The Annals of Statistics
1980, Vol. 8, No. 1, 217–220

A CHARACTERIZATION OF THE EXPONENTIAL AND RELATED DISTRIBUTIONS BY LINEAR REGRESSION

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Let $X_1, \cdots, X_n$ ($n \geq 2$) be a random sample on a rv X, and $Y_1 < \cdots < Y_n$ be the corresponding order statistics. Define

$$Z_k = \frac{1}{n-k}\Sigma_{i=k+1}^{n}(Y_i - Y_k),\ 1 \leq k \leq n-1,\ W_k = \frac{1}{k-1}\Sigma_{i=1}^{k-1}(Y_k - Y_i),$$

$2 \leq k \leq n$. Using the properties $E(Z_k | Y_k = y) = \alpha y + \beta$ and $E(W_k | Y_k = y) = \alpha y + \beta$, a.e. (dF), where α and β are constants, we obtain characterizations of several distributions which include the exponential, the Pearson (type I) and the Pareto (of the second kind) distributions.

1. Introduction. Let $X_1, \cdots, X_n$ be $n \geq 2$ independent observations on a random variable X having a two-parameter exponential distribution defined by

$$(1)\qquad F(x) = 1 - e^{-\frac{1}{\theta}(x-\mu)}, \qquad x > \mu > -\infty, \theta > 0,$$
$$= 0, \qquad \text{elsewhere.}$$

Also let $Y_1 < \cdots < Y_n$ denote the corresponding order statistics, $U_k = Y_k - Y_{k-1}$, with $Y_0 \equiv 0$, be the corresponding sample spacings and $Z = 1/(n-1) \cdot \Sigma_{i=2}^{n}(Y_i - Y_1)$. Many characterization theorems of the exponential distribution F, based on certain properties of the functions of Y_k's, U_k's and Z, have been obtained in recent years. For recent surveys of the literature in this area, we refer the reader to Kotz [5] and Galambos [4]. The following two papers, not quoted in the articles by Kotz and Galambos, are directly related to the development of this paper.

In [2] Dallas showed that if the distribution F of X is continuous with finite first moment, then $E(Z | Y_1 = y) =$ constant a.e. (dF) is a characterizing property of the exponential distribution (1). Assuming the above conditions of Dallas, Beg and Kirmani [1] later proved the more general results as obtained by Ferguson in [3] by using the property $E(Z | Y_1 = y) = \alpha y + \beta$, a.e. (dF), where α and β are constants. In both articles, the authors claimed that their characterization theorem of the exponential distribution (1) based on the constant regression of Z on Y_1 is stronger than the result obtained by using the independence of Z and Y_1, which is not true because in the later case it is not necessary to assume the existence of the first moment of F (see Rossberg [6]).

Received November 1976; revised May 1978.
[1]Research partially sponsored by NRC of Canada, Grant No. A8792.
AMS 1970 *subject classification.* Primary 62E10.
Key words and phrases. Characterization, linear regression, order statistics, exponential distribution, Pearson type I distribution, Pareto distribution of second kind.

In this paper, we intend to generalize the results of Dallas and Beg and Kirmani further. Besides the exponential distribution (1), our result also includes the Pearson type I distribution as defined by

$$(2) \qquad F(x) = 1 - \left(\frac{\nu - x}{\nu - \mu}\right)^{\theta}, \qquad -\infty < \mu < x < \nu < \infty, \qquad \theta > 0$$
$$= 0, \qquad \text{elsewhere,}$$

and the Pareto distribution of the second kind as defined by

$$(3) \qquad F(x) = 1 - \left(\frac{\mu + \gamma}{x + \gamma}\right)^{\theta}, \qquad x > \mu > -\infty, \qquad \theta > 0, \gamma > -\infty$$
$$= 0, \qquad \text{elsewhere.}$$

2. Characterization by linear regression. Define

$$(4) \qquad Z_k = \frac{1}{n-k}\Sigma_{i=k+1}^{n}(Y_i - Y_k), \qquad \text{for} \quad k = 1, \cdots, n-1$$
$$W_k = \frac{1}{k-1}\Sigma_{i=1}^{k-1}(Y_k - Y_i), \qquad \text{for} \quad k = 2, \cdots, n.$$

THEOREM 1. *Suppose the distribution function F of X is continuous with finite first moment. Then for some $1 \leqslant k \leqslant n-1$,*

$$(5) \qquad E(Z_k | Y_k = y) = \alpha y + \beta, \qquad \text{a.e. } (dF)$$

where α and β are constants, if and only if,

(a) *$\alpha = 0$ and F is given by* (1) *with $\theta = \beta > 0$ and some $\mu > -\infty$;*

(b) *$-1 < \alpha < 0$ and F is given by* (2) *with $\theta = -(\alpha + 1)/\alpha > 0$ and $\mu = (1 - \beta)/\alpha < \nu = -\beta/\alpha$;*

(c) *$\alpha > 0$ and F is given by* (3) *with $\theta = (\alpha + 1)/\alpha > 0$, $\mu = (1 - \beta)/\alpha$ and $\gamma = -\beta/\alpha$.*

For $\alpha \leqslant -1$, there exists no F possessing property (5).

PROOF. Suppose (5) is true. First we note that the integration on the left-hand side of (5) is over the $(n-1)$ dimensional space $\{Y_k = y\}$. Fix $1 \leqslant k \leqslant n-1$. Let $s \in \{1, \cdots, n\}$ and σ be a subset of size k of $\{1, \cdots, n\}$ with $s \in \sigma$. We denote $\sigma_0 = \{1, \cdots, k\}$. Define

$$(6) \qquad A_{\sigma, s} = \big\{(X_1, \cdots, X_n) : X_j < y \quad \text{for } j \in \sigma \quad \text{and } j \neq s; X_s = y \quad \text{and}$$
$$X_j > y \quad \text{for } j \notin \sigma\big\}.$$

For each fixed s, there are $\binom{n-1}{k-1}$ disjoint $A_{\sigma, s}$'s and $\cup_{\sigma, s} A_{\sigma, s} = \{Y_k = y\}$ a.e. $P(\cdot | Y_k = y)$. And on $A_{\sigma, s}$ we can write $\Sigma_{j=k+1}^{n}(Y_j - Y_k) = \Sigma_{j \notin \sigma}(X_j - y)$. Therefore, using the fact that $P(A_{\sigma, s} | Y_k = y) = 1/\left(\binom{n-1}{k-1}n\right)$ is constant with respect to σ and s, we have

$$(7)\qquad \begin{aligned} E(Z_k|Y_k = y) &= \frac{1}{n-k} E\{\Sigma_{i=k+1}^{n}(Y_i - Y_k)|Y_k = y\} \\ &= \frac{1}{n-k}\Sigma_{\sigma,s} E\{\Sigma_{j\notin\sigma}(X_j - y)|A_{\sigma,s}\} P(A_{\sigma,s}|Y_k = y) \\ &= \frac{1}{n-k}\Sigma_{j=k+1}^{n} E\{(X_j - y)|A_{\sigma_0,k})\} \\ &= E(X_n - y|A_{\sigma_0,k}) \\ &= E(X_n - y|X_n > y). \end{aligned}$$

Consequently

$$(8)\qquad E(Z_k|Y_k = y) = \textstyle\int_y^\infty (w - y)\dfrac{F(dw)}{1 - F(y)} = \int_y^\infty \dfrac{wF(dw)}{1 - F(y)} - y.$$

From (8) it follows that the condition $E(Z_k|Y_k = y) = \alpha y + \beta$ is equivalent to

$$(9)\qquad [(\alpha + 1)y + \beta](1 - F(y)) = \textstyle\int_y^\infty wF(dw) \qquad \text{for almost all} \quad y(dF).$$

By following the proof of the lemma on page 268 in Ferguson [3], it can be shown that

$$(10)\qquad \int_y^\infty \frac{wF(dw)}{1 - F(y)}$$

is an almost sure (dF) strictly increasing function of y. Therefore we must have $\alpha > -1$. Also from (9), there exists a $\mu > -\infty$ such that $F(\mu) = 0$, otherwise letting $y \to -\infty$, the left-hand side of (9) tends to $-\infty$ while the right-hand side tends to the first moment of F which is finite by assumption. By writing the right-hand side of (9) as $\int_y^\infty wF(dw) = \int_y^\infty \int_0^w dt\, F(dw)$ and interchanging the order of integration (this operation is permissible by the Fubini's theorem because $|\int_y^\infty wF(dw)| \leqslant E|X| < \infty$), we obtain

$$(11)\qquad [\alpha y + \beta](1 - F(y)) = \textstyle\int_y^\infty (1 - F(t))\, dt \qquad \text{for almost all} \quad y(dF).$$

Denote $H(y) = \int_y^\infty (1 - F(t))\, dt$. Then H is a nonnegative differentiable function with $H'(y) = -(1 - F(y))$. Therefore, we can rewrite equation (10) as

$$(12)\qquad \frac{d}{dy}[\ln H(y)] = \frac{-1}{\alpha y + \beta} \qquad \text{for almost all} \quad y(dF).$$

It follows from (11) and (12) that $F(y)$ is strictly increasing whenever $F(y) > 0$. The solutions of the differential equation (12) are: if $\alpha = 0$, then $\beta = E(Z_k) > 0$. By letting $\theta = \beta$ we obtain (1). If $\alpha \neq 0$, we integrate both sides of (12) to get

$$(13)\qquad F(y) = 1 - (\alpha y + \beta)^{-(\alpha+1)/\alpha}$$

It can be easily checked that (13) equals to (2) or (3) depending on $-1 < \alpha < 0$ or $\alpha > 0$, with the parameters μ, ν and γ as defined in the theorem.

For the necessary condition. It can be verified by a straightforward calculation that with F as defined by (1), (2) and (3), the equation (9) holds, depending on the value of α. This completes the proof.

For completeness, we state the following theorem which follows immediately from Theorem 1.

THEOREM 2. *With the same assumptions as Theorem* 1, *then for some* $2 \leqslant k \leqslant n$

$$E(W_k | Y_k = y) = \alpha y + \beta, \qquad \text{a.e. } (dF) \tag{14}$$

where α *and* β *are constants, if and only if, the distribution function of* $-X$ *is as specified in Theorem* 1 *for* $\alpha > -1$. *For* $\alpha \leqslant -1$, *there exists no* F *possessing property* (14).

Acknowledgment. We are indebted to S. Kotz for bringing the reference [2] to our attention and both referees for their valuable criticisms.

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