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ON ALMOST SURE LINEARITY THEOREMS FOR SIGNED RANK ORDER STATISTICS[1]

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Almost sure asymptotic linearity of signed rank order statistic in shift parameter is studied under suitable regularity conditions and the results are extended to stationary ϕ-mixing processes as well.

1. Introduction. Let $\{X_i, i \geqslant 1\}$ be a sequence of independent and identically distributed random variables (i.i.d. rv's) with a continuous distribution function (df) F, defined on the real line $R = (-\infty, \infty)$. For every $n(\geqslant 1)$, let $\mathbf{X}_n = (X_1, \cdots, X_n)'$ and consider the usual one sample *(signed-) ranked order statistic*

$$T_n = T(\mathbf{X}_n) = n^{-1}\Sigma_{i=1}^n a_n(R_{ni}^+)\operatorname{sgn} X_i, \tag{1.1}$$

where R_{ni}^+ = rank of $|X_i|$ among $|X_1|, \cdots, |X_n|$, for $i = 1, \cdots, n$, the *scores* $a_n(i)$ are defined by

$$a_n(i) = E\phi(U_{ni}) \quad \text{or} \quad \phi(i/(n+1)), \qquad i = 1, \cdots, n, \tag{1.2}$$

$U_{n1} < \ldots < U_{nn}$ are the ordered rv's of a sample of size n from the uniform (0, 1) df and the *score function* $\phi = \{\phi(u), 0 < u < 1\}$ is assumed to be nondecreasing, absolutely continuous and square integrable inside $I = [0, 1]$. For every $n(\geqslant 1)$, let $\mathbf{1}_n = (1, \cdots, 1)'$ and based on the aligned observations $X_i - b$, $i = 1, \cdots, n$, define

$$T_n(b) = T(\mathbf{X}_n - b\mathbf{1}_n), \qquad b \in R. \tag{1.3}$$

Also, for every $K(0 < K < \infty)$ and $k(\geqslant 0)$, let

$$\Delta_n(K, k) = \{b : |b| \leqslant Kn^{-\frac{1}{2}}(\log n)^k\}, \tag{1.4}$$

$$\omega_n(\Delta_n(K, k)) = \sup\{n^{\frac{1}{2}}|T_n(b) - T_n(0) + bB(\phi, F)| : b \in \Delta_n(K, k)\}, \tag{1.5}$$

where $B(\phi, F)$ (a functional of ϕ, F) is a constant, to be defined later on.

When F has an absolutely continuous probability density function (pdf) f (symmetric about 0) and certain other regularity conditions hold, following the method of attack of Jurečková (1969), van Eeden (1972) has shown that as $n \to \infty$,

$$\omega_n(\Delta_n(K, 0)) \to 0, \text{ in probability, for every (fixed) } K; \tag{1.6}$$

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this is termed the *(weak) asymptotic linearity of signed rank statistic in shift parameter*. Earlier, under more stringent regularity conditions, it has been shown by Sen and Ghosh (1971) that for every $s > 0$ and $k \geq 0$, there exist positive constants k_s, k_s^* and a sample size n_s, such that for all $n \geq n_s$,

$$P\left\{\omega_n(\Delta_n(K, k)) \geq k_s^* n^{-\frac{1}{4}}(\log n)^{2k}\right\} \leq k_s n^{-s}, \tag{1.7}$$

and hence, on letting $s > 1$,

$$\omega_n(\Delta_n(K, k)) = 0\left(n^{-\frac{1}{4}}(\log n)^{2k}\right) \text{ almost surely (a.s.), as } n \to \infty. \tag{1.8}$$

For (1.6), the square integrability of ϕ suffices. But, for (1.7), Sen and Ghosh (1971) used a more stringent condition that

$$M(t) = \int_0^1 \exp\{t\phi(u)\}du < \infty, \forall t \leq t_0, \text{ for some } t_0 > 0. \tag{1.9}$$

In a variety of statistical applications (viz., Sen and Ghosh (1971, 1974), Steyn and Geertsema (1974)), one actually needs a somewhat intermediate result, namely that

$$\omega_n(\Delta_n(K, k)) \to 0 \text{ a.s., as } n \to \infty, \tag{1.10}$$

and for this (1.9) may not be that necessary.

The object of the present investigation is to study suitable regularity conditions pertaining to (1.10) and, in this context, we have not confined ourselves to the case where F is symmetric about 0. A key to this investigation is a later paper of Sen and Ghosh (1973) where an a.s. representation for T_n has been considered. Also, certain basic results on the weighted empirical processes due to Ghosh (1972), Csaki (1977) and others are incorporated in the derivation of our main results. Along with the preliminary notions, the basic theorems are stated in Section 2. The proofs of these theorems are presented in Section 3. The last section deals with some general observations and a discussion on (1.10) for some stationary ϕ-mixing processes.

2. The main theorems. In the customary fashion, we assume that

$$\phi(0) = 0 \quad \text{and} \quad \phi(u) \nearrow \text{ in } \quad u \in (0, 1); \tag{2.1}$$

the theorems to follow hold even if $\phi(u)$ is the difference of two monotone functions. Also, we impose the usual Chernoff-Savage type condition (viz., Puri and Sen (1971, Ch. 4) or Sen and Ghosh (1973)) that ϕ is twice differentiable inside $I = [0, 1]$, denote by $\phi^{(r)}(u) = (d^r/du^r)\phi(u)$, $r = 0, 1, 2$ and assume that there exist positive constants C and δ (both finite), such that

$$|\phi^{(r)}(u)| \leq C[1 - u]^{-\frac{1}{2}-r+\delta}, \text{ for every } u \in (0, 1) \text{ and } r = 0, 1, 2. \tag{2.2}$$

Let $\mathcal{F}$ be the class of all absolutely continuous df's F (need not be symmetric about 0) admitting density function f and its first derivative f' (both continuous) for almost all x (a.a. x) and, for every $\delta(>0)$, let

(2.3)

$$\mathscr{F}_\delta = \left\{F \in \mathscr{F} : \sup_x f(x)\{F(x)[1 - F(x)]\}^{-\frac{1}{2}+\eta} < \infty \text{ for some } \eta : 0 < \eta < \delta\right\}.$$

$$\mathscr{F}_\delta^* = \{F \in \mathscr{F}_\delta : \sup_x |f'(x)| < \infty\}. \tag{2.4}$$

For every $b \in R$, let $F_b(x) = F(x + b)$, $f_b(x) = f(x + b)$, $x \in R$ and let $H_b(x) = F(x + b) - F(-x + b)$, $h_b(x) = f(x + b) + f(-x + b)$ and $h_b^*(x) = f(x + b) - f(-x + b)$, $x \geqslant 0$. Let then

$$\mu_b(\phi, F) = \int_0^\infty \phi(H_b(x))dF_b(x) = \int_b^\infty \phi(F(x) - F(-x + 2b))dF(x), b \in R. \tag{2.5}$$

Note that $(\partial/\partial b)\{\phi(H_b(x))f_b(x)\} = f_b'(x)\phi(H_b(x)) + f_b(x)\phi^{(1)}(H_b(x))h_b^*(x)$ is continuous in $x(\geqslant 0)$ and $b(\in R)$ and, by (2.2),

$$|f_b(x)\phi^{(1)}(H_b(x))h_b^*(x)| \leqslant C[1 - H_b(x)]^{-3/2+\delta}f_b(x)[f_b(x) + f_b(-x)]$$

(2.6)

$$\leqslant C[1 - F_b(x)]^{-3/2+\delta}f_b^2(x) + C[1 - F_b(x)]^{-1+\eta}f_b(x)[F_b(-x)]^{-\frac{1}{2}+\delta-\eta}f_b(-x)$$

$$\leqslant C[1 - F_b(x)]^{-1+\eta}f_b(x)\left\{2 \sup_x f(x)\{F(x)[1 - F(x)]\}^{-\frac{1}{2}+\delta-\eta}\right\}$$

as $1 - H_b(x) = 1 - F_b(x) + F_b(-x) \geqslant [1 - F_b(x)] \vee F_b(-x)$. Hence, for every $c > 0$, $\int_0^\infty \phi^{(1)}(H_b(x)h_b^*(x)f_b(x)dx$ converges uniformly in $b : |b| \leqslant c$ when $F \in \mathscr{F}_\delta$. Similarly, by (2.2) and (2.3), $|\phi(H_b(x))f_b(x)| \leqslant Cf_b(x)[1 - F_b(x)]^{-\frac{1}{2}+\delta}$, so that by partial integration, it follows that for every $c > 0$, $\int_0^\infty \phi(H_b(x))f_b'(x)dx$ converges uniformly in $b : |b| \leqslant c$, when $F \in \mathscr{F}_\delta$. Hence, for every $c > 0$, under (2.2) and (2.3),

$$\int_0^\infty (\partial/\partial b)\{\phi(H_b(x))f_b(x)\}dx \text{ converges uniformly in } b : |b| \leqslant c. \tag{2.7}$$

Therefore, by (2.7), for every $F \in \mathscr{F}_\delta$ and $c > 0$,

$$\begin{aligned}\gamma_b(\phi, F) &= -(\partial/\partial b)\mu_b(\phi, F) \\ &= 2\int_0^\infty \phi^{(1)}(F(x + b) - F(-x + b))f(x + b)f(-x + b)dx\end{aligned} \tag{2.8}$$

exists and is continuous in b, for all $b : |b \leqslant c$. Further we define,

$$\begin{aligned}\omega_n^*(\Delta_n(K, k)) = \sup\{n^{\frac{1}{2}}|T_n(b) &- T_n(0) - 2\mu_b(\phi, F) \\ &+ 2\mu_0(\phi, F)| : |b| \leqslant Kn^{-\frac{1}{2}}(\log n)^k\}.\end{aligned} \tag{2.9}$$

Then, we have the following

THEOREM 1. *Under* (2.1) *and* (2.2), *for every* $F \in \mathscr{F}_\delta$ *and finite* K, *as* $n \to \infty$,

$$\omega_n(\Delta_n(K, 0)) \to 0 \text{ a.s., where } B(\phi, F) = 2\gamma_0(\phi, F). \tag{2.10}$$

THEOREM 2. *Suppose that* (2.1) *and* (2.2) *hold for some* $\delta > \frac{1}{4}$. *Then, for every finite* K, k, *there exists positive numbers* α, β, d, q *and a sample size* n^* (*all possibly dependent on* δ, K, k), *such that for every* $F \in \mathscr{F}_\delta$,

$$P\{\omega_n^*(\Delta_n(K, k)) \geqslant dn^{-\alpha}(\log n)\} \leqslant qn^{-1-\beta}, \qquad \forall n \geqslant n^*. \tag{2.11}$$

If, in addition, $F \in \mathfrak{F}_\delta^*$, *then,* (2.11) *also holds for* $\omega_n^*(\Delta_n(K, k))$ *being replaced by* $\omega_n(\Delta_n(K, k))$, *where* $B(\phi, F) = 2\gamma_0(\phi, F)$, *and hence,* (1.10) *holds.*

The proofs of these theorems are considered in Section 3. In the remainder of this section, we present some results on the empirical df which will be used in the sequel. Let $\{U_i, i \geqslant 1\}$ be a sequence of i.i.d. rv's with the uniform (0, 1) df and let $c(t)$ be equal to 1 or 0 according as t is $\geqslant$ or < 0. For every $n(\geqslant 1)$, let $G_n(t) = n^{-1}\Sigma_{i=1}^n c(t - U_i)$, $t \in I$ be the empirical df. Then, we have the following result due to Ghosh (1972): for every $\theta > 0$, there exist $K(0 < K < \infty)$, $\tau(= (1 + \theta)/2(2 + \theta))$ and n_0 (all dependent on θ), such that for every $n \geqslant n_0$,

$$(2.12) \qquad P\left\{\sup_{t\in I}|G_n(t) - t|/\{t(1 - t)\}^{\frac{1}{2}-\tau} \geqslant Kn^{-\frac{1}{2}}(\log n)\right\} \leqslant 2n^{-1-\theta}.$$

By (2.12) and the Borel-Cantelli lemma, we obtain that as $n \to \infty$,

$$(2.13) \qquad n^{\frac{1}{2}}(\log n)^{-1}\sup_{t\in I}|G_n(t) - t|/\{t(1 - t)\}^{\frac{1}{2}-\tau} = 0(1) \text{ a.s. } \left(\text{for } \tau > \tfrac{1}{4}\right).$$

However, (2.13) is not sharp and the following result (due to Csaki (1977)) is worth mentioning: for every $\varepsilon > 0$, as $n \to \infty$,

$$(2.14) \qquad n^{\frac{1}{2}}(\log\log n)^{-\frac{1}{2}}\sup_{t\in I}|G_n(t) - t|/\{t(1 - t)\}^{\frac{1}{2}-\varepsilon} = 0(1) \text{ a.s.}$$

Both (2.12) and (2.14) are useful for our manipulations.

3. Proofs of the theorems. Note that $\operatorname{sgn} x = 2c(x) - 1$ for every real x. Hence, if we let

$$(3.1) \qquad T_n^* = T^*(\mathbf{X}_n) = n^{-1}\Sigma_{i=1}^n c(X_i)a_n(R_{ni}^+),\ T_n^*(b) = T^*(\mathbf{X}_n - b\mathbf{1}_n), \quad b \in R,$$

then,

$$(3.2) \qquad T_n(b) = 2T_n^*(b) - \bar{a}_n,\ \forall b \in R, \text{ where } \bar{a}_n = n^{-1}\Sigma_{i=1}^n a_n(i).$$

Also, under (1.2) and (2.1),

$$(3.3) \qquad T_n^*(b) \quad \text{and} \quad \mu_b(\phi, F) \text{ are both } \searrow \text{ in } b \in R.$$

Thus, for every $a \leqslant b \leqslant c$,

$$(3.4) \qquad T_n^*(c) - \mu_a(\phi, F) \leqslant T_n^*(b) - \mu_b(\phi, F) \leqslant T_n^*(a) - \mu_c(\phi, F).$$

Moreover, by (2.8), for every $K(< \infty)$ and $\eta(> 0)$, there exist positive integers $m(= m(K, \eta))$ and n_0 and a set of numbers $b_j, j = 0, \pm 1, \cdots, \pm m$, such that $b_{-m} \leqslant -K < b_{-m+1}$, $b_{m-1} < K \leqslant b_m$ and on letting $b_{nj} = n^{-\frac{1}{2}}b_j$, $j = 0, \pm 1, \cdots, \pm m$,

$$(3.5) \qquad 0 \leqslant \mu_{b_{nj}}(\phi, F) - \mu_{b_{nj+1}}(\phi, F) \leqslant \tfrac{1}{4}\eta n^{-\frac{1}{2}}, \qquad \forall -m \leqslant j \leqslant m - 1, n \geqslant n_0.$$

Thus, by (2.9), (3.2), (3.3), (3.4) and (3.5), for every $n \geqslant n_0$,

$$\omega_n^*(\Delta_n(K, 0))$$

$$(3.6) \qquad \leqslant 2\max_{|j|\leqslant m}\left\{n^{\frac{1}{2}}|T_n^*(b_{nj}) - T_n^*(0) + \mu_0(\phi, F) - \mu_{b_{nj}}(\phi, F)|\right\} + \eta/2.$$

Further, by (2.8),

(3.7) $\sup\{n^{\frac{1}{2}}|\mu_b(\phi, F) - \mu_0(\phi, F) + b\gamma_0(\phi, F)| : |b| \leq n^{-\frac{1}{2}}K\} \to 0$ as $n \to \infty$.

Hence, to prove Theorem 1, it suffices to show that for each $j : |j| \leq m$,

$$n^{\frac{1}{2}}|T_n^*(b_{nj}) - T_n^*(0) + \mu_0(\phi, F) - \mu_{b_{nj}}(\phi, F)| \to 0 \text{ a.s., as } n \to \infty. \tag{3.8}$$

The subscript j in (3.8) will be omitted in the sequel for notational convenience. We now appeal to Theorem 4.1 of Sen and Ghosh (1973) and obtain that for the special case of i.i.d. rv's, for every real $b \in R$, their (2.10)–(2.13) simplifies to

$$n^{\frac{1}{2}}\{T_n^*(bn^{-\frac{1}{2}}) - \mu_{n^{-\frac{1}{2}}b}(\phi, F)\} = B_{n1}(b) - B_{n2}(b) + R_n(b), \tag{3.9}$$

where

$$B_{n1}(b) = \int_0^\infty n^{\frac{1}{2}}\left[F_n(x + n^{-\frac{1}{2}}b) - F(x + n^{-\frac{1}{2}}b)\right]\phi^{(1)}\left(F(x + n^{-\frac{1}{2}}b) - F(-x + n^{-\frac{1}{2}}b)\right)dF(-x + n^{-\frac{1}{2}}b), \tag{3.10}$$

$$B_{n2}(b) = \int_0^\infty n^{\frac{1}{2}}\left[F_n(-x + n^{-\frac{1}{2}}b) - F(-x + n^{-\frac{1}{2}}b)\right]\phi^{(1)}\left(F(x + n^{-\frac{1}{2}}b) - F(-x + n^{-\frac{1}{2}}b)\right)dF(x + n^{-\frac{1}{2}}b), \tag{3.11}$$

$$R_n(b) = 0(n^{-\eta}) \text{ a.s., as } n \to \infty, \text{ for some } \eta > 0, \tag{3.12}$$

and $F_n(x) = n^{-1}\sum_{i=1}^n c(x - X_i)$, $x \in R$. It may be remarked that though Sen and Ghosh (1973) considered the case of a fixed F, their treatment remains valid for the translation case considered here. In fact, (3.9) through (3.12) hold uniformly in b in any bounded interval. For simplicity of proof, we assume that $b > 0$; a similar proof holds for $b < 0$. We may rewrite

$$B_{n1}(b) = \int_{-n^{-\frac{1}{2}}b}^\infty n^{\frac{1}{2}}\left[F_n(Y + 2n^{-\frac{1}{2}}b) - F(y + 2n^{\frac{1}{2}}b)\right] \times \phi^{(1)}\left(F(Y + 2n^{-\frac{1}{2}}b) - F(-y)\right)dF(-y). \tag{3.13}$$

Note that by (2.3), $F(x) - F(-x + 2n^{-\frac{1}{2}}b) = F(x) - F(-x) + O(n^{-\frac{1}{2}})$, for every $x \in R$. Hence, defining $\{x_n\}$ by $F(x_n) - F(-x_n) = 1 - n^{-1/6}$, we obtain from (2.2) ($r = 1$) and (2.14) with $\varepsilon = \delta/2$, where $\delta(> 0)$ is defined by (2.2), that as $n \to \infty$,

$$|\int_{x_n}^\infty n^{\frac{1}{2}}\left[F_n(y + 2n^{-\frac{1}{2}}b) - F(y + 2n^{-\frac{1}{2}}b)\right]\phi^{(1)}\left(F(y + 2n^{-\frac{1}{2}}b) - F(-y)\right)dF(-y)|$$

$$\leq \left[0\left((\log\log n)^{\frac{1}{2}}\right)\right].\int_{x_n}^\infty\left\{F(y + 2n^{-\frac{1}{2}}b)\left[1 - F(y + 2n^{-\frac{1}{2}}b)\right]\right\}^{(1-\delta)/2}.$$

$$\left[1 - F(y + 2n^{-\frac{1}{2}}b) + F(-y)\right]^{-3/2+\delta}dF(-y) \tag{3.14}$$

$$\leq \left[0\left((\log\log n)^{\frac{1}{2}}\right)\right]\int_{x_n}^\infty[F(-y)]^{-1+\delta/2}dF(-y)$$

$$= \left[0\left((\log\log n)^{\frac{1}{2}}\right)\right][F(-x_n)]^{\delta/2} \qquad [\text{as } F(-x_n) \leq n^{-1/6}]$$

$$= 0\left(n^{-\delta/12}(\log\log n)^{\frac{1}{2}}\right) \text{ a.s.}$$

Further, by (2.1), (2.2), (2.3) and (2.14),

$$(3.15)\quad \int_{-n^{-\frac{1}{2}}b}^{0} n^{\frac{1}{2}}\Big[F_n\big(y + 2n^{-\frac{1}{2}}b\big) - F\big(y + 2n^{-\frac{1}{2}}b\big)\Big] \phi^{(1)}\big(F\big(y + 2n^{-\frac{1}{2}}b\big) - F(-y)\big)dF(-y)$$
$$= 0\big((\log\log n)^{\frac{1}{2}}\big).0\big(n^{-\frac{1}{2}}\big) = 0\Big(\big(n^{-1}\log\log n\big)^{\frac{1}{2}}\Big) \text{ a.s., as } n \to \infty.$$

Hence,

$$\begin{aligned} B_{n1}(b) - B_{n1}(0) = {} & \int_0^{x_n} n^{\frac{1}{2}}\Big[F_n\big(y + 2n^{-\frac{1}{2}}b\big) - F_n(y) - F\big(y + 2n^{-\frac{1}{2}}b\big) + F(y)\Big] \\ & \times \phi^{(1)}\big(F\big(y + 2n^{-\frac{1}{2}}b\big) - F(-y)\big)dF(-y) \\ & + \int_0^{x_n} n^{\frac{1}{2}}\big[F_n(y) - F(y)\big]\Big\{\phi^{(1)}\big(F\big(y + 2n^{-\frac{1}{2}}b\big) - F(-y)\big) \\ & - \phi^{(1)}(F(y) - F(-y))\Big\}dF(-y) \\ & + 0\Big(\big(n^{-\delta/2}(\log\log n\big)^{\frac{1}{2}}\Big) + 0\Big(\big(n^{-1}\log\log n\big)^{\frac{1}{2}}\Big) \text{ a.s., as } n \to \infty. \end{aligned} \tag{3.16}$$

Since, by (2.2), $|\phi^{(2)}(u)| \leqslant C[1 - u]^{-5/2+\delta}$ for every $u \in (0, 1)$, $F(x_n) - F(-x_n) = 1 - n^{-1/6}$ and $\sup\{|n^{\frac{1}{2}}[F_n(x + 2n^{-\frac{1}{2}}b) - F_n(x) - F(x + 2n^{-\frac{1}{2}}b) + F(x)]| : x \in R\} = 0(n^{-\frac{1}{4}}\log n)$ a.s. as $n \to \infty$ (cf. Sen and Ghosh (1971)), by using (2.2), (2.3) and (2.14) for $\varepsilon = \delta/2$, the right-hand side of (3.16) can easily be shown to be bounded by

$$(3.17)\quad 0\Big(\big(n^{-(1+\delta)/6}\big[(\log n) + n^{-1/12}(\log\log n)^{\frac{1}{2}}\big]\big)\Big) + 0\Big(\big(n^{-\delta/2}(\log\log n\big)^{\frac{1}{2}}\Big)$$
$$+ 0\Big(\big(n^{-1}\log\log n\big)^{\frac{1}{2}}\Big) = 0((n^{-\rho}(\log n)) \text{ a.s., as } n \to \infty, \text{ where } \rho > 0.$$

A similar case holds for $B_{2n}(b) - B_{2n}(0)$. Hence, the proof of Theorem 1 follows from (3.8), (3.9), (3.10), (3.11), (3.12), (3.16), (3.17) and the above argument.

For Theorem 2, in (3.5)–(3.6), we replace m by $m_n = n^{\alpha}(\log n)^k$ and note that (3.6) holds with K being replaced by $K(\log n)^k$. In this case, we show that for every j: $|j| \leqslant m_n$, for every $\varepsilon > 0$, there exist positive numbers a, b and a sample size n_0, such that for $n \geqslant n_0$,

(3.18)

$$P\Big\{n^{\frac{1}{2}}|T_n^*(b_{nj}) - T_n^*(0) + \mu_0(\phi, F) - \mu_{b_{nj}}(\phi, F)| > dn^{-\alpha}(\log n)^k\Big\} \leqslant an^{-1-b}.$$

For this purpose, we appeal to (2.2) with $\delta > \frac{1}{4}$ and to (2.12), under which, there exist positive numbers η, c_1 and c_2, such that for all n, sufficiently large,

$$(3.19)\qquad P\{|R_n(b)| \geqslant c_1 n^{-\eta}\} \leqslant c_2 n^{-1-\theta}, \text{ for some } \theta > 0,$$

where $R_n(b)$ is defined by (3.9) and (3.12). The proof of (3.19) is implicit in the proof of Theorem 4.1 of Sen and Ghosh (1983); an explicit proof of (3.19) is also contained in Müller-Funk (1977). Also, in the treatment of (3.13) through (3.17), we replace everywhere the use of (2.14) by (2.12), whereby, we replace the a.s. statements by a statement with a probability greater than $1 - 2n^{-1-b'}$, for some $b' > 0$. This will lead to (3.18) where $b < b'$. Since $m_n = 0((\log n)^k)$ for some $k > 0$, (3.6), (3.18) and the Borel-Cantelli lemma insure that (2.11) holds. For the rest of the theorem, it suffices to show that (3.7) holds for K being replaced by $K(\log n)^k$ when $F \in \mathcal{F}_\delta^*$. For this purpose, we make use of (2.8) and hence, it suffices to show that for n sufficiently large,

$$\sup\left\{|\gamma_b(\phi, F) - \gamma_0(\phi, F)| : |b| < Kn^{-\frac{1}{2}}(\log n)^k\right\} < n^{-\xi} \tag{3.20}$$

for some $\xi > 0$. Defining $\{x_n\}$ as in after (3.13), we write

$$\begin{aligned}\gamma_b(\phi, F) &= \int_b^{x_n} + \int_{x_n}^\infty \phi^{(1)}(F(x) - F(-x + 2b))f(-x + 2b)f(x)dx \\ &= I_{n1}(b) + I_{n2}(b), \text{ say.}\end{aligned} \tag{3.21}$$

Note that by (2.2) and (2.3),

$$\begin{aligned}|I_{n2}(b)| &< C\int_{x_n}^\infty\left[1 - F(x) + F(-x + 2b)\right]^{-3/2+\delta}f(x)f(-x + 2b)dx \\ &< C\left\{\sup_x f(x)\{F(x)[1 - F(x)]\}^{-\frac{1}{2}+\eta}\right\}\int_{x_n}^\infty[1 - F(x)]^{-1+\delta-\eta}dF(x) \\ &= 0\left([1 - F(x_n)]^{\delta-\eta}\right) = 0(n^{-(\delta-\eta)/6}).\end{aligned} \tag{3.22}$$

Similarly, for every $b : |b| < Kn^{-\frac{1}{2}}(\log n)^k$,

$$\begin{aligned}\int_0^b \phi^{(1)}(F(x) - F(-x))f(-x)f(x)dx &= 0\left([F(b) - F(0)]^{\delta-\eta}\right) \\ &= 0\left(\left(n^{-\frac{1}{2}}(\log n)^k\right)^{\delta-\eta}\right).\end{aligned} \tag{3.23}$$

Hence, by (3.21), (3.22) and (3.23), for n sufficiently large,

$$\begin{aligned}\gamma_b(\phi, F) - \gamma_0(\phi, F) = &\int_b^{x_n}\{\phi^{(1)}(F(x) - F(-x + 2b)) - \phi^{(1)}(F(x) - F(-x))\} \\ &\times f(-x + 2b)dF(x) + \int_b^{x_n}\phi^{(1)}(F(x) - F(-x)) \\ &[f(-x + 2b) - f(-x)]dF(x) + 0(n^{-\xi}), \text{ where } \xi > 0.\end{aligned} \tag{3.24}$$

Using (2.2) (for $r = 2$), (2.3) and the fact that $1 - F(x_n) < n^{-1/6}$, it follows that the first term on the right-hand side of (3.24) is $[0(n^{-\frac{1}{2}}(\log n)^k)][0(n^{(1-\delta+\eta)/6})] = 0(n^{-\xi})$, where $\xi > 0$. Similarly, by (2.2) (for $r = 1$) and (2.4), for $F \in \mathcal{F}_\delta^*$, the second term on the right-hand side of (3.24) is $[0(n^{-\frac{1}{2}}(\log n)^k)][0(n^{(\frac{1}{2}-\delta)/6})] = 0(n^{-\xi})$, where $\xi > 0$. Hence, (3.20) is proved and the theorem follows.

4. Some general remarks. We may note that if F is symmetric about 0, then $F(x) - F(-x) = 2F(x) - 1$, $x > 0$, so that if we let $\phi(u) = \phi^*((1 + u)/2)$, $0 < u < 1$ where $\phi^*(u) + \phi^*(1 - u) = 0$ for every $u \in I$ and assume that f has a finite

Fisher information $I(f) = \int (f'/f)^2 dF$, then

$$\gamma_0(\phi, F) = \int_0^\infty \phi^{*(1)}(F(x))f(-x)dF(x) = \int_0^\infty \phi^*(F(x))[-f'(x)/f(x)]dF(x)$$

$$= \tfrac{1}{2}\left(\int_{-\infty}^\infty \phi^*(F(x))[-f'(x)/f(x)]dF(x)\right) = \tfrac{1}{2}\left(\int_0^1 \phi^*(u)\psi(u)du\right) \tag{4.1}$$

where $\psi(u) = -f'(F^{-1}(u))/f(F^{-1}(u))$, $u \in I$. In this case, the results are in agreement with van Eeden (1972), though we have a.s. convergence under slightly more stringent regularity conditions. On the other hand, (2.2), even for $\delta > \frac{1}{4}$ and the other regularity conditions of Theorem 2 are less stringent than the ones in Sen and Ghosh (1971).

One of the important uses of the asymptotic linearity theorems is the estimation of $\gamma_0(\phi, F)$ from the aligned statistics $T_n(a)$. Almost sure convergence of such estimates follows readily from Theorem 1. It may be of interest to note that the current Theorem 1 throws light on the behavior of such estimates when the underlying df F is not necessarily symmetric.

One advantage of using Theorem 4.1 of Sen and Ghosh (1973) lies in its flexibility for adaptation for certain stationary stochastic processes. Actually, Sen and Ghosh (1973) considered stationary ϕ-mixing processes, and, under diverse mixing-conditions, studied the feasibility of (3.9) through (3.12), under conditions parallel to (2.2) and (2.3). For mixing processes, we may not have a strong result as in (2.14). Nevertheless, certain a.s. orders for the sup-norm of empirical processes are studied in Lemma 3.1 of Sen and Ghosh (1973) and these may be used with advantage for the study of results parallel to (2.10) for such processes. Let $\{X_i, -\infty < i < \infty\}$ be a stationary ϕ-mixing process, defined on a probability space $(\Omega, \mathcal{Q}, P)$. Thus, if $\mathfrak{M}_{-\infty}^k$ and $\mathfrak{M}_{k+n}^\infty$ be respectively the σ-fields generated by the X_i, $i \leqslant k$ and X_i, $i \geqslant k + n$ and if $E_1 \in \mathfrak{M}_{-\infty}^k$ and $E_2 \in \mathfrak{M}_{k+n}^\infty$, then for all $k(-\infty < k < \infty)$ and $n(\geqslant 1)$,

$$|P(E_2|E_1) - P(E_2)| \leqslant \phi(n), \qquad \phi(n) \searrow 0 \text{ as } n \to \infty. \tag{4.2}$$

Consider the following *mixing conditions*:

$$\text{(i) for some } k \geqslant 0,\ A_k(\phi) = \Sigma_{n=1}^\infty n^k \phi^{\frac{1}{2}}(n) < \infty, \tag{4.3}$$

$$\text{(ii) for some } t > 0,\ \Sigma_{n=1}^\infty e^{tn}\phi(n) < \infty. \tag{4.4}$$

Parallel to (2.2), let us assume here that

$$|\phi^{(r)}(u)| \leqslant C(1-u)^{-\alpha - r + \delta}, \qquad 0 < u < 1, r = 0, 1, 2, \tag{4.5}$$

where α and δ are positive numbers. Then, we have the following

THEOREM 3. *Suppose that* (2.1), (2.3) *and* (2.5) *hold and either one of the following three conditions holds*:

(i) *for some* $k > 1$, $A_k(\phi) < \infty$ *and* (4.5) *holds with* $\alpha = (2k-1)/2(2k+1)$,

(ii) *for some* $k > 3$, $A_k(\phi) < \infty$ *and* (4.5) *holds with* $\alpha = (k-2)/2k$, *or*

(iii) (4.4) *holds and* (4.5) *holds with* $\alpha = \frac{1}{2}$.

Then, (2.10) *holds*.

The proof is quite analogous to that of Theorem 1 in Section 3. Wherever, in that proof, we have used (2.14), we may use Lemma 3.1 of Sen and Ghosh (1973) and noting that by virtue of Theorem 4.1 of Sen and Ghosh (1973), the representation in (3.9) through (3.12) remains valid under the hypothesis of Theorem 3, the rest of the proof may practically be repeated with minor variations. Hence, the details are omitted.

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