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CONTINUED FRACTIONS FOR THE INCOMPLETE BETA FUNCTION: ADDITIONS AND CORRECTIONS[1]

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This article corrects some additional errors in Aroian's article and preseents some new results on two associated continued fraction representations of the incomplete beta function.

1. Introduction. Aroian (1941) presented a new continued fraction (cf) for computing the incomplete beta function,

$$I_x(p, q) = B(p, q)^{-1} \int_0^x u^{p-1}(1 - u)^{q-1}\, du,$$

and compared it to the existing continued fraction of Mueller. The purpose of this article is threefold: (1) to point out some uncorrected errors in the paper by Aroian (1941); (2) to present the associated cf derived from Mueller's corresponding cf; and (3) to show how a rearrangement of the associated cf's of Aroian and Mueller can have a dramatic effect on their numerical stability, and computational usefulness.

2. Definitions and terminology. Aroian (1941) gives an algorithm for converting the infinite power series, $1 + \sum_{n=1}^{\infty} d_n x^n$, into what is often referred to as the corresponding cf. A much simpler algorithm produces the same corresponding cf when the series is a special form of hypergeometric series (Slater (1966), page 15). Both series used by Mueller and Aroian have this special hypergeometric form. The term, nth convergent, is used to refer to the truncated infinite cf,

$$\frac{1}{/1 +}\frac{c_1 x}{/1 +}\frac{c_2 x}{/1 +}\frac{c_3 x}{/1 +} \cdots \frac{c_n x}{/1 +}. \tag{2.1}$$

If the nth convergent of a corresponding cf is, itself, expanded in an infinite power series, this series will "correspond" term for term with the original power series up to the term containing x^n (Henrici (1977), page 518; Khovanskii (1963), pages 26–27). The analytic nature of corresponding cf's is not completely understood. In many cases, including the cf's of Aroian and Mueller, the corresponding cf converges much more rapidly than the original series (Henrici (1977)).

The convergents of the associated cf are just the odd convergents of the

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corresponding cf. For the inifinite form of (2.1) the associated cf is,

$$1 - \frac{c_1 x}{/1 + (c_1 + c_2)x} - \frac{c_2 c_3 x^2}{/1 + (c_3 + c_4)x} - \frac{c_4 c_5 x^2}{/1 + (c_5 + c_6)x} - \cdots$$

(Wall (1948), page 21; Perron (1913), page 332).

3. Corrections. Aroian (1959) corrected some of the errors in his (1941) paper, but evidently missed the following: page 220 equations (2.4):

$$l_1 = -(p + q + 1)/(p + 2)$$

$$k_{s+1} = [s(q - s)(p + s)(p + q + s)]/[(p + 2s - 1)(p + 2s)^2(p + 2s + 1)]$$

$$l_{s+1} = [(s + 1)(q - (s + 1))]/[(p + 2s + 1)(p + 2s + 2)]$$
$$-[(p + s)(p + q + s)]/[(p + 2s)(p + 2s + 1)].$$

4. Mueller's associated continued fraction. The associated cf derived from Mueller's corresponding cf is:

$$I_x(p, q) = [\Gamma(p + q)]/[\Gamma(p + 1)\Gamma(q)]x^p(1 - x)^{q-1}$$

$$\left\{1 + \frac{k_1 w}{/1 + l_1 w} + \frac{k_2 w^2}{/1 + l_2 w} + \frac{k_3 w^2}{/1 + l_3 w} + \cdots \right\}, \text{ where } w = x/(1 - x),$$

$$k_1 = (q - 1)/(p + 1),$$
$$l_1 = -(q - 2)/(p + 2),$$
$$k_s = [(s - 1)(p + q + s - 2)(p + s - 1)(q - s)]/$$
$$[(p + 2s - 3)(p + 2s - 2)^2(p + 2s - 1)], \text{ and}$$
$$l_s = [(p + s - 1)(s - q)]/[(p + 2s - 2)(p + 2s - 1)]$$
$$+[s(p + q + s - 1)]/[(p + 2s - 1)(p + 2s)].$$

This representation has not appeared elsewhere in the literature.

5. Improvement of numerical stability by rearrangement of Aroian's and Mueller's associated continued fractions. The motivation for the following results comes from the desire to compute $I_x(p, q)$ over the full range of all parameters. The present problem was uncovered in checking a program that evaluates the non-central t distribution in terms of $I_x(p, q)$, when $p \gg q$ and $x \simeq 1$, i.e., $p = .5 \times 10^{20}$, $q = .5 \times 10^4$ and $(1 - x) = .1 \times 10^{-16}$.

Existing series representations and recursive techniques are of little use in the significant digit computation of the above example. In most cases they are slowly converging. The corresponding and associated cf's are known to dramatically accelerate the convergence of their original series. However, in computing the above forms by forward recursion—see equation 5.1 Aroian (1941)—all digits are lost to subtraction before convergence occurs. Thus, for the example, the cf's are numerically unstable.

Representations that are numerically unstable can sometimes be made stable by performing analytic subtraction. The only forms suitable for this approach were the partial denominators, $1 + l_s x$, of the associated cf's. An analytic subtraction was obtained by writing the partial denominators in terms of the variable $F = qx/p(1 - x)$, expanding and simplifying. (Note that the relationship between x and F is the same as the relationship between the beta and the usual F random variable.)

The partial denominator terms in Aroian's associated cf become:

$$1 + l_{s+1}x = 1 + l_{s+1}[pF/(pF + q)].$$

After expansion and simplification, we have:

$$1 + l_{s+1}x = [2(2q + pF)s^2 + 2(p + 1)(2q + pF)s + p(((1 - F)p + 2)q + pF)] / [(pF + q)(p + 2s)(p + 2s + 2)].$$

Similarly, the Mueller associated cf has partial denominator terms $1 + l_s x/(1 - x) = 1 + l_s pF/q$, which on simplification become:

$$[2pFs^2 + 4qs(s - 1) + 2p(p - 1)Fs + 2pq(2s - 1) + p^2q(1 - F)] / [q(p + 2s - 2)(p + 2s)].$$

In both cases many common terms have been eliminated. These simplifications, if performed by hand, are extremely cumbersome. The results here were easily obtained in one step using the powerful algebraic manipulation system of M.I.T.'s MACSYMA. With the elimination of subtraction it was found that the associated cf's give rather astounding results. For the above example, 25 significant digits were obtained in only 21 convergents.

6. Comments on the usefulness of associated continued fractions for computing $I_x(p, q)$. A very robust (with respect to range of argument and accuracy) algorithm for significant digit computation of $I_x(p, q)$ has been constructed by the authors. The algorithm only uses the Mueller associated cf. The troublesome j-shaped forms of the density are handled by a second application of analytic subtraction to the cumulative distribution function itself.

Mueller's associated cf was chosen over Aroian's because of the availability and simplicity of error bounds. The corresponding cf of Mueller is bounded above and below by the even and odd convergents (not necessarily in that order) (Peizer and Pratt (1968), page 1452). Since the convergents of the associated cf are the odd convergents of the corresponding form, bounds on the truncation error of the Mueller associated form are readily obtained. The convergents of Aroian's corresponding form do not behave as nicely (Aroian (1941)), and thus, the associated cf is not easily bounded. Mueller's cf is slightly more efficient than Aroian's (Peizer and Pratt (1968), page 1453), but this is a minor consideration. Aroian (1941) was not far wrong when he visualized the potential usefulness of cf's for computing $I_x(p, q)$.

7. Summary. Continued fractions have and should receive more attention for computing purposes, especially for statistical distribution functions and related special functions. They can clearly improve the convergence of related series. There is still much to be learned about the role and behavior of cf's in computing statistical distribution functions.

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