The Annals of Statistics
1981, Vol. 9, No. 6, 1258–1266

INVARIANT TESTS ON COVARIANCE MATRICES[1]

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Minimal complete classes of invariant tests are presented for modifications of the problem of testing the independence of Y and X, where $(Y, X) \equiv (Y, X_1, \cdots, X_p)$ is a multivariate normal random vector. One modification involves having extra independent observations on Y. Others involve extra variates $Z \equiv (Z_1, \cdots, Z_q)$ such that (Y, X, Z) is multivariate normal. Among other results, locally most powerful invariant tests and asymptotically most powerful invariant tests are found; it is shown that for some problems the likelihood ratio test is admissible among invariant tests only for levels less than a specified one; and it is shown that for the problem of testing the independence of Y and X when it is known that Y and Z are independent, the test based on the sample multiple correlation coefficient of Y and (X, Z) is inadmissible.

1. Introduction. Suppose that the random vector $(Y, X_1, \cdots, X_p) \equiv (Y, X)$ has a $(1 + p)$-dimensional normal distribution. If $n \geq p + 1$ observations are taken, and one desires to test the independence of Y and X, it is well known (cf. Lehmann (1959), page 320) that the test based on the sample multiple correlation coefficient is the uniformly most powerful (UMP) invariant test. Here, and throughout this paper, invariance is relative to the largest linear group which leaves the problem invariant. In this paper we consider several modifications of this testing problem, involving cases where either q additional observations have been taken on Y, or where a third set of variates $Z \equiv (Z_1, \cdots, Z_q)$ is present such that (Y, X, Z) has a $(1 + p + q)$-dimensional normal distribution. In contrast to the problem above, some of these problems have no UMP invariant test, so that it is of interest to characterize admissibility among invariant tests. Section 2 contains a minimal complete class theorem which we apply to demonstrate admissibility or inadmissibility of specific tests in the problems now described.

Eaton and Kariya (1975) consider the problem when there are extra observations on Y. These extra data have no effect on the likelihood ratio test (LRT), but they do cause there to be no UMP invariant test. Using the theorem of Section 2 we exhibit the locally most powerful (LMP) invariant test found by Eaton and Kariya (1975), find the asymptotically most powerful (AMP) invariant test, and show that the level α LRT is inadmissible for $\alpha^* < \alpha < 1$, but admissible among invariant tests for $0 < \alpha \leq \alpha^*$, where α^* depends on (n, p). We deal briefly with the case in which the variance of Y is known, corresponding roughly to having $q = \infty$.

Next we look at a set of problems based on $n + q$ observations of (Y, X, Z). Das Gupta (1977), Giri (1977), Banerjee and Giri (1977), Sinha and Giri (1976) and R. A. Wijsman (personal communication) considered several problems based on such data. Let the covariance matrix Λ be given as

$$\Lambda = \begin{bmatrix} \lambda_{yy} & \Lambda_{yx} & \Lambda_{yz} \\ \Lambda_{xy} & \Lambda_{xx} & \Lambda_{xz} \\ \Lambda_{zy} & \Lambda_{zx} & \Lambda_{zz} \end{bmatrix}, \tag{1.1}$$

Received May 1978; revised December 1980.

[1] Support for this research was provided in part by National Science Foundation Grant No. MCS72-04364 A04 and by U. S. Energy Research and Development Administration Contract No. EY-76-S-02-2751 *000.

AMS 1970 *subject classifications.* Primary 62C07, 62C10, 62C15, 62H15; secondary 62H20.

Key words and phrases. Multivariate normal distribution, invariant tests, multiple correlation coefficient, complete class, admissible tests, Bayes tests, likelihood ratio tests.

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where λ_{yy} is a scalar, Λ_{xx} is $p \times p$, Λ_{zz} is $q \times q$. The problems treated are

$$\begin{aligned}
&P_1, \quad H_0 : \Lambda_{xy} = 0 \quad \text{versus} \quad H_A : \Lambda_{xy} \neq 0;\\
&P'_1, \quad H_0 : \Lambda_{xy\cdot z} \equiv \Lambda_{xy} - \Lambda_{xz}\Lambda_{zz}^{-1}\Lambda_{zx} = 0 \quad \text{versus} \quad H_A : \Lambda_{xy\cdot z} \neq 0;\\
&P_2, \quad H_0 : \Lambda_{xy} = 0, \quad \Lambda_{yz} = 0 \quad \text{versus} \quad H_A : \Lambda_{xy} \neq 0, \quad \Lambda_{yz} = 0;\\
&P_3, \quad H_0 : \Lambda_{xy} = 0, \quad \Lambda_{yz} = 0, \quad \Lambda_{xz} = 0\\
&\qquad \text{versus} \quad H_A : \Lambda_{xy} \neq 0, \quad \Lambda_{yz} = 0, \quad \Lambda_{xz} = 0.
\end{aligned} \tag{1.2}$$

Das Gupta's (1977) problems (iv), (iii) and (ii) are P_1, P'_1 and P_2 respectively, and problem (C) in Banerjee and Giri (1977) is P'_1. The UMP invariant tests for P_1 and P_3 are based on the sample multiple correlation coefficient of X and Y, and the UMP invariant test for P'_1 is based on the sample conditional correlation coefficient of X and Y, conditioning on Z. There is no UMP invariant test for P_2, but we can use the results of Section 2 to find a minimal complete class of invariant tests. We find the LMP and AMP invariant tests, and show that the LRT, which is based on the conditional correlation coefficient, is admissible among invariant tests if and only if $0 < \alpha \leq \alpha^*$. Also, the test based on the sample multiple correlation coefficient of Y and (X, Z) is inadmissible (Remark 3.1).

The ordered set of problems $\{P_1, P_2, P_3\}$ ($\{P'_1, P_2, P_3\}$) forms a hierarchy in the sense that each problem tests $\Lambda_{xy} = 0$ ($\Lambda_{xy\cdot z} = 0$) but the problems have increasing amounts of information on $(\Lambda_{yz}, \Lambda_{xz})$. We look at the potential loss in power resulting from using one problem when another with more information actually obtains.

2. Extra observations on Y. The data for this section consists of $n \geq p + 1$ independent observations on (Y, X), plus another q independent observations on Y alone. We assume (Y, X) has mean zero and nonsingular covariance matrix, and reduce the data by sufficiency to S, the sum of squares and cross products matrix of the complete observations, and V_{yy}, the sum of squares of the extra observations. Thus

$$S \sim W_{p+1}(n, \Sigma) \quad \text{and} \quad V_{yy} \sim \sigma_{yy}\chi_q^2,$$

where

$$\Sigma = \begin{bmatrix} \sigma_{yy} & \Sigma_{yx} \\ \Sigma_{xy} & \Sigma_{xx} \end{bmatrix};$$

i.e., S is a central Wishart variable on n degrees of freedom which is independent of the scaled chi-squared variable V_{yy} on q degrees of freedom, σ_{yy} is a scalar, Σ_{xx} is $p \times p$, and S is partitioned as Σ. We test

$$H_0 : \Sigma_{xy} = 0 \quad \text{versus} \quad H_A : \Sigma_{xy} \neq 0 \tag{2.1}$$

based on (S, V_{yy}). This problem is invariant under multiplication of the Y's by a nonzero scalar and the X's by a $p \times p$ nonsingular matrix. Eaton and Kariya (1975) show that the invariance-reduced problem tests

$$H_0 : \Delta = 0 \quad \text{versus} \quad H_A : \Delta > 0 \tag{2.2}$$

based on (L, M), where

$$\Delta = \sigma_{yy}^{-1}\Sigma_{yx}\Sigma_{xx\cdot y}^{-1}\Sigma_{xy}, \qquad L = S_{yy}^{-1}S_{yx}S_{xx\cdot y}^{-1}S_{xy} \qquad \text{and} \qquad M = S_{yy}^{-1}V_{yy}.$$

The statistic (L, M) has the distribution given by

$$L \mid S_{yy} \sim \frac{\chi_p^2(\Delta S_{yy})}{\chi_{n-p}^2}, \quad M \mid S_{yy} \sim \sigma_{yy}S_{yy}^{-1}\chi_q^2, \quad S_{yy} \sim \sigma_{yy}\chi_n^2. \tag{2.3}$$

Here, $\chi_p^2(\Delta s_{yy})$ is a noncentral chi-squared variable with noncentrality parameter ΔS_{yy} which is independent of the central chi-squared variable χ_{n-p}^2. The variables L and M are conditionally independent given S_{yy}, and are unconditionally independent under H_0. The distribution of M does not depend on Δ, hence M is an ancillary statistic.

If there is no extra information, i.e., $q = 0$, then the UMP invariant level α test for problem (2.1) rejects H_0 when

$$L > \{p/(n-p)\}F_{p,n-p,\alpha}, \tag{2.4}$$

where $F_{a,b,\alpha}$ is the upper α point of the $F_{a,b}$ distribution. When $q > 0$, there is no UMP invariant test. However, Eaton and Kariya (1975) have shown that for each fixed value of M, this test is the UMP conditional level α test for problem (2.2) and is also the LRT for problem (2.1). Two distinct tests which are admissible among invariant tests are the level α locally most powerful (LMP) test as $\Delta \to 0$, found by Eaton and Kariya (1975), which rejects H_0 when

$$\{(n/p - 1)L - 1\}(1 + L)^{-1}(1 + M)^{-1} > d_\alpha, \tag{2.5}$$

and the asymptotically most powerful (AMP) test as $\Delta \to \infty$, which rejects H_0 when

$$R^*_\infty(L, M) \equiv (1 + M)^c \sum_{k=0}^\infty \frac{d_k}{k!}\left(\frac{L}{1+L}\right)^k > g_\alpha, \tag{2.6}$$

where d_α and g_α are chosen to provide the desired level, and

$$d_k = (a)_k(c)_k/(b)_k, \quad (Z)_k = Z(Z+1)\cdots(Z+k-1), \quad a = n/2, \quad b = p/2, \quad c = (m+q)/2.$$

Thus there are potentially many admissible tests for problem (2.2).

Define the functions $R_\Delta(\ell, m)$, $R^*_\Delta(\ell, m)$ and $d(\ell, m)$ as follows:

$$R_\Delta(\ell, m) = f_\Delta(\ell, m)/f_0(\ell, m), \tag{2.7}$$

where $f_\Delta(\ell, m)$ is the density of (ℓ, m) when Δ obtains, $R^*_\Delta(\ell, m) = (1 + \Delta)^c R_\Delta(\ell, m)$, and

$$d(\ell, m) \equiv d(\ell, m; \pi^0, \pi^1) = \int_0^1 [\{R^*_\Delta(\ell, m) - 1\}/\Delta]\pi^0(d\Delta) + \int_1^\infty R^*_\Delta(\ell, m)\pi^1(d\Delta),$$

where π^0 and π^1 are finite measures on $[0, 1]$ and $[1, \infty]$ respectively. The definition of $(R^*_\Delta - 1)/\Delta$ is continuously extended to $\Delta = 0$ by setting

$$(R^*_\Delta - 1)/\Delta\,|_{\Delta=0} = \frac{\partial}{\partial\Delta} R^*_\Delta\,|_{\Delta=0} = c + c\{(a/b - 1)\ell - 1\}/(1 + \ell)(1 + m)$$

and the definition of R^*_Δ is continuously extended to $\Delta = \infty$ by defining R^*_∞ to be as in (2.6). Unless otherwise specified, an integral over an interval is assumed to include the endpoints of the interval. Define Φ to be the class of all tests ϕ of the form

$$\phi = I_{\{d(\ell,m)>t\}} \quad \text{a.e. [Lebesgue]} \tag{2.8}$$

for some π^0, π^1 and $|t| < \infty$, where for any set C, I_C represents the indicator function of that set. We now state the first theorem, which is proved in Section 4.

THEOREM 2.1. *The class of tests Φ is minimal complete for problem* (2.2).

Lemma 4.2 shows that $0 \le d(\ell, m) < \infty$ for all (ℓ, m). Note that the function d is linear in R_Δ. Any monotonicity or convexity property in (ℓ, m) which holds for R_Δ will thus also hold for d. By investigating such properties, useful necessary conditions for a test to be admissible can be found. Following Eaton and Kariya (1975), from (2.7),

$$R_\Delta(\ell, m) = \left(1 + \frac{\Delta}{1+m}\right)^{-c} \sum_{k=0}^\infty \frac{d_k}{k!}\left(\frac{\Delta}{1+m}\cdot\frac{\ell}{1+\ell}\right)^k\left(1 + \frac{\Delta}{1+m}\right)^{-k}. \tag{2.9}$$

It is convenient to use the homeomorphic transformation

$$(\ell, m) \leftrightarrow (v, w), \tag{2.10}$$

where $w = (1 + \ell)^{-1}(1 + m)^{-1}$, and $v = \ell w$. The range of (v, w) is $\Omega \equiv \{(v, w) \mid v > 0, w > 0, v + w < 1\}$. The function R_Δ can be written in terms of (v, w) as

$$R_\Delta(v, w) = \{1 + \Delta(w + v)\}^{-c} \sum_{k=0}^{\infty} \frac{d_k}{k!} (\Delta v)^k \{1 + \Delta(v + w)\}^{-k}, \tag{2.11}$$

and we define $R_\Delta^*(v, w)$ and $d(v, w)$ in the obvious way.

In the proof of Theorem 2.2, it is shown that R_Δ, $\Delta > 0$, is strictly increasing in v for fixed w, and is strictly decreasing in v along line segments $w(v) = (n/p - 1)v + \gamma$ contained in Ω. Hence in order for a set $\{(v, w) \mid d(v, w) \le t\}$ to be the acceptance region for an admissible test, it must be contained in the class of sets $\bar{A}$ consisting of all sets $A \subseteq \Omega$ satisfying the following two conditions:

(i) For all $v < v_0$, $(v_0, w_0) \in A$ and $(v, w_0) \in \Omega$ implies that $(v, w_0) \in A$.

(ii) For all $v > v_0$, γ real, $(v_0, (n/p - 1)v_0 + \gamma) \in A$ and $(v, (n/p - 1)v + \gamma) \in \Omega$ implies that $(v, (n/p - 1)v + \gamma) \in A$.

Now we present Theorem 2.2.

THEOREM 2.2. *A necessary condition for a test ϕ defined on Ω to be admissible for problem* (2.2) *is that* $\phi = 1 - I_A$ a.e. $[\mu]$ *for some $A \in \bar{A}$, where μ is Lebesgue measure on Ω.*

PROOF. In view of the foregoing discussion, it suffices to prove the monotonicity properties of $R_\Delta(v, w)$. Set $w(v) = \beta v + \gamma$ for fixed γ. Now $(\partial/\partial v)R_\Delta(v, w(v))$ has the same sign as

$$\{1 - (\beta + 1)Z\} \sum_{k=0}^{\infty} \frac{d_{k+1}}{k!} Z^k - c(\beta + 1) \sum_{k=0}^{\infty} \frac{d_k}{k!} Z^k, \tag{2.12}$$

where

$$Z = \Delta v / \{1 + \Delta v + \Delta w(v)\}. \tag{2.13}$$

If $\beta = 0$, then the coefficient of Z^k for each k is positive since $a > b$. If $\beta = a/b - 1$, then the coefficient of Z^0 is zero, and of Z^k for $k \ge 1$ is negative. Thus R_Δ is strictly increasing in v for fixed w, and strictly decreasing along lines $w = (n/p - 1)v + \gamma$, noting that $a/b = n/p$.

Theorem 2.2 shows that the boundary between the rejection region and acceptance region in Ω of any admissible test, when considered as a function $w(v)$, must have slope between 0 and $n/p - 1$ for all v at which the slope is defined. We now turn our attention to specific tests for problem (2.1).

The LMP test (2.5) can be seen to be in Φ by taking $\pi^0 = \delta_0$ (point mass at 0), $\pi^1 = 0$, and $t = c(d_\alpha + 1)$ in (2.8). In terms of (V, W), this test rejects H_0 when

$$(n/p - 1)V - W > d_\alpha. \tag{2.14}$$

The slope of the boundary of the acceptance region in Ω is exactly the largest allowed under condition (ii).

The boundary of the acceptance region of the LRT (2.4) in Ω is

$$W = V\left(\frac{p}{n - p} F_{p,n-p,\alpha}\right)^{-1}. \tag{2.15}$$

If $\alpha > \alpha^*$, where $\alpha^* \equiv \alpha^*(p, n - p)$ is given by

$$F_{p,n-p,\alpha^*} = 1, \tag{2.16}$$

then the slope in (2.15) is greater than $p/(n-p)$. Thus Theorem 2.2 shows the test inadmissible among invariant tests, hence among all tests. Some representative values of α^* can be found in Table 3.1 of Marden and Perlman (1980). Comparing (2.14) to (2.15), it can be seen that the level α^* LRT is identical to the level α^* LMP, i.e., $d_{\alpha^*}=0$ in (2.14). Thus the LRT is admissible. To show the LRT is admissible for problem (2.2) when $\alpha<\alpha^*$, we define π_δ^0, π_δ^1 and t_δ for $\delta\in(0,1)$ by

$$\pi_\delta^0=(1+\Delta)^{-c}\Delta^{-\delta}I_{(0,1)}\,d\Delta,$$

$$\pi_\delta^1=(1+\Delta)^{-c}\Delta^{-1-\delta}I_{[1,\infty)}\,d\Delta,$$

$$t_\delta=\int_0^1\{(1+\Delta)^c-1\}\Delta^{-1}\pi^0(d\Delta)+\int_1^\infty\Delta^{-1-\delta}\,d\Delta.$$

It can be shown that $d(\ell,m;\pi_\delta^0,\pi_\delta^1)>t_\delta$ is equivalent to

$$\sum_{k=1}^\infty\frac{(a)_k(1-\delta)_k}{(b)_k}\left(\frac{\ell}{1+\ell}\right)^k>\delta^{-1}. \tag{2.17}$$

Thus the test $I_{\{d(\ell,m)>t_\delta\}}$ is admissible for problem (2.2) by Theorem 2.1, and is clearly a LRT for some level α. An argument identical to the one in Section 3.4 of Marden and Perlman (1980) shows that for each $\alpha\in(0,\alpha^*)$ there exists a $\delta\in(0,1)$ such that test (2.17) is the level LRT.

The level α AMP test (2.6) for problem (2.2) is seen to be admissible by taking $\pi^0=0$, $\pi^1=\delta_\infty$, and $t=g_\alpha$ in (2.8). The proof of Theorem 4.5 shows that any other level α invariant test which is essentially different will have lower power than the level α AMP for Δ in some neighborhood of ∞.

REMARK 2.1. Suppose that σ_{yy} is known, corresponding roughly to the case $m=\infty$. Take $\sigma_{yy}=1$. Sufficiency allows us to ignore V_{yy}. Problem (2.1) is now invariant under the group of $p\times p$ nonsingular matrices A which act on S by taking S_{xx} to $AS_{xx}A'$. The maximal invariant statistic is (L,S_{yy}), which is distributed as in (2.3). Now

$$R'_\Delta(\ell,s_{yy})=f_\Delta(\ell,s_{yy})/f_0(\ell,s_{yy})=e^{-\Delta s_{yy}/2}\sum_{k=0}^\infty\frac{(a)_k}{(b)_k}\frac{1}{k!}\left(\frac{s_{yy}\ell}{1+\ell}\right)^k.$$

This is exactly the R_Δ function we had in Marden and Perlman (1980), where we identify $(S_{yy},S_{yy}L/(1+L))$ here with the (U,V) in that paper (the ranges of the statistics differ). The LRT (2.4) behaves as before. The LMP invariant test rejects H_0 when

$$nS_{yx}S_{xx}^{-1}S_{xy}-pS_{yy}>d_\alpha,$$

and the AMP invariant test rejects H_0 when

$$S_{yy\cdot x}<\chi^2_{n-p,\alpha}.$$

3. Extra variates Z. The problems in (1.2) are based on $n+q$ $(n\ge p+1)$ independent observations on the normal vector (Y,X,Z), assumed to have zero mean and covariance matrix Λ (1.1). A sufficient statistic is the sum of squares and cross products matrix T, distributed as a $W_{1+p+q}(n+q,\Lambda)$ variable. The invariance groups for the problems (1.2) are G_1, G_1', G_2 and G_3, respectively, defined as follows. The group G_1 contains all $(1+p+q)\times(1+p+q)$ nonsingular matrices A of the form

$$A=\begin{bmatrix}a_{yy}&0&0\\0&A_{xx}&0\\A_{zy}&A_{zx}&A_{zz}\end{bmatrix} \tag{3.1}$$

partitioned as Λ, $G_1'=\{A\mid A'\in G_1\}$, $G_2=\{A\in G_1'\mid A_{yz}=0\}$ and $G_3=\{A\in G_2\mid A_{xz}=0\}$.

In each case the group acts on T via $A: T \to ATA'$. The maximal invariant statistic and parameter for P_1 and P_3 are

$$L_{xy} = T_{yy}^{-1} T_{yx} T_{xx\cdot y}^{-1} T_{xy} \quad \text{and} \quad \Delta_{xy} = \lambda_{yy}^{-1} \Lambda_{yx} \Lambda_{xx\cdot y}^{-1} \Lambda_{xy},$$

for P_1' are

$$L_{xy\cdot z} = T_{yy\cdot z}^{-1} T_{yx\cdot z} T_{xx\cdot yz}^{-1} T_{xy\cdot z} \quad \text{and} \quad \Delta_{xy\cdot z} = \lambda_{yy\cdot z}^{-1} \Lambda_{yx\cdot z} \Lambda_{xx\cdot yz}^{-1} \Lambda_{xy\cdot z},$$

and for P_2 are $(L_{xy\cdot z}, M_{yz})$ and $\Delta_{xy\cdot z}$, where $M_{yz} = T_{yy\cdot z}^{-1} T_{yz} T_{zz}^{-1} T_{zy}$. To show the result for P_3, note that sufficiency allows one to disregard (T_{yz}, T_{xz}) before applying invariance arguments. It is interesting that the groups corresponding to the hierarchy $\{P_1', P_2, P_3\}$ are nested $(G_3 \subseteq G_2 \subseteq G_1')$, but the groups corresponding to the other are not $(G_2 \not\subseteq G_1)$.

The LRT for P_1 and P_3 is based on L_{xy}, and for P_1' and P_2 on $L_{xy\cdot z}$. See Das Gupta (1977). Except for P_2, the LRT is the UMP invariant test. Since $(L_{xy\cdot z}, M_{xz})$ given $\Delta_{xy\cdot z}$ has the same distribution as (L, M) given Δ in Section 2, the class Φ of Theorem 2.2 with (L, M, Δ) replaced by $(L_{xy\cdot z}, M_{xz}, \Delta_{xy\cdot z})$ is the minimal complete class of invariant tests for P_2. In particular, tests (2.5) and (2.6) are the LMP invariant and AMP invariant tests, and the LRT is admissible among invariant tests if and only if $\alpha \le \alpha^*$ (2.16). We show in Remark 3.1 that the test based on

$$R^2 \equiv T_{yy}^{-1}(T_{yx}, T_{yz}) \begin{bmatrix} T_{xx} & T_{xz} \\ T_{zx} & T_{zz} \end{bmatrix}^{-1} \begin{bmatrix} T_{xy} \\ T_{zy} \end{bmatrix}, \tag{3.2}$$

the sample multiple correlation coefficient of Y and (X, Z), is inadmissible.

For each hierarchy of problems mentioned in the Introduction we consider the effect of using one problem when a more informative one obtains. The distributions of L_{xy} and $L_{xy\cdot z}$ are given by

$$L_{xy} \mid T_{yy} \sim \frac{\chi_p^2(\Delta_{xy} T_{yy})}{\chi_{n+q-p}^2}, \qquad T_{yy} \sim \lambda_{yy} \chi_{n+q}^2; \tag{3.3}$$

$$L_{xy\cdot z} \mid T_{yy\cdot z} \sim \frac{\chi_p^2(\Delta_{xy\cdot z} T_{yy\cdot z})}{\chi_{n-p}^2}, \qquad T_{yy\cdot z} \sim \lambda_{yy\cdot z} \chi_n^2, \tag{3.4}$$

where the chi-square variables in each ratio are independent. First look at hierarchy $\{P_1, P_2, P_3\}$. Assume $\Lambda_{xz} = 0$ and $\Lambda_{yz} = 0$, i.e., P_3 holds, but we ignore the information that $\Lambda_{xz} = 0$, i.e., we consider P_2. We consequently would use the level α test based on $L_{xy\cdot z}$ rather than that based on L_{xy}. When $\Lambda_{xz} = 0$ and $\Lambda_{yz} = 0$, the former test has strictly smaller power than the latter for all values of the parameter. To see this, note that $\Delta_{xy} = \Delta_{xy\cdot z} \equiv \tau$ and $\lambda_{yy} = \lambda_{yy\cdot z}$. Letting c_1 and c_2 be the cutoff points for the L_{xy} and $L_{xy\cdot z}$ tests, the powers when $\tau > 0$ obtains are

$$\int_0^\infty P_{n+q-p}(L_{xy} > c_1 \mid x\tau) g_{n+q}(x)\, dx \quad \text{and} \quad \int_0^\infty P_{n-p}(L_{xy\cdot z} > c_2 \mid x\tau) g_n(x)\, dx \tag{3.5}$$

respectively, where $P_k(\cdot \mid x\tau)$ is the nonnormalized noncentral F distribution with degrees of freedom (p, k) and noncentrality parameter $x\tau$, and $g_k(x)$ is the density of a $\lambda_{yy}\chi_k^2$ variable. The power of the level α F test is strictly increasing in the denominator degrees of freedom (see Das Gupta and Perlman, 1974) and the noncentrality parameter, and $g_k(x)$ has strict increasing monotone likelihood ratio in the parameter k. Thus the first term in (3.5) is strictly larger than the second when $\tau > 0$.

Now suppose $\Lambda_{yz} = 0$ but we ignore that information, i.e., we consider P_1 instead of P_2. In this case, $\Delta_{xy} < \Delta_{xy\cdot z}$ as long as $\Lambda_{xz} \ne 0$. If $\Lambda_{xz} = 0$, then we do better using the test based on L_{xy}, which is not G_2 invariant, than using that based on $L_{xy\cdot z}$, as in the previous paragraph. However, if Λ_{xz} is "large", $\Delta_{xy\cdot z}$ could be substantially larger than Δ_{xy}, and $L_{xy\cdot z}$ would be a preferable test statistic. Thus we may or may not lose power in this situation, though P_2 does have a richer class (Φ) of tests admissible among invariant tests. No power is lost using the LRT for P_1 when P_3 obtains since they have identical LRT's. However, the $L_{xy\cdot z}$ test could be reasonable for P_1 if Λ_{xz} or Λ_{yz} is large, but not P_3. Thus knowing $\Lambda_{xz} = 0$ and $\Lambda_{yz} = 0$ would prevent us from using a suboptimal test.

Next we turn to the hierarchy $\{P_1', P_2, P_3\}$. If we compare P_1' and P_2, we see that the LRT's are identical. Thus the advantage of knowing that $\Lambda_{yx} = 0$ lies in having a richer class of tests which are admissible among invariant tests. Using P_1' or P_2 when P_3 obtains leads to a definite loss of power, as seen from the earlier comparison of P_2 and P_3.

REMARK 3.1. Problem P_2 based on T conditioned on T_{yy} is identical to the problem found in Marden and Perlman (1980). According to their results, conditioned on a fixed T_{yy}, the asymptotically most powerful test as $\Delta_{xy\cdot z} \to \infty$ rejects H_0 for large values of R^2 (3.2). This is the LRT for the problem of testing

$$H_0: (\Lambda_{yx}\Lambda_{yz}) = 0 \quad \text{versus} \quad H_A: (\Lambda_{yx}\Lambda_{yz}) \neq 0.$$

We now show that unconditionally, the test based on R^2 is inadmissible among invariant tests for P_2. Assume it is admissible. By Theorem 2.1, there exist π^0, π^1, and t such that $d(v, w) > t$ if and only if $R^2 > R^2_\alpha$, where v and w are defined as in (2.10) by identifying (L, M) with $(L_{xy\cdot z}, M_{xz})$ and R^2_α is chosen to attain the desired level. Since $W = 1 - R^2$, it must be that for all v, $d(v, w_\alpha) = t$, where $w_\alpha = 1 - R^2_\alpha$. By Equation 15.3.4 of Abramowitz and Stegun (1964),

$$R^*_\Delta(v, w) = (1 + \Delta)^c(1 + \Delta w)^{-c} \sum \frac{(c)_k(a - c)_k}{k!(b)_k}\left(\frac{\Delta v}{1 + \Delta w}\right)^k.$$

Thus $d(v, w_\alpha)$ is a power series in v, so that the coefficients of v^k for $k > 0$ must be zero. The coefficient for v is zero if and only if

$$\text{(3.6)} \quad \int_0^1 (1 + \Delta)^c(1 + \Delta w_\alpha)^{-(c+1)}\pi^0(d\Delta) + \int_1^\infty \Delta(1 + \Delta)^c(1 + \Delta w_\alpha)^{-(c+1)}\pi^1(d\Delta) = 0.$$

Since the integrands in (3.6) are strictly positive over $[0, 1]$ and $[1, \infty]$, respectively, it must be that $\pi^1 = \pi^0 = 0$. That choice of measures, however, does not yield the R^2 test. Thus we have a contradiction, hence the test is not admissible among invariant tests.

4. Proof of Theorem 2.1. Theorem 2.1 is implied by Theorems 4.3 and 4.4 below as follows. Suppose $\psi \notin \Phi$. Theorem 4.3 guarantees that there exists a test $\phi \in \Phi$ such that $E_0(\phi) \le E_0(\psi)$ and $E_\Delta(\phi) \ge E_\Delta(\psi)$ for all $\Delta > 0$. If $E_0(\phi) = E_0(\psi)$, then $E_\Delta(\phi) > E_\Delta(\psi)$ for some $\Delta > 0$ by Theorem 4.4. Otherwise $E_0(\phi) < E_0(\psi)$. Thus ψ is inadmissible, showing that Φ is complete. Since Theorem 4.4 proves all tests in Φ are admissible, Φ is minimal complete.

Before stating and proving Theorems 4.3 and 4.4, we present two lemmas without proof. For definitions used below we refer the reader tó Section 5 of Marden and Perlman (1980).

LEMMA 4.1. (Ghia, 1976, Theorem 2.1). *If F_n, $n \ge 1$, F, are functions on Ω such that $F_n \to F$ pointwise, and*

$$I_{\{F_n > 1\}} \to_{w^*} \phi$$

for some measurable function ϕ, $0 \le \phi \le 1$, then there exists a measurable function χ on Ω, $0 \le \chi \le 1$, such that

$$\phi = I_{\{F>1\}} + \chi \cdot I_{\{F=1\}} \qquad \text{a.e. } [\mu].$$

PROOF. See Marden and Perlman (1980, Lemma 6.2).

LEMMA 4.2. *Consider $R^*_\Delta(v, w)$ defined in Section 2.*

a) $R_\Delta^*(v, w)$ *is bounded for* $\Delta \in [0, \infty]$ *for each fixed* $(v, w) \in \Omega$.
b) *There exists a constant* B, $0 < B \le \infty$, *such that for all* $(v, w, \Delta) \in \Omega \times [0, 1]$, $0 < \{R_\Delta^*(v, w) - 1\}/\Delta \le B$.

THEOREM 4.3. *If* ϕ *is the weak* limit of a sequence of proper Bayes tests* $\{\phi_n\}$ *defined on* Ω, *then* ϕ *is of the form* (2.8). *Thus* Φ *is essentially complete for problem* (2.2) *by Theorem* 5.8 *of Wald* (1950).

PROOF. Any proper Bayes test ψ for problem (2.2) is of the form

$$\psi = I_{\left\{\int_0^\infty R_\Delta \pi(d\Delta) > 1\right\}} \qquad \text{a.e. } [\mu]$$

for some proper measure π on $(0, \infty)$. Let $\{\pi_n\}$ be the sequence of proper measures on $(0, \infty)$ corresponding to $\{\phi_n\}$. Now, $\int R_\Delta \pi_n(d\Delta) > 1$ if and only if

$$\int_0^{1-} (R_\Delta^* - 1)\pi_n^*(d\Delta) + \int_{1-}^\infty R_\Delta^* \pi_n^*(d\Delta) > 1 - \pi_n^*(0, 1), \tag{4.1}$$

where $\pi_n^*(d\Delta) = (1 + \Delta)^{-c}\pi_n(d\Delta)$. Let $u_n = r_n^* + s_n^* + |t_n^*|$ where

$$r_n^* = \int_0^{1-} \Delta \pi_n^*(d\Delta), \qquad s_n^* = \pi_n^*[1, \infty), \quad \text{and} \quad t_n^* = 1 - \pi_n^*(0, 1).$$

Since at least one of r_n^* and t_n^* is nonzero, u_n is always positive. Divide both sides of (4.1) by u_n to obtain

$$r_n \int_0^{1-} \{(R_\Delta^* - 1)/\Delta\}\bar{\pi}_n^0(d\Delta) + s_n \int_{1-}^\infty R_\Delta^* \bar{\pi}_n^1(d\Delta) > t_n, \tag{4.2}$$

where $r_n = r_n^*/u_n$, $s_n = s_n^*/u_n$, $t_n = t_n^*/u_n$,

$$\bar{\pi}_n^0(d\Delta) = \begin{cases} (r_n^*)^{-1}\Delta I_{(0,1)}\pi_n^*(d\Delta) & \text{if } r_n^* > 0 \\ \nu^0(d\Delta) & \text{if } r_n^* = 0, \end{cases} \tag{4.3}$$

and

$$\bar{\pi}_n^0(d\Delta) = \begin{cases} (s_n^*)^{-1} I_{[1,\infty)}\pi_n^*(d\Delta) & \text{if } s_n^* > 0 \\ \nu^1(d\Delta) & \text{if } s_n^* = 0, \end{cases}$$

where ν^0 and ν^1 are arbitrary fixed probability measures on $[0, 1)$ and $[1, \infty)$ respectively. Note that $r_n \ge 0$, $r_n + s_n + |t_n| = 1$ (so that (r_n, s_n, t_n) ranges over a compact space), and $\bar{\pi}_n^0$ and $\bar{\pi}_n^1$ can be extended to probability measures on the compact spaces $[0, 1]$ and $[1, \infty]$, respectively, by defining $\bar{\pi}^0(\{0\}) = \bar{\pi}^0(\{1\}) = \bar{\pi}^1(\{\infty\}) = 0$. Thus there exists a subsequence $\{m\} \subseteq \{n\}$, a point (r, s, t) with $r \ge 0$, $s \ge 0$, $r + s + |t| = 1$, and probability measures $\bar{\pi}^0$ and $\bar{\pi}^1$ on $[0, 1]$ and $[1, \infty]$ respectively such that $(r_m, s_m, t_m) \to (r, s, t)$, $\bar{\pi}_m^0 \to \bar{\pi}^0$ weakly, and $\pi_m^{-1} \to \bar{\pi}^1$ weakly. Lemma 4.2 shows that for each $(v, w) \in \Omega$,

$$\int_0^1 \{(R_\Delta^* - 1)/\Delta\}\bar{\pi}_n^0(d\Delta) \to \int_0^1 \{(R_\Delta^* - 1)/\Delta\}\bar{\pi}^0(d\Delta)$$

and

$$\int_1^\infty R_\Delta^* \bar{\pi}_n^1(d\Delta) \to \int_1^\infty R_\Delta^* \bar{\pi}^1(d\Delta).$$

Thus Lemma 4.1 applied to (4.2) implies that

$$\phi = I_{\{d(v,w) > t\}} + \chi I_{\{d(v,w) = t\}} \qquad \text{a.e. } [\mu]$$

for some $0 \le \chi \le 1$, where $\pi^\circ(d\Delta) = r\bar{\pi}^0(d\Delta)$ and $\pi^1(d\Delta) = s\bar{\pi}^1(d\Delta)$ in (4.3).

If $r = s = 0$, then $t = 1$ or -1, and $d(v, w) \equiv 0$, so that the set $\{d(v, w) = t\}$ is empty. If r or s is positive,then $d(v, w)$ is strictly increasing in w for each fixed value of v because $R_\Delta(v, w)$ is (see (2.11)). Hence $\mu(\{d(v, w) = t\}) = 0$, and so ϕ is of the form (2.8). The proof that Φ is essentially complete is complete.

THEOREM 4.4. *Suppose $\phi \in \Phi$ and ψ is any other test such that $E_0(\phi) = E_0(\psi) \equiv \alpha$ but $\mu(\{\phi \neq \psi\}) > 0$. There exists a $\Delta > 0$ such that $E_\Delta(\phi) > E_\Delta(\psi)$. Hence ϕ is admissible.*

PROOF. Let π^0 and π^1 be as in equation (2.8) for ϕ. Define the sequence of proper measures $\{\pi_n\}$ on $(0, \infty)$ by $\pi_n(d\Delta) = (1 + \Delta)^{-c}\nu_n(d\Delta)$, where

$$\nu_0(d\Delta) = n\gamma_0\delta_{n^{-1}} + \Delta^{-1}I_{(n^{-1},1)}\pi^0(d\Delta) + I_{[1,\infty)}\pi^1(d\Delta) + \gamma_1\delta_n,$$

$$\gamma_0 = \pi^0(\{0\}) \qquad \text{and} \qquad \gamma_1 = \pi^1(\{\infty\}).$$

An argument similar to that in Theorem 5.11 of Marden and Perlman (1980) shows that

$$\lim_{n\to\infty} \int (E_\Delta(\phi) - E_\Delta(\psi))\pi_n(d\Delta) > 0,$$

proving the theorem.

Acknowledgments. I wish to acknowledge the incalculable help given by Professor Michael Perlman. I also wish to thank Professor Robert Wijsman for his suggestions and corrections, and a referee for his careful reading of the paper and his detailed comments.

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