The Annals of Statistics
1979. Vol. 7, No. 4, 869–881

CHARACTERIZATION OF A FAMILY OF DISTRIBUTIONS BY THE INDEPENDENCE OF SIZE AND SHAPE VARIABLES

BY IAN R. JAMES

C.S.I.R.O., Melbourne, Australia.

Let $X_1, \cdots, X_n$ be $n \geqslant 2$ positive random variables and $G(\mathbf{X})$ a positive variable satisfying $G(a\mathbf{X}) = aG(\mathbf{X})$ for all $a > 0$. Then G is a size variable, and $\mathbf{X}/G$ is a shape vector. If $X_1, \cdots, X_n$ are independent, then the independence of shape and the size variable $G(\mathbf{X})$ characterizes (i) the lognormal distribution if $G(\mathbf{X}) = \Pi X_i^{1/n}$, (ii) the generalized gamma distribution if $G(\mathbf{X}) = (\Sigma X_i^b)^{1/b}$, (iii) the Pareto distribution or its discrete analogue if $G(\mathbf{X}) = \min(\mathbf{X})$, and (iv) the power-function distribution or its discrete analogue if $G(\mathbf{X}) = \max(\mathbf{X})$. It is shown here that if $X_1, \cdots, X_n$ have piecewise continuous density functions and G is a continuous function then these four size variables are effectively the only ones for which such independence properties are attainable. A connection with the theory of sufficient statistics for a scale parameter is also considered.

1. Introduction, size and shape variables. Let $\mathbf{X} = (X_1, \cdots, X_n)$ be a vector of $n \geqslant 2$ positive random variables, and suppose $G(\mathbf{X})$ is a random variable with G a function from P^n into P, where P denotes the set of positive real numbers. Following Mosimann (1970, 1975a, 1975b) we define $G(\mathbf{X})$ to be a *size variable* if G satisfies the homogeneity condition $G(a\mathbf{x}) = aG(\mathbf{x})$ for all $a \in P$, $\mathbf{x} \in P^n$. The *shape vector* associated with G is defined by $\mathbf{Z}(\mathbf{X}) = \mathbf{X}/G(\mathbf{X})$, so that $\mathbf{Z}(a\mathbf{X}) = \mathbf{Z}(\mathbf{X})$ for all $a \in P$. The following important result is given by Mosimann (1970):

THEOREM 1.1. *Let $G_1(\mathbf{X})$ be a size variable and $\mathbf{Z}_1(\mathbf{X})$ a nondegenerate (at a point) shape vector, not necessarily associated with $G_1(\mathbf{X})$. If $G_1(\mathbf{X})$ is independent of $\mathbf{Z}_1(\mathbf{X})$, then*

(a) *any other shape vector $\mathbf{Z}_2(\mathbf{X})$ must be independent of $G_1(\mathbf{X})$; and*

(b) *no shape vector can be independent of any other size variable $G_2(\mathbf{X})$ unless $G_2(\mathbf{X})/G_1(\mathbf{X})$ is a degenerate random variable.*

In view of (a) we can talk unambiguously about the independence of size $G(\mathbf{X})$ and shape generally. In later sections it will often be convenient to choose shape vectors of the form $(X_2/X_1, \cdots, X_n/X_1)$ when studying size-shape independence.

Suppose now that $X_1, \cdots, X_n$ are mutually independent, and that size $G(\mathbf{X})$ is independent of shape. Then Mosimann (1970) shows that the lognormal and generalized gamma distributions are characterized by taking $G(\mathbf{X}) = (\Pi X_i)^{1/n}$ and $G(\mathbf{X}) = (\Sigma X_i^b)^{1/b}$ respectively, where b was taken to be positive, although it is only necessary that $b \neq 0$. In Theorem 3.1 we slightly generalize this result and include

Received August 1977; revised June 1978.

AMS 1970 *subject classifications.* Primary 62E10; secondary 62B05.

Key words and phrases. Size and shape, generalized gamma, lognormal, Pareto, power-function, sufficient statistics for scale parameters.

also the size variables $G(\mathbf{X}) = \max(X_1, \cdots, X_n)$, and $G(\mathbf{X}) = \min(X_1, \cdots, X_n)$, which respectively characterize the power function and Pareto distributions and their discrete analogues. In the main result of this paper, Theorem 3.2, we show that, if the X_i have piecewise continuous density functions, and G is continuous, then these four size variables are effectively the only possible ones for which $X_1, \cdots, X_n$ can be independent together with independent shape and size. Theorem 3.2 in a sense generalizes a result of Klebanov (1973), who considers only identically distributed variables, but then one needs only assume that the homogeneous function (size variable) is positive in a neighborhood of $\mathbf{x} = \mathbf{1}$, and that $G(\mathbf{X})$ is independent of $\{\min(X_j/X_1, j \neq 1), \max(X_j/X_1, j \neq 1)\}$. As a corollary to Theorem 3.2 we obtain Klebanov's result without assuming the differentiability conditions he imposes upon the size variable, thereby slightly weakening his restrictions.

The generalized gamma, lognormal, power-function and Pareto distributions were considered as a family by Ferguson (1962), who showed that the last three are limiting cases of the generalized gamma. In Section 2 we introduce his convenient notation and briefly review some properties of the family, before giving the main characterization results in Section 3. In the final section, alternative characterizations of the generalized gamma and lognormal distributions are obtained by showing that if size and shape are independent, then the size variable can be regarded as a sufficient statistic for a scale parameter.

From the point of view of practical allometric studies the characterizations given here are unlikely to be important, since measurement variables $X_1, \cdots, X_n$ will rarely be independent. Nevertheless, in searching for families of multivariate distributions suitable for the study of size-shape relationships, it is desirable that the richness of the families be sufficient to include the possibility that size and shape be independent together with independent $X_1, \cdots, X_n$, and our results indicate the limitations imposed. For size variables of the form $G(\mathbf{X}) = (\Pi X_i)^{1/n}$, the multivariate lognormal distribution is both rich and easily handled (Mosimann (1970, 1975b)), but the natural multivariate analogues of the generalized gamma, power-function and Pareto distributions are less obvious.

For a detailed discussion of the present size and shape definitions and the concept of *related sequences* of size variables, the reader is referred to the papers of Mosimann (1970, 1975a, 1975b).

2. Ferguson's family of distributions.

Let Y be a positive random variable with gamma distribution and let $X = Y^{1/\gamma}$, $\gamma \neq 0$. Then X has a generalized gamma distribution with density function

$$p(x) = |\gamma|/(\Gamma(\alpha)\beta^{\alpha})x^{\alpha\gamma-1}\exp(-x^{\gamma}/\beta), \quad x > 0 \tag{1}$$
$$= 0 \qquad x \leqslant 0$$

where $\alpha, \beta > 0$. The generalized gamma distribution given by (1) has been discussed by a number of people, including Ferguson (1962), Stacy (1962), Stacy and

Mihram (1965) and Hager and Bain (1970). Following Ferguson we define new positive parameters θ and σ^2 by

$$\gamma \log \theta = \log \alpha\beta \qquad (2)$$
$$\sigma^2\gamma^2 = \psi'(\alpha),$$

where ψ' is the trigamma function, and denote the distribution (1) by $L(\theta, \sigma^2, \gamma)$. The other members of Ferguson's family are now obtained as limiting cases of $L(\theta, \sigma^2, \gamma)$ as $\gamma \to 0, \pm\infty$, with θ, σ^2 fixed. Specifically, Ferguson (1962) shows that the distribution $L(\theta, \sigma^2, \gamma)$ tends to the lognormal distribution with density

$$(3) \qquad \begin{aligned} p(x) &= (2\pi\sigma^2)^{-\frac{1}{2}}x^{-1}\exp\left(-(\log(x/\theta))^2/2\sigma^2\right), & x > 0 \\ &= 0 & x \leqslant 0 \end{aligned}$$

as $\gamma \to 0$; to the Pareto distribution with density

$$(4) \qquad \begin{aligned} p(x) &= (\theta\sigma)^{-1}(x/\theta)^{-1-1/\sigma}, & x > \theta \\ &= 0 & x \leqslant \theta \end{aligned}$$

as $\gamma \to -\infty$; and to the power-function distribution with density

$$(5) \qquad \begin{aligned} p(x) &= (\theta\sigma)^{-1}(x/\theta)^{-1+1/\sigma}, & 0 < x < \theta \\ &= 0 & \text{otherwise} \end{aligned}$$

as $\gamma \to +\infty$. The distributions (3), (4) and (5) may thus be denoted by $L(\theta, \sigma^2, 0)$, $L(\theta, \sigma^2, -\infty)$ and $L(\theta, \sigma^2, \infty)$, respectively, and Ferguson's family by $L(\theta, \sigma^2, \gamma)$ where γ takes any values on the extended real line.

Now let $X_1, \cdots, X_n$ be independent random variables with $X_i \sim L(\theta_i, \sigma_i^2, \gamma)$, $i = 1, \cdots, n$ where γ is finite and nonzero, and $\sim$ denotes "distributed as". Then the size variable

$$A(\mathbf{X}; \boldsymbol{\beta}, \gamma) = (\Sigma_{i=1}^n X_i^\gamma/\beta_i^*)^{1/\gamma}$$

is independent of shape, where $\beta_i^* = \beta_i\Sigma_{j=1}^n 1/\beta_j$. (The purpose of the normalization will become apparent later when we consider limit size variables). Similarly, if each $X_i \sim L(\theta_i, \sigma_i^2, 0)$, then the size variable

$$M(\mathbf{X}; \boldsymbol{\sigma}^2) = \left(\Pi_{i=1}^n X_i^{1/\sigma_i^2}\right)^{1/\Sigma_{i=1}^n 1/\sigma_i^2}$$

is independent of shape; if each $X_i \sim L(\theta_i, \sigma_i^2, -\infty)$, then the size variable

$$\text{Min}(\mathbf{X}; \boldsymbol{\theta}) = \min(X_1/\theta_1, \cdots, X_n/\theta_n)$$

is independent of shape; and if each $X_i \sim L(\theta_i, \sigma_i^2, \infty)$, then the size variable

$$\text{Max}(\mathbf{X}; \boldsymbol{\theta}) = \max(X_1/\theta_1, \cdots, X_n/\theta_n)$$

is independent of shape.

Corresponding to the limit properties in the family $L(\theta, \sigma^2, \gamma)$, one can show, using the identities (2), that with $\boldsymbol{\theta}$ and $\boldsymbol{\sigma}^2$ fixed, $A(\mathbf{X}; \boldsymbol{\beta}, \gamma)$ converges to $M(\mathbf{X}; \boldsymbol{\sigma}^2)$ as $\gamma \to 0$, to $\text{Min}(\mathbf{X}; \boldsymbol{\theta})$ as $\gamma \to -\infty$, and to $\text{Max}(\mathbf{X}; \boldsymbol{\theta})$ as $\gamma \to \infty$. Consequently, if we denote by $G(\mathbf{X}; \boldsymbol{\theta}, \boldsymbol{\sigma}^2, \gamma)$ the size variables A, M, Min and Max, we see that G is continuous in γ on the extended real line, and we can summarize the above as follows: if each $X_i \sim L(\theta_i, \sigma_i^2, \gamma)$ then size $G(\mathbf{X}; \boldsymbol{\theta}, \boldsymbol{\sigma}^2, \gamma)$ is independent of shape. However, since M does not depend on $\boldsymbol{\theta}$, and Min and Max do not depend on $\boldsymbol{\sigma}^2$, it is convenient to retain their separate identities.

If we were to take logs of our variables and assume they were identically distributed, then the above size variables correspond to the means of order γ defined by Ferguson (1962). A more comprehensive discussion of the family of $L(\theta, \sigma^2, \gamma)$ distributions, including a summary of some important characterizations, is provided by Ferguson.

3. Characterizations by the independence of size and shape. The following result extends Theorem 4 of Mosimann (1970) to include the size variables $\operatorname{Min}(\mathbf{X}; \boldsymbol{\theta})$ and $\operatorname{Max}(\mathbf{X}; \boldsymbol{\theta})$.

THEOREM 3.1. *Let $X_1, \cdots, X_n$ be $n \geqslant 2$ nondegenerate, positive, independent random variables. Then all shape vectors $\mathbf{Z}(\mathbf{X})$ are independent of the size variable $G(\mathbf{X})$ equal to*

(a) $A(\mathbf{X}; \boldsymbol{\beta}, \gamma)$ *if and only if each $X_i \sim L(k\theta_i, \sigma_i^2, \gamma)$ for some constant $k > 0$, where θ_i and σ_i^2 are related to β_i through* (2);

(b) $M(\mathbf{X}; \boldsymbol{\sigma}^2)$ *if and only if each $X_i \sim L(\theta_i, k\sigma_i^2, 0)$ for some $\theta_i > 0$ and $k > 0$;*

(c) $\operatorname{Min}(\mathbf{X}; \boldsymbol{\theta})$ *if and only if each $X_i \sim L(k\theta_i, \sigma_i^2, -\infty)$ for some $k > 0$, $\sigma_i^2 > 0$, or each X_i is discrete with mass function*

$$\text{(6)} \qquad \begin{aligned} \Pr[X_i = k\theta_i \exp(cn)] &= (1 - p_i)p_i^n, \quad n = 0, 1, 2 \cdots \\ &= 0 \quad \text{otherwise} \end{aligned}$$

for some $c > 0$, $k > 0$, $0 < p_i < 1$;

(d) $\operatorname{Max}(\mathbf{X}; \boldsymbol{\theta})$ *if and only if each $X_i \sim L(k\theta_i, \sigma_i^2, \infty)$ for some $k > 0$, $\sigma_i^2 > 0$, or each X_i is discrete with mass function*

$$\text{(7)} \qquad \begin{aligned} \Pr[X_i = k\theta_i \exp(-cn)] &= (1 - p_i)p_i^n, \quad n = 0, 1, 2, \cdots \\ &= 0 \qquad \text{otherwise,} \end{aligned}$$

for some $c > 0$, $k > 0$, $0 < p_i < 1$.

PROOF. The "if" results are easily checked, and have already been mentioned for continuous variables in the previous section. We need only consider the "only if" parts. Parts (a) and (b) are trivial generalizations of Theorem 4 of Mosimann (1970), and are also mentioned in the two variate case by Ferguson (1962). Consider (c). For $n = 2$ the result follows immediately from Crawford (1966) by taking logs of our variables and taking shape to be X_2/X_1. For $n > 2$ we see from our assumptions and Theorem 1.1 that

$$(X_i/\theta_i)/\operatorname{Min}(\mathbf{X}; \boldsymbol{\theta}) \quad \text{and} \quad \operatorname{Min}(\mathbf{X}; \boldsymbol{\theta})$$

are independent, $i = 1, \cdots, n$. Now since $\operatorname{Min}(\mathbf{X}; \boldsymbol{\theta}) = \min(X_i/\theta_i, \min(X_j/\theta_j; j \neq i))$, it follows from the result for $n = 2$ that for each i, $X_i \sim L(k\theta_i, \sigma_i^2, -\infty)$ and $\min(X_j/\theta_j; j \neq i) \sim L(k, b_i^2, -\infty)$ for some k, σ_i^2, b_i^2 (that k does not depend on i follows by considering the distribution of $\text{Min}(\mathbf{X}; \boldsymbol{\theta})$), or X_i is discrete with mass function (6) and $\min(X_j/\theta_j; j \neq i)$ is discrete with mass function of the same form. It readily follows that the X_i's are either all continuous, or all discrete, which completes the proof of (c). Part (d) follows from (c) simply by noting that $\max(X_1/\theta_1, \cdots, X_n/\theta_n) = 1/\min(\theta_1/X_1, \cdots, \theta_n/X_n)$. □

As noted by Mosimann (1970), for $n > 2$ only the properties that $X_i/G(\mathbf{X})$ be independent of $G(\mathbf{X})$ for each $i = 1, \cdots, n$ are used in the proof of Theorem 3.1, rather than the stronger assumption of independence of size and shape.

The distributions given by (6) and (7) are the discrete analogues of the Pareto and power-function distributions, $\log X_i$ having a geometric distribution if X_i has distribution (6). Note that if $c \to 0$, and $p_i \to 1$ such that $(1 - p_i)/c \to 1/\sigma_i$, then the law (6) tends to $L(k\theta_i, \sigma_i^2, -\infty)$, while law (7) tends to $L(k\theta_i, \sigma_i^2, \infty)$ (cf. Johnson and Kotz (1969) page 123).

It is natural now to ask whether other size variables might characterize different distributions from those given in Theorem 3.1 under the same conditions. In the remainder of this section we show that if $X_1, \cdots, X_n$ have piecewise continuous densities and the size variable G is continuous, then A, M, Min and Max are the only size variables for which the independence properties can hold. We hope to consider the case of discrete variables separately.

THEOREM 3.2. *Let $X_1, \cdots, X_n$ be $n \geqslant 2$ positive, independent random variables with piecewise continuous density functions. Let $G(\mathbf{X})$ be a size variable with $G(\cdot)$ continuous as a function from P^n into P, and suppose that $G(\mathbf{X})$ is independent of shape. Then each $X_i \sim L(\theta_i, \sigma_i^2, \gamma)$ for some $\theta_i > 0$, $\sigma_i^2 > 0$, and γ on the extended real line.*

Klebanov (1973) proves a result very similar to Theorem 3.2 when $X_1, \cdots, X_n$ are assumed identically distributed, but then it is only necessary to assume the homogeneous function (size variable) is positive at $\mathbf{x} = \mathbf{1}$, and that $G(\mathbf{X})$ is independent of $\{\min(X_j/X_1, j \neq 1), \max(X_j/X_1, j \neq 1)\}$. His proof does not appear to carry over to the nonidentically distributed case. In proving Theorem 3.2 we shall obtain Klebanov's result as a corollary, under weaker conditions than he assumed.

Before proving Theorem 3.2 we require some preliminary results. The first, Lemma 3.1, is a generalization of Theorem 4.1 of Brown (1964) to the case of unequal functions r_1 and r_2. While the result is not needed in its full generality, it is stated completely analogously to Brown's theorem and can be proved by following the same steps with only obvious minor changes necessary to account for the different functions. Lemma 3.2 is a special case of Theorem 3 of Ferguson (1962) and is stated for reference.

In the following Lemma 3.1 and its corollary we denote by $I_1, \cdots, I_n$ intervals of the real line and by V a subset of the real line. For each $j = 1, \cdots, n$ and $\nu \in V$, $r_j(\cdot, \nu)$ and $u_j(\cdot, \nu)$ are continuous functions from I_j into the one-point compactification of the real line, and we assume that both $r_j(\cdot, \nu)$ and $u_j(\cdot, \nu)$ are equivalent to Lebesgue measure on I_j. Furthermore, we shall suppose that *at least two* of the u_j cannot be factored as $u_j(x, \nu) = f_j(x)d_j(\nu)$ for any f_j, d_j.

LEMMA 3.1. *Let $B \subset I_1$ be a set containing a limit point and ϕ a continuous function from $B \times I_2$ into the real line. Then, if, for some function w,*

$$r_1(x_1, \nu)r_2(x_2, \nu) = w(\phi(x_1, x_2), \nu) \tag{8}$$

for all $(x_1, x_2) \in B \times I_2$, $\nu \in V$, we have either $r_2(x_2, \nu) = d(\nu)$ for some function d, or there exist functions s_1 and s_2 and an interval $K \subset I_1$, with $K \cap B$ nonempty, such that

$$r_1(x_1, \nu) = s_1(\nu)(r_1(x_1, \nu_0))^{s_2(\nu)} \tag{9}$$

for all $x_1 \in K \cap B$, $\nu \in V$, where $\nu_0 \in V$ is fixed.

COROLLARY. *Let ϕ be a continuous function from $I_1 \times \cdots \times I_n$, $n \geqslant 2$, into the real line, such that, for some function w,*

$$\Pi_{j=1}^n u_j(x_j, \nu) = (\Pi_{j=1}^n u_j(x_j, \nu_0))w(\phi(x_1, \cdots, x_n), \nu), \tag{10}$$

$\nu \in V$, $(x_1, \cdots, x_n) \in I_1 \times \cdots \times I_n$. Then, for each $j = 1, \cdots, n$,

$$u_j(x, \nu) = C_j(\nu)H_j(x)\exp(Q_j(\nu)T_j(x)) \tag{11}$$

for some functions C_j, H_j, Q_j, T_j, $x \in I_j$, $\nu \in V$.

PROOF. Following the proof of Theorem 4.2 of Brown (1964), put

$$r_j(x_j, \nu) = u_j(x_j, \nu)/u_j(x_j, \nu_0),$$

so, from (10), we have

$$\Pi_{j=1}^n r_j(x_j, \nu) = w(\phi(x_1, \cdots, x_n), \nu). \tag{12}$$

For any $j = 1, \cdots, n$, there is an $i \neq j$ such that $u_i(x_i, \nu)$ cannot be factored as $f_i(x_i)d_i(\nu)$, so that for some $\nu \in V$, $r_i(x_i, \nu)$ is not constant for $x_i \in I_i$. Fix x_k; $k \neq i, j$ in (12), then, for any point $x \in I_j$, it follows from Lemma 3.1 that r_j has the form (9) on an interval containing x, and it is readily seen that it must have the form (9) on I_j. The form (11) then follows immediately for each $j = 1, \cdots, n$. □

A family $\{u_j(\cdot, \nu); \nu \in V\}$ satisfying (11) is a *one-parameter exponential family*.

Let $\{p(\cdot, \nu); \nu \in V\}$ be a family of probability density functions. Then the family is a *scale parameter family* if $p(x, \nu) = \nu g(x\nu)$ for some g.

LEMMA 3.2. (Ferguson (1962)). *A scale parameter family of densities $\{p(\cdot, \nu);$ $\nu \in V\}$, with $p(x, \nu) = 0$ for $x \leqslant 0$, $\nu \in V$, is also a one-parameter exponential family of distributions in ν if and only if for fixed $\nu = \nu_0$ say, $p(\cdot, \nu_0)$ is an $L(\theta, \sigma^2, \gamma)$ distribution for finite γ.*

PROOF OF THEOREM 3.2. We begin by using initial steps similar to those of Klebanov (1973). Let $\mathbf{s} = (s_2, \cdots, s_n)$ and $\Delta\mathbf{s} = (\Delta s_2, \cdots, \Delta s_n)$ where the s_i are positive and the Δs_i are positive increments. Then for $t > 0$, $\Delta t > 0$ we have

$$(13)\quad \begin{aligned}&\Pr[t \leqslant X_1 G(1, X_2/X_1, \cdots, X_n/X_1) \\ &\qquad \leqslant t + \Delta t, s_i \leqslant X_i/X_1 \leqslant s_i + \Delta s_i, i = 2, \cdots, n] \\ &= \int_{s_n}^{s_n+\Delta s_n} dy_n \cdots \int_{s_2}^{s_2+\Delta s_2} dy_2 \int_{t/G^*}^{(t+\Delta t)/G^*} p_1(y_1)p_2(y_1y_2)\cdots p_n(y_1y_n)y_1^{n-1}\,dy_1,\end{aligned}$$

where $G^* = G(1, y_2, \cdots, y_n)$ and p_j is a piecewise continuous density function of $X_j, j = 1, \cdots, n$. Dividing (13) by $\Delta t \Delta s_2 \cdots \Delta s_n$ and letting $\Delta \mathbf{s} \to \mathbf{0}$, then $\Delta t \to 0$, we obtain using the independence of $G(\mathbf{X})$ and $\{X_j/X_1; j \neq 1\}$,

$$(14)\qquad p_1(tG_1(\mathbf{s}))\Pi_{j=2}^n p_j(ts_jG_1(\mathbf{s})) = f(t)h^*(\mathbf{s})$$

for some functions f and h^*, where $G_1(\mathbf{s}) = 1/G(1, s_2, \cdots, s_n)$. Now change variables to $x = tG_1(\mathbf{s}), y_j = ts_jG_1(\mathbf{s})$ and choose $\psi_2, \cdots, \psi_n$ such that $q(y) = \Pi_{j=2}^n p_j(\psi_j y)$ is piecewise continuous and positive on some interval. Let $\phi(x, y) = G(x, \psi_2 y, \cdots, \psi_n y)$. Then we have, from (14), on putting $y_j = \psi_j y$ for each j,

$$(15)\qquad p_1(x)q(y) = f(\phi(x, y))h(y/x), \quad x > 0, \quad y > 0.$$

Note that $\phi(x, y) = x\phi(1, y/x) = y\phi(x/y, 1)$. We shall break the rest of the proof into a number of steps so that they can be referenced.

(i) Suppose that $p_1(x) = 0$ for some $x \in (0, \infty)$. Then, using piecewise continuity of p_1, we may assume that $x_0 \in (0, \infty)$ is a point such that $p_1(x_0) = 0$ but $p_1(x) > 0$ for $x \in K$, where K is one of the intervals $(x_0, x_0 + \delta)$ or $(x_0 - \delta, x_0)$ for some $\delta > 0$. Let L be an open interval on which $q(y) > 0$. Then for $x \in K, y \in L$, both $f(\phi(x, y))$ and $h(y/x)$ are positive, and so for $y \in L$, $h(y/x_0) > 0$. Thus, from (15) we have, for $y \in L$

$$(16)\qquad \begin{aligned}&f(x_0\phi(1, y/x_0)) = 0\\ &f(x\phi(1, y/x)) > 0 \qquad \text{for} \quad x \in K.\end{aligned}$$

Suppose that $\phi(1, y/x_0)$ is not constant for $y \in L$. Then, since ϕ is continuous, there exist points $y_1, y_2 \in L$ and $x' \in K$ such that $x_0\phi(1, y_1/x_0) = x'\phi(1, y_2/x')$. But then substitution into (16) provides a contradiction. Thus we must have $\phi(1, y/x_0)$ constant for $y \in L$, and, using a similar argument, we see that either $q(y) > 0$ for all $y \in (0, \infty)$ or $\phi(z, 1)$ is constant on an interval.

(ii) Let intervals K and L be as defined in (i). Let y^* be any point in L, and suppose that there exist points $y_0, y_1 \in (0, \infty)$ such that y_0 is the first zero of q to the left of y^*, and y_1 the first zero to the right of y^*. We may redefine $L = (y_0, y_1)$. Now from (i) we have $\phi(1, y/x_0) = c_1$ (constant) for $y \in L$, and, by similar arguments, we have both $\phi(x/y_0, 1) = c_2$ and $\phi(x/y_1, 1) = c_3$ for $x \in K$. Thus for all $x \in K, y \in L$, $\phi(x_0, y) = c_1x_0$, $\phi(x, y_0) = c_2y_0$, and $\phi(x, y_1) = c_3y_1$. Since ϕ is continuous, we then have

$$\lim_{y \to y_0} \phi(x_0, y) = \phi(x_0, y_0) = c_1x_0$$

and

$$\lim_{x \to x_0} \phi(x, y_0) = \phi(x_0, y_0) = c_2y_0,$$

so that $c_1x_0 = c_2y_0$, and we find similarly that $c_1x_0 = c_3y_1$. Thus, for all $x_1, x_2 \in K$, we have $\phi(x_1, y_0) = \phi(x_2, y_1) = c$ (constant), and it follows that for δ small enough and $x \in K$,

$$\phi(x, y) = cy/y_0 \qquad \text{for} \quad y_0 < y < y_0 + \delta,$$

and

$$\phi(x, y) = cy/y_1 \qquad \text{for} \quad y_1 - \delta < y < y_1.$$

So, for any $x \in K$, $\phi(x, \cdot)$ is an increasing function at both y_0^+ and y_1^- and takes the same value at these points. It follows, by continuity of ϕ, that there must be a point $y' \in L = (y_0, y_1)$ such that $\phi(x, y') = \phi(x, y_0) = \phi(x, y_1)$. But then we must have $f(\phi(x, y'))$ simultaneously zero and positive, so that the points y_0 and y_1 as defined cannot exist. We conclude, therefore, that if $p_1(x) = 0$ for some $x \in (0, \infty)$, then either $q(y) > 0$ for all $y \in (0, \infty)$, or $q(y) > 0$ for $y \in (0, y_0)$, $q(y) = 0$ for $y \in [y_0, \infty)$, or $q(y) > 0$ for $y \in (y_0, \infty)$, $q(y) = 0$ for $y \in (0, y_0]$, for some $y_0 \in (0, \infty)$. An analogous conclusion is reached for p_1 if q takes a zero value.

(iii) Again suppose that $p_1(x) = 0$ for some $x \in (0, \infty)$. Then from (i) there exists an interval (α, β), say, such that $\phi(1, z) = c$ and $h(z) > 0$ for $z \in (\alpha, \beta)$. Now from (15) we have

$$p_1(x)q(xz) = f(x\phi(1, z))h(z), \qquad x, z \in (0, \infty) \tag{17}$$

and so, for $z \in (\alpha, \beta)$ and x in some interval on which $f(cx) > 0$,

$$q(xz) = h(z)f(xc)/p_1(x). \tag{18}$$

Hence for such x, z,

$$\begin{aligned} q(xz) &= ab(xz)^d \\ f(cx)/p_1(x) &= ax^d \\ h(z) &= bz^d \end{aligned} \tag{19}$$

for some constants a, b, d. It can be seen from (i) that if $q(y) > 0$ for all $y \in (0, \infty)$, then the interval (α, β) can be taken as $(0, \infty)$, so from (19), $q(y) = ky^d$ for all $y \in (0, \infty)$. But recalling that $q(y) = \Pi_{j=2}^n p_j(\psi_j y)$, where the p_j's are density functions, it follows from Hölder's inequality that $q(y)^{1/(n-1)}$ must be integrable over $(0, \infty)$, so we have a contradiction. Hence $q(y) = 0$ for some $y \in (0, \infty)$ and from (i), $\phi(z, 1) = r$ (constant) for $z \in (\tau, \delta)$ say. Then analogously to (19) we obtain for $z \in (\tau, \delta)$ and y in any interval on which $f(ry) > 0$,

$$\begin{aligned} p_1(yz) &= tu(yz)^v \\ f(ry)/q(y) &= ty^v \\ h(z) &= uz^v, \qquad \text{for some constants } t, u, v. \end{aligned} \tag{20}$$

Furthermore, from (ii) it follows that $p_1(x) > 0$ for $x \in I_1$, $p_1(x) = 0$ for $x \notin I_1$ where $I_1 = (0, x_0)$ or (x_0, ∞), $x_0 > 0$, and $q(y) > 0$ for $y \in I_2$, $q(y) = 0$ for $y \notin I_2$, where $I_2 = (0, y_0)$ or (y_0, ∞), $y_0 > 0$. It is then easily checked from (17) that $p_1(x) = 0$ if and only if $f(cx) = 0$, and $q(y) = 0$ if and only if $f(ry) = 0$. From (19) and (20) it then follows that

$$\begin{aligned} p_1(x) &= c_1 x^v, && x \in I_1 \\ &= 0, && x \notin I_1 \\ q(y) &= c_2 y^d, && y \in I_2 \\ &= 0, && y \notin I_2, \end{aligned}$$

where $I_1 = (0, x_0)$, $I_2 = (0, y_0)$, or $I_1 = (x_0, \infty)$, $I_2 = (y_0, \infty)$. By an analogous argument the same conclusion is reached if we begin by assuming $q(y) = 0$ for some $y \in (0, \infty)$.

(iv) The remaining case to be considered is when both $p_1(x) > 0$ and $q(y) > 0$ for all $x, y \in (0, \infty)$. Suppose also that p_1 is discontinuous at $x_0 \in (0, \infty)$, and let (α, β) be any interval for which $q(x_0 z)$ is continuous, $z \in (\alpha, \beta)$. From (17) it follows that $f(\cdot)$ is discontinuous at the point $x_0\phi(1, z)$ for each $z \in (\alpha, \beta)$, so that if $\phi(1, z)$ is not constant for $z \in (\alpha, \beta)$, the continuity of ϕ ensures that f is discontinuous everywhere on some interval. But by fixing z in (17) we see that f must be piecewise continuous, therefore providing a contradiction. Hence $p_1(x)$ is continuous for $x \in (0, \infty)$ or $\phi(1, z) = c$ (constant) for $z \in (\alpha, \beta)$. But in the latter case we obtain (18) and then (19) for all $x \in (0, \infty)$, and again, by Hölder's inequality, this is not possible. Thus $p_1(x)$ is continuous. Similarly we see that $q(y)$ is continuous for $y \in (0, \infty)$. Then from (15) we have for $\mu > 0$,

$$p_1(\mu x) q(\mu y) = f(\mu\phi(x, y)) h(y/x), \qquad x, y \in (0, \infty)$$

so that

$$p_1(\mu x) q(\mu y) = p_1(\mu_0 x) q(\mu_0 y) f(\mu\phi(x, y))/f(\mu_0\phi(x, y)).$$

It is easily checked that the conditions of the corollary to Lemma 3.1 are satisfied by p_1, q and ϕ, so that the families $\{\mu p_1(\mu x); \mu > 0\}$ and $\{\mu q(\mu y); \mu > 0\}$ are one-parameter exponential families in the scale parameter μ. Hence from Lemma 3.2, p_1 is an $L(\theta_1, \sigma_1^2, \gamma_1)$ distribution for some positive θ_1, σ_1^2, and finite γ_1.

(v) By repeating all of the above arguments with the subscripts 1 and i interchanged $i = 2, \cdots, n$, we see from (iii) and (iv) that either;

(a) $X_i \sim L(\theta_i, \sigma_i^2, -\infty)$, $i = 1, \cdots, n$

or

(b) $X_i \sim L(\theta_i, \sigma_i^2, \infty)$, $i = 1, \cdots, n$

or

(c) $X_i \sim L(\theta_i, \sigma_i^2, \gamma_i)$, $i = 1, \cdots, n$,

γ_i finite, for some θ_i, $\sigma_i^2 > 0$. It remains to show that the γ_i's in (c) are all equal. Suppose that $m \geqslant 1$ of the X_i's are lognormals. Without loss of generality we may assume that $X_i \sim L(\theta_i, \sigma_i^2, 0)$, $i = 1, \cdots, m$, and $X_i \sim L(\theta_i, \sigma_i^2, \gamma_i)$, $\gamma_i \neq 0$ for $i = m + 1, \cdots, n$. Then from (1), (3) and (15), with $\psi_2 = \cdots = \psi_n = 1$ we find that

$$\begin{aligned} \rho(\phi(x, y)) &= \frac{\partial}{\partial\mu} \log f(\mu\phi(x, y))|_{\mu=1} \\ &= k \text{ (constant)} - \log x/\sigma_1^2 - \log y \Sigma_{j=2}^m 1/\sigma_j^2 \\ &\quad - \Sigma_{j=m+1}^n \gamma_j y^{\gamma_j}/\beta_j. \end{aligned}$$

Similarly

$$\eta(\phi(x, y)) = (\partial/\partial\mu)\rho(\mu\phi(x, y))|_{\mu=1}$$
$$= r\text{ (constant)} - \Sigma_{j=m+1}^{n}\gamma_j^2 y^{\gamma_j}/\beta_j.$$

Since the right-side is a function only of y, while the left-side is a function of $y\phi(x/y, 1)$, and $\phi(z, 1)$ is not constant on any interval, by (iv), it follows that each side must be constant, and thus $m = n$. Hence the X_i's are all lognormals or all generalized gammas. In the latter case we find that

$$\rho(\phi(x, y)) = k\text{ (constant)} - \gamma_1 x^{\gamma_1}/\beta_1 - \Sigma_{j=2}^{n}\gamma_j y^{\gamma_j}/\beta_j$$

and

$$\eta(\phi(x, y)) = -\gamma_1^2 x^{\gamma_1}/\beta_1 - \Sigma_{j=2}^{n}\gamma_j^2 y^{\gamma_j}/\beta_j.$$

Thus $\gamma_1\rho(\phi(x, y)) - \eta(\phi(x, y)) = k\gamma_1 - \Sigma_{j=2}^{n}\gamma_j(\gamma_1 - \gamma_j)y^{\gamma_j}/\beta_j$, and again each side must be constant, from which we see that $\gamma_j = \gamma_1$ for $j = 2, \cdots, n$. This now completes the proof of Theorem 3.2. □

COROLLARY. (Klebanov (1973)). *Let $X_1, \cdots, X_n$ be positive, independent, identically distributed random variables with piecewise continuous density function $p(x)$. Let H be a continuous, homogeneous function from P^n into the real line, and suppose that $H(1, \cdots, 1) > 0$. If $H(X_1, \cdots, X_n)$ is independent of the pair $\{Y = \min(X_j/X_1, j = 2, \cdots, n), Z = \max(X_j/X_1, j = 2, \cdots, n)\}$, then each $X_i \sim L(\theta, \sigma^2, \gamma)$ for some $\theta, \sigma^2 > 0$ and γ on the extended real line.*

PROOF. If $p(x) = 0$ for some $x \in (0, \infty)$, then from Corollaries 2 and 3 of Klebanov (1973) it follows that each $X_i \sim L(\theta, \sigma^2, -\infty)$ or each $X_i \sim L(\theta, \sigma^2, \infty)$. Further, (15) holds for all x, y such that y/x is sufficiently close to 1, where $\phi(x, y) = H(x, y, \cdots, y)$. By following the argument in (iv) of the proof of Theorem 3.2, we see that if $p(x) > 0$ for all $x \in (0, \infty)$, then it must be continuous. It will therefore suffice to show that $\phi(1, z) > 0$ for all z, since then all of our arguments in (iv) are valid. Now from equation (3) in Klebanov (1973), we have

$$(21)\quad p(t/\phi(1, z))q(tz/\phi(1, z))/\{p(\alpha/\phi(1, z))q(\alpha z/\phi(1, z))\}$$
$$= dp^n(ct), \qquad t \in (0, \infty), \quad z \text{ near } 1,$$

where α, d, c are positive constants. Suppose that $z_0 \neq 0, \infty$ is the first zero of $\phi(1, z)$ to the right or left of $z = 1$. Then, using (21) and taking the limit as $z \to z_0$ in the direction away from $z = 1$, we obtain

$$p^n(ct_1t_2) = dp^n(ct_1)p^n(ct_2),$$

(cf. Feller (1966) page 268), from which we see $p^n(t) = c^* t^{d^*}$ for some $c^*, d^*, t \in (0, \infty)$. But then p is not integrable on $(0, \infty)$. Hence no such z_0 exists and $\phi(1, z) > 0$ for all z, proving the corollary. □

We may note here that Theorem 3.2 remains valid if G is replaced by any homogeneous, positive function H of degree $k \neq 0$, for then $H^{1/k}$ is a size variable and is independent of shape.

4. Size, shape and sufficient statistics for scale parameters. In Theorem 3.2 continuity assumptions were made about both the size variable and the density functions. Some different regularity conditions are assumed in this section; in particular in Theorem 4.1 we assume nothing about G other than it be positive and measurable, but we place stronger conditions on the densities. In so doing we characterize the generalized gamma and lognormal distributions. Other possible sets of regularity conditions will be mentioned later.

LEMMA 4.1. *Let $X_1, \cdots, X_n$ be positive random variables, not necessarily independent nor identically distributed, and suppose that size $G(\mathbf{X})$ is independent of shape. Let $Y_j = X_j/\mu$, $j = 1, \cdots, n$, where $\mu > 0$. Then $G(\mathbf{Y})$ is sufficient for the scale parameter μ.*

PROOF. Consider the conditional characteristic function

$$(22)\quad E\big[\exp(i\Sigma_{j=1}^n t_j \log Y_j)|G(\mathbf{Y}) = g\big]$$
$$= E\big[\exp\big(i\{\Sigma_{j=2}^n t_j(\log Y_j - \log Y_1) + (t_1 - \Sigma_{j=2}^n t_j)\log Y_1\}\big)|G(\mathbf{Y}) = g\big].$$

Since $Y_1 = G(\mathbf{Y})/G(1, Y_2/Y_1, \cdots, Y_n/Y_1)$ and $G(\mathbf{Y})$ is independent of $(Y_2/Y_1, \cdots, Y_n/Y_1)$, (22) equals

$$\exp\big(i(t_1 - \Sigma_{j=2}^n t_j)\log g\big)E\big[\exp\big(i\{\Sigma_{j=2}^n t_j \log(X_j/X_1)$$
$$- (t_1 - \Sigma_{j=2}^n t_j)\log G(1, X_2/X_1, \cdots, X_n/X_1)\}\big)\big].$$

Thus (22) does not depend on μ, which proves the sufficiency of $G(\mathbf{Y})$ for μ. (Rao (1973) page 130). $\square$

THEOREM 4.1. *Let $X_1, \cdots, X_n$ be positive, independent random variables with densities p_j, $j = 1, \cdots, n$. Suppose that each p_j is positive and continuously differentiable on $(0, \infty)$. Then, if size $G(\mathbf{X})$ is independent of shape, $X_j \sim L(\theta_j, \sigma_j^2, \gamma)$, $j = 1, \cdots, n$ for some θ_j, $\sigma_j^2 > 0$ and γ finite.*

PROOF. Let $Y_j = X_j/\mu$, $j = 1, \cdots, n$, $\mu > 0$, so, by Lemma 4.1, $G(\mathbf{Y})$ is sufficient for μ. By Corollary 3 to Theorem 3 of Zhuravlev (1966) we see that for each j, $\{\mu p_j(\mu y); \mu > 0\}$ is a one-parameter exponential family of distributions in the scale parameter μ, and the proof is completed using Lemma 3.2 and (v) in the proof of Theorem 3.2. $\square$

Note that Theorem 4.1 shows there exist multivariate distributions for which no size variable is independent of shape; for example, X_1, X_2 independent with X_1 gamma and X_2 lognormal.

Zhuravlev (1966) assumes slightly less than continuous differentiability of each p_j, but his results generalize those of Dynkin (1951), and Brown (1964) has pointed out that the stronger conditions are necessary.

The only regularity conditions imposed in Theorem 4.1 are those used to conclude, from the existence of a single sufficient statistic, that the densities belong to one-parameter exponential families. Many such results under different conditions are available when $X_1, \cdots, X_n$ are identically distributed; e.g. Dynkin (1951), Brown (1964), Denny (1970) and Hipp (1974). Hipp, for example, requires the probability measure to be equivalent to Lebesgue measure and the sufficient statistic to be a locally Lipschitz function, while Brown, Theorem 2.1, places some rather nonintuitive restrictions on the sufficient statistic. The incorporation of such results into a modification of Theorem 4.1 is obvious. Both Brown (1964) and Denny (1969) show that some regularity conditions of the type mentioned here are necessary to deduce exponential families from the existence of a sufficient statistic.

For nonidentically distributed variables $X_1, \cdots, X_n$, the results connecting sufficient statistics and exponential families are fewer, and, apart from Zhuravlev (1966), we mention Barankin and Maitra (1963), who place harsher differentiability conditions on the density functions. It seems likely that some of the results mentioned above for identically distributed variables could be generalized to the nonidentically distributed case (cf. Lemma 3.1 and its corollary above) and, if so, the appropriate changes in regularity conditions in Theorem 4.1 can be made.

If $X_1, \cdots, X_n$ are assumed to be identically distributed then Theorem 4.1 can be extended to include the distributions $L(\theta, \sigma^2, \pm\infty)$ by assuming $p(x) > 0$ for $x \in I$, $p(x) = 0$ for $x \notin I$, where I is an interval. This follows from Theorem 9 of Dynkin (1951). Again it seems likely that a generalization of this result to the case of nonidentically distributed variables, along the lines of Zhuravlev (1966), may be possible. However, we shall not pursue the matter here.

Finally, we note that Ferguson (1962) conjectured that the existence of a complete sufficient statistic for a scale parameter for any $n \geqslant 2$, when $X_1, \cdots, X_n$ are identically distributed, positive and dominated by a σ-finite measure, would characterize the distributions $L(\theta, \sigma^2, \gamma)$ for γ on the extended real line. The truth or otherwise of this, without further assumptions, appears still to be unknown.

Acknowledgments. I am grateful to Professor E. J. Williams for bringing the paper by Ferguson to my attention, thereby stimulating the work reported here, and to Dr. J. E. Mosimann and Professor J. N. Darroch for many helpful discussions.

REFERENCES

[1] Barankin, E. W. and Maitra, A. P. (1963). Generalization of the Fisher–Darmois–Koopman–Pitman theorem on sufficient statistics. *Sankhyā. Ser. A* **25** 217–244.

[2] Brown, L. D. (1964). Sufficient statistics in the case of independent random variables. *Ann. Math. Statist.* **35** 1456–1474.

[3] Crawford, G. B. (1966). Characterization of geometric and exponential distributions. *Ann. Math. Statist.* **37** 1790–1795.

[4] Denny, J. L. (1969). Note on a theorem of Dynkin on the dimension of sufficient statistics. *Ann. Math. Statist.* **40** 1474–1476.

[5] Denny, J. L. (1970). Cauchy's equation and sufficient statistics on arcwise connected spaces. *Ann. Math. Statist.* **41** 401–411.

[6] DYNKIN, E. B. (1951). Necessary and sufficient statistics for a family of probability distributions. *Select. Transl. Math. Statist. Probability* **1** 23–41.

[7] FELLER, W. (1966). *An Introduction to Probability Theory and its Applications, Vol. II*. Wiley, New York.

[8] FERGUSON, T. S. (1962). Location and scale parameters in exponential families of distributions. *Ann. Math. Statist.* **33** 986–1001. Correction *Ann. Math. Statist.* **34** 1603.

[9] HAGER, W. H. and BAIN, L. J. (1970). Inferential procedures for the generalized gamma distribution. *J. Amer. Statist. Assoc.* **65** 1601–1609.

[10] HIPP, C. (1974). Sufficient statistics and exponential families. *Ann. Statist.* **2** 1283–1292.

[11] JOHNSON, N. L. and KOTZ, S. (1969). *Distributions in Statistics: Discrete Distributions*. Houghton Mifflin, Boston.

[12] KLEBANOV, L. B. (1973). On the characterization of a family of distributions by the property of independence of statistics. *Theor. Probability Appl.* **18** 608–611.

[13] MOSIMANN, J. E. (1970). Size allometry: size and shape variables with characterizations of the lognormal and generalized gamma distributions. *J. Amer. Statist. Assoc.* **65** 930–945.

[14] MOSIMANN, J. E. (1975a). Statistical problems of size and shape. I. Biological applications and basic theorems. In *Statistical Distributions in Scientific Work*, Vol. 2. (G. P. Patil, S. Kotz, J. K. Ord, eds.), 187–217, D. Reidel Publishing Company, Dordrecht.

[15] MOSIMANN, J. E. (1975b). Statistical problems of size and shape. II. Characterizations of the lognormal, gamma and Dirichlet distributions. In *Statistical Distributions in Scientific Work*, Vol. 2. (G. P. Patil, S. Kotz, J. K. Ord, eds.), 219–239, D. Reidel Publishing Company, Dordrecht.

[16] RAO, C. R. (1973). *Linear Statistical Inference and its Applications, 2nd ed.* Wiley, New York.

[17] STACY, E. W. (1962). A generalization of the gamma distribution. *Ann. Math. Statist.* **33** 1187–1192.

[18] STACY, E. W. and MIHRAM, G. A. (1965). Parameter estimation for a generalized gamma distribution. *Technometrics* **7** 349–358.

[19] ZHURAVLEV, O. G. (1966). Minimal sufficient statistics for a sequence of independent random variables. *Theor. Probability Appl.* **11** 282–291.

DEPARTMENT OF MATHEMATICS
THE UNIVERSITY OF WESTERN AUSTRALIA
NEDLANDS, WESTERN AUSTRALIA 6009
AUSTRALIA