes of distributions including elliptically symmetric distributions. On the other hand, in a class of elliptically symmetric distributions, Chmielewski (1980) showed the uniqueness of the null distributions of some invariant tests for equality of scale matrices, for sphericity, etc. However, as far as the distributions of multivariate test statistics are concerned, simple conditions for the uniqueness (or invariance) in those classes can be obtained. This paper gives such conditions and applies them to some multivariate tests. These results help us to check whether the critical point of a test derived under normality is stable against the departure from normality to a left $\mathscr{O}(n)$-invariant distribution or an elliptically symmetric distribution. We remark that the uniqueness is essentially dependent on the uniqueness of the uniform distribution on a Stiefel manifold.

In the latter part of this paper, a robustness property of the usual MANOVA tests is studied. As is well known, in certain special cases, the likelihood ratio test, Roy's test, Lawley-Hotelling's test and Pillai's test are identical and under normality they are UMPI (uniformly most powerful invariant). Here it is shown that these tests are still UMPI in a class of left $\mathscr{O}(n)$-invariant distributions including elliptically symmetric distributions. The argument is rather similar to Kariya (1981) where the robustness of Hotelling's T^2-test is considered. However, the class of distributions treated in this paper is broader, and as a special case, also for the Hotelling T^2-problem, a stronger result is obtained.

2. Main Results. Let $\mathscr{O}(n)$ and $\mathscr{S}(p)$ denote the set of $n \times n$ orthogonal matrices and the set of $p \times p$ positive definite matrices respectively. For an $n \times p$ random matrix X, let $\mathscr{L}(X)$ denote the distribution of X. We shall call X left $\mathscr{O}(n)$-invariant about M if $\mathscr{L}\{Q(X-M)\} = \mathscr{L}(X-M)$ for all $Q \in \mathscr{O}(n)$. Also, we shall call X elliptically symmetric about M with scale matrix $\Sigma \in \mathscr{S}(p)$ if $\mathscr{L}(gy) = \mathscr{L}(y)$ for all $g \in \mathscr{O}(np)$, where $y = (y_1, \cdots, y_n)'$, y_i is the ith row of $Y = (X-M)\Sigma^{-1/2}$. Let $\mathscr{X} = \{X: n \times p \mid \operatorname{rank}(X) = p\}$ and throughout the paper $n \geq p$ is assumed. Further, let $\mathscr{F}_L(M)$ and $\mathscr{F}_E(M, I_n \otimes \Sigma)$ denote respectively the class of np-dimensional left $\mathscr{O}(n)$-invariant distributions about M such that $P(X - M \in \mathscr{X}) = 1$ and the class of np-dimensional elliptically symmetric distributions

---

Received October 2, 1980; revised March 1981.

[1] This research was supported by an Andrew-Mellon Fellowship and in part by Air Force Office of Scientific Research, Bolling Air Force Base, F-49620-79-C-0161.

AMS 1970 *subject classification.* Primary 62H15; secondary 62C07.

Key words and phrases. Left $\mathscr{O}(n)$-invariant distribution, elliptic symmetry, spherical symmetry, Stiefel manifold, MANOVA and GMANOVA problem, tests of independence, robustness.

about M with scale matrix $\Sigma \in \mathscr{S}(p)$ such that $P(X - M \in \mathscr{X}) = 1$. Clearly

$$\mathscr{F}_E(M, I_n \otimes \Sigma) \subseteq \mathscr{F}_L(M) \qquad \text{for all} \quad M: n \times p \quad \text{and} \quad \Sigma \in \mathscr{S}(p).$$

If $\mathscr{L}(X) \in \mathscr{F}_E(M, I_n \otimes \Sigma)$ has a density, it is expressed as

$$f(X \mid M, \Sigma) = |\Sigma|^{-n/2}\phi(\operatorname{tr} \Sigma^{-1}(X - M)'(X - M)), \tag{2.1}$$

where $\phi: [0, \infty) \to [0, \infty)$, and if $\mathscr{L}(X) \in \mathscr{F}_L(M)$ has a density, it is expressed as

$$f(X \mid M) = \psi((X - M)'(X - M)), \tag{2.2}$$

where $\psi: \mathscr{S}(p) \to [0, \infty)$. A left $\mathscr{O}(n)$-invariant distribution which is not elliptically symmetric is the matrix variate t-distribution, whose density is given by

$$f_0(X) = c\,|I_p + (X - M)'(X - M)|^{-(m+n)/2}, \tag{2.3}$$

where c is a normalizing constant. Frazer and Ng (1980) treated a multivariate regression model with this distribution for the error term.

In most multivariate hypothesis testing problems, $\mathscr{L}(X)$ is assumed to be normal:

$$\mathscr{L}(X) = N(M, \Omega) \qquad M: n \times p, \quad \Omega \in \mathscr{S}(np). \tag{2.4}$$

Here $E(X) = M$ and $\operatorname{Cov}(x) = \Omega$, where $x = (x_1, \cdots, x_n)'$ and x_i is the ith row of X. After transformation, many null hypotheses can be stated as

$$H: (M, \Sigma) \in \Theta_0 \times \Lambda_0, \tag{2.5}$$

where in (2.4), $\Omega = I_n \otimes \Sigma$, $\Sigma \in \mathscr{S}(p)$, and $\Theta_0 \times \Lambda_0 \subset R^{np} \times \mathscr{S}(p)$. It is noted that when $\Omega = I_n \otimes \Sigma$ in (2.4), the rows of X are independent with common covariance matrix Σ. Here we assume

$$0 \in \Theta_0, \qquad \text{or} \quad \Theta_0 \text{ contains } 0 \text{ in } R^{np}. \tag{2.6}$$

Usually, this assumption is satisfied; if necessary, replace X by $X - M_0$ and Θ_0 by $\Theta_0 - M_0$, where M_0 is a fixed point in Θ_0. Typical problems of the form (2.5) with (2.6) are the MANOVA, GMANOVA problems, the problems of testing independence or equality of covariance matrices or sphericity. In these problems, except for some special cases, there exist no UMP (uniformly most powerful) tests and usually many tests are proposed in each problem. A feature that these tests have in common is similarity, which is often implied by invariance. In fact, under the null hypotheses in invariant problems, the groups leaving the problems invariant often act transitively on the parameter space $\Theta_0 \times \Lambda_0$ so that the null distributions of these tests do not depend on $(M, \Sigma) \in \Theta_0 \times \Lambda_0$, and the tests become similar. Now, let us consider the uniqueness of the null distributions in $\mathscr{F}_L$ and $\mathscr{F}_E$, where

$$\mathscr{F}_L = \cup\{\mathscr{F}_L(M) \mid M \in \Theta_0\} \tag{2.7}$$

and

$$\mathscr{F}_E = \cup\{\mathscr{F}_E(M, I_n \otimes \Sigma) \mid (M, \Sigma) \in \Theta_0 \times \Lambda_0\}. \tag{2.8}$$

Let $\mathscr{Z} = \{Z \in \mathscr{X} \mid Z'Z = I_p\}$. Let $G\ell(p)$ denote the set of $p \times p$ nonsingular matrices and let $GU(p)$ (or $GT(p)$) denote the set of $p \times p$ nonsingular upper (or lower) triangular matrices with positive diagonal elements.

LEMMA 1. (Eaton, 1979, Proposition 7.4) *Suppose $X \in \mathscr{X}$ is a random matrix with a left $\mathscr{O}(n)$-invariant distribution about* 0, *i.e., $\mathscr{L}(X) \in \mathscr{F}_L(0)$, and write $X = ZA$ where $Z \in \mathscr{Z}$ and $A \in \mathscr{S}(p)$. Then Z and A are independent, and Z has the unique uniform distribution on the Stiefel manifold $\mathscr{Z}$.*

Below, $t(X)$ denotes a test statistic for the problem (2.5) with (2.6).

THEOREM 1. *A necessary and sufficient condition for* $\mathscr{L}(t(X))$ *to remain the same for all* $\mathscr{L}(X) \in \mathscr{F}_L$ *is that when* $\mathscr{L}(X) = N(M, I_n \otimes \Sigma)$, *the following conditions* (i) *and* (ii) *hold*:

(i) $\mathscr{L}\{t(X - M)\} = \mathscr{L}\{t(X)\}$ *for all* $M \in \Theta_0$ *and all* $\Sigma \in \mathscr{S}(p)$.

(ii) $\mathscr{L}\{t(X)\} = \mathscr{L}\{t(Z)\}$ *for* $M = 0$ *and all* $\Sigma \in \mathscr{S}(p)$, *where* $X = ZA$ *with* $Z \in \mathscr{Z}$ *and* $A \in \mathscr{S}(p)$.

PROOF. (Sufficiency) Assume (i) and (ii) under $\mathscr{L}(X) = N(M, I_n \otimes \Sigma)$, where $M \in \Theta_0$. Let $Y = X - M$. Since $\mathscr{L}(Y) = N(0, I_n \otimes \Sigma)$, from (i) and (ii), $\mathscr{L}\{t(X)\} = \mathscr{L}\{t(Y)\} = \mathscr{L}\{t(Z)\}$, where $Y = ZA$ with $Z \in \mathscr{Z}$ and $A \in \mathscr{S}(p)$. But by Lemma 1, $\mathscr{L}(Z)$ does not depend on $\mathscr{L}(Y)$ provided Y is left $\mathscr{O}(n)$-invariant or $\mathscr{L}(Y) \in \mathscr{F}_L(0)$. Since $\mathscr{L}(Y) \in \mathscr{F}_L(0)$ is equivalent to $\mathscr{L}(X) \in \mathscr{F}_L(M)$ with $M \in \Theta_0$, this implies $\mathscr{L}\{t(X)\} = \mathscr{L}\{t(Z)\}$ for all $\mathscr{L}(X) \in \mathscr{F}_L$.

(Necessity) Since $0 \in \Theta_0$ by (2.6), $\mathscr{F}_L(0) \subseteq \mathscr{F}_L$. Since $\mathscr{L}(X - M) \in \mathscr{F}_L(0)$ for all $M \in \Theta_0$, the uniqueness of $\mathscr{L}\{t(X)\}$ in $\mathscr{F}_L$ implies (i). On the other hand, since $\mathscr{L}(X) \in \mathscr{F}_L(0)$ implies $\mathscr{L}(XC) \in \mathscr{F}_L(0)$ for any $C \in G\ell(p)$, the uniqueness also implies $\mathscr{L}\{t(XC)\} = \mathscr{L}\{t(X)\}$ for all $C \in G\ell(p)$ and $\mathscr{L}(X) \in \mathscr{F}_L$. Since $\mathscr{N} = \{N(0, I_n \otimes \Sigma) \mid \Sigma \in \mathscr{S}(p)\} \subset \mathscr{F}_L$, in particular

$$\mathscr{L}\{t(ZAC)\} = \mathscr{L}\{t(ZA)\} \quad \text{for all } C \in G\ell(p) \quad \text{and} \quad \mathscr{L}(X) \in \mathscr{N}, \tag{2.9}$$

where by Lemma 1, $X = ZA$ with $Z \in \mathscr{Z}$ and $A \in \mathscr{S}(p)$, and $\mathscr{L}(Z)$ is the uniform distribution over $\mathscr{Z}$. Since $A^2 = A'Z'ZA = X'X$ and $\mathscr{L}(X'X) = W_p(\Sigma, n)$ under $\mathscr{L}(X) = N(0, I_n \otimes \Sigma)$, A^2 is a complete statistic for $\{W_p(\Sigma, n) \mid \Sigma \in \mathscr{S}(p)\}$ where $W_p(\Sigma, n)$ denotes the Wishart distribution with mean $n\Sigma$ and degrees of freedom n. Further, since A is the unique root of A^2, A is a complete statistic. Therefore, together with the independence of Z and A and the completeness of A, (2.9) implies that for any Borel set $B \subset R^1$ and for any $C \in G\ell(p)$

$$E^Z[I_B\{t(ZAC)\}] = E^Z[I_B\{t(ZA)\}] \quad \text{a.e. } (A), \tag{2.10}$$

where $P\{t(ZA) \in B\} = E^A_\Sigma E^Z[I_B\{t(ZA)\}]$ is used. Here $I_B(\cdot)$ is the indicator function of B, and E^Z and E^A_Σ denote the expectations under $\mathscr{L}(Z)$ and $\mathscr{L}(A)$ respectively. Taking $C = A^{-1}$ in (2.10) yields $E^Z[I_B\{t(Z)\}] = E^Z[I_B\{t(ZA)\}]$ a.e. (A), and so taking the expectation with respect to A produces (ii). This completes the proof.

We remark that in the conditions (i) and (ii) of Theorem 1, the part "all $\Sigma \in \mathscr{S}(p)$" can be replaced by "all $\Sigma \in S$" where S is a nonempty open subset of $\mathscr{S}(p)$. In fact, in the proof of the necessity, even if $\mathscr{S}(p)$ is replaced by a nonempty open subset of $\mathscr{S}(p)$, the argument holds as it stands.

COROLLARY 1.1. *The null distribution of* $t(X)$ *is unique in* $\mathscr{F}_L$ *if the following conditions* (i)′ *and* (ii)′ *hold*:

(i)′ $t(X - M) = t(X)$ *for all* $M \in \Theta_0$.

(ii)′ $t(XC) = t(X)$ *for all* $C \in \mathscr{S}(p)$, *or for all* $C \in GU(p)$ *or for all* $C \in GT(p)$.

PROOF. Clearly (i)′ implies (i) and from Lemma 1, (ii) is implied by the fact that $t(XC) = t(X)$ for all $C \in \mathscr{S}(p)$. On the other hand, by the Gram-Schmidt orthogonalization, $X = ZT$ where $Z \in \mathscr{Z}$ and $T \in GU(p)$ (or $T \in GT(p)$ when the orthogonalization process starts from the last column of X). Further, when $\mathscr{L}(X) = N(0, I_n \otimes \Sigma)$, $\mathscr{L}(Z)$ is the unique uniform distribution over Z; see Nachbin (1967), or Eaton (1979, Proposition 7.3). Hence, taking $C = T^{-1}$ in $t(ZTC) = t(ZT)$ yields $t(Z) = t(X)$, implying (ii). This completes the proof.

COROLLARY 1.2. *If* (i)′ *and* (ii)′ *hold*, $\mathscr{L}\{t(X)\}$ *is unique in the class* $\mathscr{F}_E$.

PROOF. This is obvious from $\mathscr{F}_E \subset \mathscr{F}_L$.

To apply Corollary 1.1 to a specific test, conditions (i)′ and (ii)′ need to be verified. Usually (i)′ is satisfied, but (ii)′ is not in most problems. In invariant problems, if the groups leaving the problems invariant contain as a subgroup $\mathscr{S}(p)$ or $GU(p)$ or $GT(p)$ acting on X by $X \to XC$, then the condition (ii)′ is clearly satisfied. The MANOVA and GMANOVA problems are typical examples which satisfy (ii)′.

Next, we consider the uniqueness of the null distribution of a test $t(X)$ in $\mathscr{F}_E$. Let $\mathscr{U} = \{U \in \mathscr{X} \mid \operatorname{tr} U'U = 1\}$ and $\|X\| = (\operatorname{tr} X'X)^{1/2}$.

THEOREM 2. *A necessary and sufficient condition for $\mathscr{L}(t(X))$ to remain the same for all $\mathscr{L}(X) \in \mathscr{F}_E$ is that when $\mathscr{L}(X) = N(M, I_n \otimes \Sigma)$, the following conditions* (iii) *and* (iv) *hold:*

$$\text{(iii) } \mathscr{L}\{t((X-M)\Sigma^{-1/2})\} = \mathscr{L}\{t(X)\} \qquad \textit{for all} \quad (M, \Sigma) \in \Theta_0 \times \Lambda_0.$$

$$\text{(iv) } \mathscr{L}\{t(X)\} = \mathscr{L}\{t(X/\|X\|)\} \qquad \textit{for } M = 0 \quad \textit{and} \quad \Sigma = \sigma^2 I, \sigma^2 > 0.$$

PROOF. (Sufficiency) Let $Y = (X - M)\Sigma^{-1/2}$ where $(M, \Sigma) \in \Theta_0 \times \Lambda_0$. Then from (iii) and (iv), $\mathscr{L}\{t(X)\} = \mathscr{L}\{t(Y)\} = \mathscr{L}\{t(Y/\|Y\|)\}$ since $\mathscr{L}(Y) = N(0, I_n \otimes I_p)$. But $\mathscr{L}(Y/\|Y\|)$ is the unique uniform distribution over $\mathscr{U}$ provided $\mathscr{L}(Y) \in \mathscr{F}_E(0, I_n \otimes I_p)$ or equivalently $\mathscr{L}(X) \in \mathscr{F}_E(M, I_n \otimes \Sigma)$ with $(M, \Sigma) \in \Theta_0 \times \Lambda_0$; see, e.g., Kariya and Eaton (1977). Hence, $\mathscr{L}\{t(X)\}$ remains the same for all $\mathscr{L}(X) \in \mathscr{F}_E$.

(Necessity) Completely analogous to the proof of the necessity part of Theorem 1.

COROLLARY 2.1. *The* (*null*) *distribution of $t(X)$ is unique in $\mathscr{F}_E$ if the following conditions* (iii)′ *and* (iv)′ *hold*:

$$\text{(iii)}' \; t((X-M)\Sigma^{-1/2}) = t(X) \qquad \textit{for all} \quad (M, \Sigma) \in \Theta_0 \times \Lambda_0.$$

$$\text{(iv)}' \; t(\alpha X) = t(X) \qquad \textit{for all} \quad \alpha > 0.$$

PROOF. Similar to the proof of Corollary 1.1.

We remark that the above results are extendable to the nonnull case provided the assumption and conditions are correspondingly modified. For example, if an alternative hypothesis is of the form (2.5) and it satisfies (2.6), the above results hold exactly.

3. Applications

1. MANOVA *problem.* A canonical form of the model is given by

$$X = (X_1', X_2', X_3')' \sim N((M_1', M_2', 0)', I_n \otimes \Sigma) \tag{3.1}$$

where X_i is $n_i \times p\,(i = 1, 2, 3)$ and $n_1 + n_2 + n_3 = n$. And the problem is to test H: $M_2 = 0$ vs K: $M_2 \neq 0$. The LRT (likelihood ratio test), Roy's test, Lawley-Hotelling's test and Pillai's test are functions of $X_2(X_3'X_3)^{-1}X_2'$ where $n_3 \geq p$. To see that these tests satisfy the conditions (i)′ and (ii)′ in Corollary 1.1, let $\Theta_0 = \{(M_1', 0, 0)' \mid M_1\colon n \times p\}$ and $\Lambda_0 = \mathscr{S}(p)$. Clearly, $0 \in \Theta_0$, and (i)′ and (ii)′ are satisfied since these tests do not depend on X_1 and since they are invariant under $X \to XC$ with $C \in G\ell(p)$. Therefore, the null distributions of these tests remain the same as far as $\mathscr{L}(\mathrm{X}) \in \mathscr{F}_L$. Based on Dempster (1969), Dawid (1977) showed this directly.

2. GMANOVA *problem.* (See Kariya, 1978). A canonical form of the model is ex-

pressed as $\mathscr{L}(X) = N(M, I_n \otimes \Sigma)$with

$$\begin{array}{cc} & \begin{array}{ccc} p_1 & p_2 & p_3 \end{array} \\ X = & \left(\begin{array}{ccc} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{array}\right) \begin{array}{l} n_1 \\ n_2, \\ n_3 \end{array} \end{array} \qquad M = \begin{pmatrix} M_{11} & M_{12} & 0 \\ M_{21} & M_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and the problem is to test H: $M_{12} = 0$ vs K: $M_{12} \neq 0$. The LBI (locally best invariant) test as well as the LRT, Roy's test, Lawley-Hotelling's test and Pillai's test do not depend on (X_{11}, X_{21}, X_{22}) and so (2.6) and (i)′ are satisfied. Further, the group leaving the problem invariant contains as a subgroup $GT(p)$, which acts on X by $X \to XC$ where $C \in GT(p)$. Therefore, (ii)′ is satisfied and these tests have the unique null distributions in $\mathscr{F}_L$. In this example, it seems complicated to express each test as a function of Z where $X = ZT$ with $Z \in \mathscr{X}$ and $T \in GT(p)$. It is noted that the MANOVA problem is a special case of the GMANOVA problem.

3. *Tests of independence.* In the model

$$X = (X_1, X_2) \sim N\left(e\mu', I_n \otimes \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \right)$$

where X_i is $n \times p_i$ $(i = 1, 2)$, $e = (1, \cdots, 1)'$ is $n \times 1$ and $\mu \in R^p$, the problem is to test $\Sigma_{12} = 0$. Invariant tests are functions of$S_{11}^{-1}S_{12}S_{22}^{-1}S_{21}$ where $S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$ is the sample covariance matrix. Since $\mathscr{L}(S)$ does not depend on μ, condition (i) in Theorem 1 is satisfied. On the other hand, (ii) is not satisfied unless a test is trivial. To see this, suppose a test statistic $t(X)$ satisfies (ii). Then, $\mathscr{L}\{t(X)\}$ remains the same not only for all $\mu \in R^p$ but also for all $\Sigma \in \mathscr{S}(p)$. Since $\Sigma_{12} = 0$ vs $\Sigma_{12} \neq 0$ is tested, this implies that the power of the test is equal to the significance level. Therefore, the LRT, Roy's test, Lawley-Hotelling's test and Pillai's test do not have unique null distributions in $\mathscr{F}_L$. However, as is easily checked, the conditions (iii)′ and (iv)′ are satisfied for invariant tests including these tests. Hence, invariant tests have unique null distributions in $\mathscr{F}_E$.

4. *Tests for equality of covariance matrices.* Let X_i's be independent normal random matrices: $X_i \sim N(e_i\mu', I_{n_i} \otimes \Sigma_i)$ where X_i is $n_i \times p$, $e_i = (1, \cdots, 1)' \in R^{n_i}$, $\mu \in R^p$, $\Sigma_i \in \mathscr{S}(p)$ and $i = 1, \cdots, k$. The problem here is to test H: $\Sigma_1 = \Sigma_2 = \cdots = \Sigma_k$. This null hypothesis is of the form (2.5) with (2.6) where $\Sigma = \Sigma_1$. In fact, $X = (X_1', \cdots, X_k')' \sim N(e\mu', I_n \otimes \Sigma)$ under H, where $e = (e_1, \cdots, e_k)'$ and $n_1 + \cdots + n_k = n$. Clearly this problem is invariant under the group of transformations $X_i \to \rho_i X_i C + e_i a'$, where $\rho_i \in \mathscr{O}(n_i)$, $C \in G\ell(p)$ and $a \in R^p$, and the group contains as a subgroup $G\ell(p)$ acting on X by $X \to XC$. Therefore, the condition (ii)′ as well as (i)′ are satisfied and so the null distributions of invariant tests are unique in $\mathscr{F}_L$. This fact was pointed out by Chmielewski (1980) after the uniqueness in $\mathscr{F}_E$ was directly shown.

Many other testing problems are treated in many textbooks as well as in Kiefer and Schwartz (1965) and Krishnaiah (1978), and for each problem, as demonstrated above, by checking the conditions for the uniqueness in $\mathscr{F}_E$ or $\mathscr{F}_L$, it is found out whether a given test has the unique null distribution in $\mathscr{F}_E$ or $\mathscr{F}_L$. We remark that, as we have seen in the problem of testing independence, if a problem concerns the structure of covariance matrix Σ in the model $N(M, I_n \otimes \Sigma)$, the condition (ii) in Theorem 1 is not satisfied. In other words, unless a null hypothesis on Σ contains a nonempty open set of $\mathscr{S}(p)$, the condition (ii) is not satisfied. On the other hand, almost all similar test we see in applications satisfy the conditions (iii)′ and (iv)′ in Corollary 2.1. Needless to say, the above results contain univariate cases. Finally, it is remarked that our formulation does not treat a testing problem on a hypothesis of the form $(M, \Sigma) \in \Delta$ where Δ is a subset of $R^{np} \times \mathscr{S}(p)$ but it cannot be expressed as $\Delta = \Theta_0 \times \Lambda_0$ for any $\Theta_0 \subset R$ and $\Lambda_0 \subset \mathscr{S}(p)$.

4. Robustness property of the MANOVA tests. As is well known, when $\min(n_2, p) = 1$, the four tests in the MANOVA problem stated in Section 3 become identical and under normality they are the UMPI (uniformly most powerful invariant). If $p = 1$, the problem is in fact the ANOVA problem, while if $n_1 = 0$ and $n_2 = 1$, the problem is the Hotelling T^2-problem. In this section, in the case of $\min(n_2, p) = 1$, a robustness property of the tests is studied. The arguments below are rather similar to Kariya (1981) where the robustness of Hotelling's T^2-test is considered. However, the framework here is more general and even for the Hotelling T^2-problem a stronger result is obtained. Let $\mathscr{Q}$ be the class of functions from the set of $p \times p$ matrices into $[0, \infty)$ such that for $q \in \mathscr{Q}$, q is convex on $\bar{\mathscr{S}}(n)$,

$$\int_{R^{np}} q(X'X)\, dX = 1 \qquad (X{:}n \times p) \tag{4.1}$$

and

$$q(BV) = q(VB) \tag{4.2}$$

for all $V \in \bar{\mathscr{S}}(p)$ and $B \in G\ell(p)$, where $\bar{\mathscr{S}}(p)$ denotes the set of $p \times p$ nonnegative definite matrices and dX is Lebesgue measure on R^{np}. Further, let

$$\mathscr{F}(M, \Sigma) = \{f \mid f(X \mid M, \Sigma) = |\Sigma|^{-n/2} q(\Sigma^{-1}(X - M)'(X - M)) \text{ for some } q \in \mathscr{Q}\}, \tag{4.3}$$

where $M \in R^{np}$ and $\Sigma \in \mathscr{S}(p)$. This is a subclass of densities of left $\mathscr{O}(n)$-invariant distributions about M on R^{np} (see (2.2)), and contains the class of elliptically symmetric densities of the form (2.1) with ϕ convex. In fact, if ϕ in (2.1) is convex on $[0, \infty)$, then $\phi(\mathrm{tr}(\cdot))$ is convex on $\bar{\mathscr{S}}(p)$, and any density of the form (2.1) with $M = 0$ and $\Sigma = I$ satisfies (4.2) as well as (4.1). In particular, (4.3) contains the density of $N(M, I_n \otimes \Sigma)$. An example of a density in $\mathscr{F}(M, \Sigma)$ which is not elliptically symmetric is the density of matrix variate t-distribution generated by $f_0(X)$ in (2.3), namely

$$f(X \mid M, \Sigma) = |\Sigma|^{-n/2} f_0((X - M)\Sigma^{-1/2}) = c|\Sigma|^{-n/2} |I + \Sigma^{-1}(X - M)'(X - M)|^{-(n+m)/2}. \tag{4.4}$$

To see that $q_0(V) = c|I + V|^{-(n+m)/2}$ is convex, it suffices to note that $q_0(X'X)$ is the density of a convex mixture of $N(0, I)$ (see Johnson and Kotz, 1972, page 151) and that the density of $N(0, I)$ is convex in $X'X$. But if $p = 1$, $\mathscr{F}(M, \Sigma)$ coincides with the class of elliptical densities of the form (2.1) with ϕ convex.

Now let h be the density of an $n \times p$ random matrix X and consider the testing problem

$$H{:}\ h \in \mathscr{F}(M, \Sigma),\ M_2 = 0,\ \Sigma \in \mathscr{S}(p) \text{ versus } K{:}\ h \in \mathscr{F}(M, \Sigma),\ M_2 \neq 0,\ \Sigma \in \mathscr{S}(p), \tag{4.5}$$

where $X = (X_1', X_2', X_3')'$ and $M = (M_1', M_2', 0)'$ are partitioned as in (3.1); i.e., X_i is $n_i \times p$, M_j is $n_j \times p\,(i = 1, 2, 3;\ j = 1, 2)$, and $n_3 \geq p$ is assumed. This is the MANOVA problem when X has a density h in $\mathscr{F}(M, \Sigma)$. Write $h \in \mathscr{F}(M, \Sigma)$ as

$$h(X \mid M, \Sigma) = |\Sigma|^{-n/2} q(\Sigma^{-1}\{(X_1 - M_1)'(X_1 - M_1) + (X_2 - M_2)'(X_2 - M_2) + X_3'X_3\}), \tag{4.6}$$

for some $q \in \mathscr{Q}$. Then using (4.2), it is easy to see that the problem (4.5) is left invariant under group $\mathscr{G} = \mathscr{O}(n_1) \times \mathscr{O}(n_2) \times \mathscr{O}(n_3) \times G\ell(p) \times R^{n_1 p}$ acting on X and (M_1, M_2, Σ) as

$$gX = (Q_1 X_1 A' + F,\ Q_2 X_2 A',\ Q_3 X_3 A')$$

(4.7) and

$$g(M_1, M_2, \Sigma) = (Q_1 M_1 A' + F,\ Q_2 M_2 A',\ A \Sigma A'),$$

where $g = (Q_1, Q_2, Q_3, A, F) \in \mathscr{G}$. When $\min(n_2, p) = 1$, a maximal invariant and a maximal invariant parameter under this group are

$$T = \operatorname{tr} X_2(X_3'X_3)^{-1}X_2' \qquad \text{and} \qquad \delta = \operatorname{tr} M_2\Sigma^{-1}M_2' \tag{4.8}$$

respectively. To obtain a formal expression of the distribution of T, we first consider the marginal distribution of $\tilde{X} \equiv (X_2', X_3')'$.

Lemma 4.1. *When h is of the form (4.6), the marginal density of $\tilde{X} \equiv (X_2', X_3')'$ is given by*

$$\tilde{h}(\tilde{X} \mid M_2, \Sigma) = |\Sigma|^{-(n_2+n_3)/2}\tilde{q}(\Sigma^{-1/2}\{(X_2 - M_2)'(X_2 - M_2) + X_3'X_3\}\Sigma^{-1/2}), \tag{4.9}$$

where $\Sigma^{-1/2} \in \mathscr{S}(p)$ and $\tilde{q}(V) = \int_{R^{n_1 p}} q(Y'Y + V)\, dY$. Further, $\tilde{q}$ is convex on $\bar{\mathscr{S}}(p)$, and independent of (M_1, M_2, Σ), and it satisfies

$$\tilde{q}(RVR') = \tilde{q}(V) \qquad \text{for any} \quad V \in \bar{\mathscr{S}}(p), \quad \text{and} \quad R \in \mathscr{O}(p). \tag{4.10}$$

Proof. In (4.6), using (4.2), transforming X_1 into $Y = (X_1 - M_1)\Sigma^{-1/2}$ and integrating it with respect to Y yields (4.9). From the procedure, $\tilde{q}$ turns out to be independent of (M_1, M_2, Σ) and the convexity of $\tilde{q}$ follows from the convexity of q. Further, by (4.2),

$$\tilde{q}(RVR') = \int q(R\{R'Y'YR + V\}R')\, dY = \int q(R'Y'YR + V)\, dY = \tilde{q}(V)$$

since the Jacobian of transformation $Y \to YR$ is 1. This completes the proof.

Next, since T is also a maximal invariant under group $\tilde{\mathscr{G}} = \mathscr{O}(n_2) \times \mathscr{O}(n_3) \times G\ell(p)$ acting on $\tilde{X}$ as $\tilde{g}\tilde{X} = (R_2X_2A', R_3X_3A')$, applying Wijsman's (1967) theorem and arguing as in Kariya (1981), we obtain

Lemma 4.2. *Let $W = w(\tilde{X})$ be a maximal invariant under $\tilde{\mathscr{G}}$ and let P_δ^W be the distribution induced by W under δ. Then when $\min(n_2, p) = 1$, the density $\tilde{h}_W$ of W with respect to P_0^W evaluated at $W = w(\tilde{X})$ is given by*

$$\tilde{h}_W(w(\tilde{X}) \mid \delta) = \frac{\displaystyle\int_{G\ell(p) \times \mathscr{O}(n_2)} \tilde{q}\{AA' + \delta ee' - v\delta^{1/2}r_{11}(a_1e' + ea_1')\}|AA'|^{k/2}\, dA\, d\nu(R)}{\displaystyle\int_{G\ell(p)} \tilde{q}(AA')|AA'|^{k/2}\, dA} \tag{4.11}$$

where $v = (T/1 + T)^{1/2}$, $k = n_2 + n_3 - p$, $e = (1, 0, \cdots, 0)' \in R^p$, a_ι is the ith column of A, $r_{\iota\jmath}$ is the (i, j) element of R and $d\nu(R)$ is the invariant probability measure over $\mathscr{O}(n_2)$.

Proof. As in Lemma 3.1 of Kariya (1981), applying Wijsman's (1967) Theorem, $\tilde{h}_W$ is given by N_δ/N_0, where

$$\begin{aligned} N_\delta = \int_{\mathscr{O}(n_2) \times G\ell(p)} & \tilde{q}\{\Sigma^{-1/2}(RX_2A' - M_2)'(RX_2A' - M_2)\Sigma^{-1/2} \\ & + \Sigma^{-1/2}AX_3'X_3A'\Sigma^{-1/2}\}|AA'|^{(n_2+n_3)/2}\, d\mu(A)\, d\nu(R) \end{aligned} \tag{4.12}$$

where $d\mu(A) = dA/|AA'|^{p/2}$. Since μ is left and right invariant, replacing A by $\Sigma^{1/2}A(X_3'X_3)^{-1/2}$ leaves the integral the same; therefore

$$N_\delta = \int_{\mathscr{O}(n_2) \times G\ell(p)} c_1\tilde{q}(Au'uA' - Au'R'\eta' - \eta RuA' + \eta\eta' + AA')|AA'|^{k/2}\, dA\, d\nu(R) \tag{4.13}$$

where $(X_3'X_3)^{-1/2} \in \mathscr{S}(p)$, $c_1 = |X_3'X_3|^{(n_2+n_3)/2}$, $u = X_2(X_3'X_3)^{-1/2}$ and $\eta = \Sigma^{-1/2}M_2'$. Suppose $n_2 = 1$ and let U_1 and U_2 be $p \times p$ orthogonal matrices with $\eta'(\eta\eta')^{-1/2}$ and $u(u'u)^{-1/2}$ as their first rows respectively. Then replacing A by $U_1'AU_2$ and using (4.10) yields

$$N_\delta = \int c_1\tilde{q}(U_1'a_1a_1'U_1T - U_1'a_1R'\eta'T^{1/2} - \eta Ra_1'U_1T^{1/2}$$

$$+ \eta\eta' + U_1'AA'U_1)|AA'|^{k/2}\,dA\,d\nu(R)$$

$$= \int c_1\tilde{q}\{(1+T)a_1a_1' + \textstyle\sum_{i=2}^{p} a_ia_i'$$

$$+ \delta ee' - T^{1/2}\delta^{1/2}R(a_1e' + ea_1')\}|AA'|^{k/2}\,dA\,d\nu(R),$$

where $T = uu' = X_2(X_3'X_3)^{-1}X_2'$ and $\delta = \eta'\eta = M_2\Sigma^{-1}M_2'$. Finally transforming a_1 into $a_1/(1+T)^{1/2}$ and taking the ratio of N_δ and N_0 produce (4.11). Next, suppose $p = 1$ and let V_1 and V_2 be $n_2 \times n_2$ orthogonal matrices with $\eta'(\eta\eta')^{-1/2}$ and $u(u'u)^{-1/2}$ as their first columns respectively. Then replacing R in (4.13) by V_1RV_2' yields

$$N_\delta = \int c_1\tilde{q}(A^2T + A^2 + \delta - 2Ar_{11}\delta^{1/2}T^{1/2})|AA'|^{k/2}\,dA\,d\nu(R),$$

where $T = u'u = X_2'X_2/X_3'X_3$. Hence, transforming A into $A/(1+T)^{1/2}$ produces (4.11). This completes the proof.

Now we shall prove our main result.

Theorem 4.1. *When* $\min(n_2, p) = 1$, *the test with critical region* $T \equiv \operatorname{tr} X_2(X_3'X_3)^{-1}X_2' > c$ *is* UMPI *for problem* (4.5), *and the null distribution of* T *under* $h \in \mathscr{F}(M, \Sigma)$ *with* $M_2 = 0$ *is the same as that under* $\mathscr{L}(X) = N(0, I_n \otimes I_p)$. *That is, under* H, $\mathscr{L}\{(n_2 + n_3 - p)T/p\} = F(p, n_2 + n_3 - p)$ *if* $n_2 = 1$ *and* $\mathscr{L}(n_3T/n_2) = F(n_2, n_3)$ *if* $p = 1$, *where* $F(\alpha, \beta)$ *denotes the F-distribution with degrees of freedom* α *and* β.

Proof. The latter part follows from the argument in Section 3. To show the first part, let $H(v)$ be the numerator of (4.11). Then transforming A into $-A$ leaves it the same and so $H(-v) = H(v)$. Hence, using the convexity of $\tilde{q}$ on $\mathscr{S}(n)$, for $½ < \alpha < 1$, we obtain

$$H(v) = \alpha H(v) + (1-\alpha)H(-v) \geq H((2\alpha - 1)v).$$

This implies that $H(v)$ is a nondecreasing function of $v \in [0, 1]$. Therefore, by applying the Neyman-Pearson Lemma to $\tilde{h}_W$ in (4.11), a MP test is given by critical region $v > c$ or $T > c$. Since this region does not depend on δ and h, the test is UMPI, which completes the proof.

By this theorem, when $\min(n_2, p) = 1$, the four tests in the MANOVA problem considered under normality are UMPI in the class $\mathscr{F}(M, \Sigma)$.

Even if $n_1 = 0$ and $n_2 = 1$, $\mathscr{F}(M, \Sigma)$ contains such left $\mathscr{O}(n)$-invariant densities as the matrix variate t-distribution in (4.4) as well as all the elliptically symmetric densities of the form (2.1) with ϕ convex. Therefore, the following corollary is stronger than the result in Kariya (1981).

Corollary 4.1. *The Hotelling* T^2*-test is* UMPI *for problem* (4.5) *where* $n_1 = 0$ *and* $n_1 = 1$.

Letting $p = 1$ in Theorem 4.1, we obtain

Corollary 4.2. *The usual F-test in the* ANOVA *problem is* UMPI *in the class of elliptically symmetric densities of* (2.1) *with* ϕ *convex.*

It is noted that when $p = 1$, $\mathscr{F}(M, \Sigma)$ is nothing but the class of elliptically symmetric densities of the form (2.1) with ϕ convex.

5. Acknowledgement. The author is very grateful to an associate editor for some suggestions on the matter of Section 4.

REFERENCES

ANDERSON, T. W. (1958). *An Introduction to Multivariate Statistical Analysis.* Wiley, New York.

CHMIELEWSKI, M. A. (1980). Invariant scale matrix hypothesis tests under elliptical symmetry. *J. Multivariate Anal.* **10** 343-350.

DAWID, A. P. (1977). Spherical matrix distributions and a multivariate model. *J. Roy. Statist. Soc. B* **39** 254-261.

DEMPSTER, A. P. (1969). *Elements of Continuous Multivariate Analysis.* Addison Wesley, Reading, Mass.

EATON, M. L. (1979). Selected topics in multivariate statistical analysis. Chapter 7. Unpublished manuscript.

FRASER, D. A. S. and NG, K. W. (1980). Multivariate regression analysis with spherical error. *Multivariate Analysis* 5 (P. R. Krishnaiah, ed.) 369-386. North-Holland, New York.

JENSEN, D. R. (1979). Linear models without moments. *Biometrika* **66** 611-617.

JOHNSON, N. L. and KOTZ, S. (1972). *Distributions in Statistics: Continuous Multivariate Distributions.* Wiley, New York.

KARIYA, T. (1978). The general MANOVA problem. *Ann. Statist.* **6** 200-214.

KARIYA, T. (1981). A robustness property of Hotelling's T^2-problem. *Ann. Statist.* **9** 210-214.

KARIYA, T. and EATON, M. L. (1977). Robust tests for spherical symmetry. *Ann. Statist.* **5** 206-215.

KIEFER, J. and SCHWARTZ, R. (1965). Admissible Bayes character of T^2-, R^2-, and other fully invariant tests for classical multivariate normal problems. *Ann. Math. Statist.* **36** 747-770.

KRISHNAIAH, P. R. (1978). Some recent developments on real multivariate distributions. *Developments in Statistics*, Vol. **1** 135-169. Academic Press, New York.

LEHMANN, E. L. (1959). *Testing Statistical Hypotheses.* Wiley, New York.

NACHBIN, L. (1965). *The Haar Integral.* Van Nostrand, New York.

WIJSMAN, R. A. (1967). Cross-sections of orbits and their applications to densities of maximal invariants. *Fifth Berk. Symp. Math. Statist. Prob.* **1** 389-400.

DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF PITTSBURGH
PITTSBURGH, PA 15260

INSTITUTE OF ECONOMIC RESEARCH
HITOTSUBASHI UNIVERSITY
KUNITACHI-SHI, TOKYO 186
JAPAN