The Annals of Statistics
1979, Vol. 7, No. 2, 459–461

ON DEPENDENT TESTS OF SIGNIFICANCE IN THE MULTIVARIATE ANALYSIS OF VARIANCE

BY RICHARD L. DYKSTRA

University of Missouri-Columbia

It is shown that in the MANOVA situation, certain dependent tests of hypothesis may be combined to obtain an overall conservative significance level. The result holds whether one uses the Wilks, Roy, Lawley-Hotelling or Pillai test criterion. This generalizes a result due to Kimball in univariate ANOVA.

1. Introduction. In univariate ANOVA, one often tests more than one hypothesis with the very same data. For example, in a two-factor analysis, one may test for "no row effect", "no column effect" and also "no interaction effect". If, however, one tries to combine these tests for the hypothesis "no row, column, or interaction effects", the lack of independence of the test statistics makes it difficult to ascertain the true significance level of the test. Kimball (1951) has shown that in situations where the test statistics are independent chi-squared random variables each of which is divided by a common chi-squared variable independent of the numerators, assuming independence among the test statistics leads to a conservative value for $\alpha = P$(Type I error). That is, if α_i denotes the significance level for the ith test, and α is the true overall significance level,

$$\alpha < 1 - \Pi(1 - \alpha_i).$$

This approach is to be preferred to the Bonferroni bound $\Sigma\alpha_i$ because it is always sharper, although if α is small, the improvement is not very large. However the improvement increases as the number of factors in the product increases even if $\Sigma\alpha_i$ remains fixed. As α increases, the improvement becomes markedly better. Neter and Wasserman (1974), pages 582 and 655, discuss this procedure.

The same general type of situation often exists in MANOVA. Under the usual normality assumption, the general sum of squares matrix often may be decomposed into the sum of independent Wishart matrices, say

$$Q_t = Q_1 + \cdots + Q_k + Q_E.$$

The individual tests are then based upon the roots of the characteristic equations,

$$|Q_i - \lambda Q_E| = 0. \tag{1.1}$$

The multivariate situation, however, is more complicated than the univariate case where the F-tests are universally accepted, because here different test criterion have been suggested by different authors. Thus if $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \lambda_p \geqslant 0$ denote the

Received December 1977; revised February 1978.
AMS 1970 *subject classification.* Primary 62H05.
Key words and phrases. Wishart distribution, characteristic roots, positive dependence, significance level, MANOVA.

characteristic roots of (1.1), some suggested test criteria are:

(a) Wilks: reject if $[\Sigma_{i=1}^{p}(1+\lambda_i)^{-1}]^{-1}$ is too large.

(b) Roy: reject if $\lambda_1/(1+\lambda_1)$ is too large.

(c) Lawley-Hotelling: reject if $\Sigma_{i=1}^{p}\lambda_i$ is too large.

(d) Pillai: reject if $\Sigma_{i=1}^{p}\lambda_i/(1+\lambda_i)$ is too large.

Anderson (1958) and Timm (1975) are good sources which discuss these various criterion.

2. The multivariate theorem. We are now ready to state the theorem which gives an analogous multivariate result to that of Kimball's in the univariate case. This theorem will allow us to combine tests and obtain conservative overall significance levels regardless of which test criterion we choose. In fact we may even use different criteria for different tests and still combine them in the same way.

THEOREM. *If $Q_1, Q_2, \cdots, Q_k, Q_E$ are independent $p \times p$ Wishart matrices with the same covariance matrix Σ, where Q_E is full rank, and if $\Lambda_i' = (\lambda_{i1}, \cdots, \lambda_{ip})$ denote the ordered characteristic roots of Q_i in the metric of Q_E, i.e., the solutions of*

$$|Q_i - \lambda Q_E| = 0, \qquad i = 1, \cdots, k,$$

then for nonnegative functions f_i, $i = 1, \cdots, k$, which are nondecreasing in each argument,

$$E\Pi_{i=1}^{k} f_i(\Lambda_i) \geqslant \Pi_{i=1}^{k} Ef_i(\Lambda_i). \tag{2.1}$$

(We may replace nondecreasing by nonincreasing in the statement of the theorem.)

PROOF. It is well known that we may assume $\Sigma = I$ w.l.o.g. Then letting $\Lambda' = (\Lambda_1', \cdots, \Lambda_k')$, we may write

$$\begin{aligned} E\Pi_{i=1}^{k} f_i(\Lambda_i) &= E_{Q_E} E_{\Lambda|Q_E} \Pi_{i=1}^{k} f_i(\Lambda_i) \\ &= E_{Q_E} \Pi_{i=1}^{k} E_{\Lambda_i|Q_E} f_i(\Lambda_i) \end{aligned} \tag{2.2}$$

since $\Lambda_1, \cdots, \Lambda_k$ are conditionally independent given Q_E. It is well known that Q_E can be expressed as $A\Psi A'$ where Ψ is a diagonal matrix consisting of the characteristic roots of Q_E and A is a random orthogonal matrix distributed independently of Ψ. Thus (2.2) is expressable as

$$E_A E_{\Psi|A} \Pi_{i=1}^{k} E_{\Lambda_i|\Psi, A} f_i(\Lambda_i). \tag{2.3}$$

Our claim is that as the diagonal elements of Ψ increase, the elements of Λ can not increase.

To see this, note that the characteristic roots of

$$Q_i Q_E^{-1} = Q_i A \Psi^{-1} A' \tag{2.4}$$

are the same as the characteristic roots of

$$\Psi^{-\frac{1}{2}} A' Q_i A \Psi^{-\frac{1}{2}} \tag{2.5}$$

where $A'Q_iA$ is semipositive definite and $\Psi^{-\frac{1}{2}}$ is positive definite. Then by Lemma 2 of Anderson and Das Gupta (1964) these roots are increasing in the diagonal elements of $\Psi^{-\frac{1}{2}}$ and hence decreasing in the diagonal elements of Ψ.

Since the elements of Λ_i are nonincreasing as the elements of Ψ increase, and since the f_i are nonnegative nondecreasing functions of Λ_i, we can say that

$$E_{\Lambda_i|\Psi, A} f_i(\Lambda_i), \qquad i = 1, \cdots, k$$

are nonnegative, nonincreasing functions of the elements of Ψ. Since Dykstra and Hewett (1978) have established that the elements of Ψ are stochastically increasing in sequence, we may use Theorem 1 of Dykstra, Hewett and Thompson (1973) to state that (2.3) is

$$\geqslant E_A \Pi_{i=1}^k E_{\Psi|A} E_{\Lambda_i|\Psi, A} f_i(\Lambda_i). \tag{2.6}$$

Finally, we need only show that the integrand for the outside expectation is free of A. In this case, (2.6) will reduce to

$$\Pi_{i=1}^k E_{\Psi|A} E_{\Lambda_i|\Psi, A} f_i(\Lambda_i) = \Pi_{i=1}^k E f_i(\Lambda_i).$$

To see this, note that

$$|Q_i - \lambda Q_E| = 0$$

is equivalent to

$$|A'Q_iA - \lambda\Psi| = 0. \tag{2.7}$$

However, since the Q_i are independent and distributed as $\Sigma_\alpha \mathbf{Z}_{\alpha i}\mathbf{Z}'_{\alpha i}$ where the $\mathbf{Z}_{\alpha i}$ are independent $N(\mathbf{0}, I)$ vectors, the distribution of $A'Q_1A, \cdots, A'Q_kA$ will be free of A, because A is orthogonal. Then recalling that A and Ψ are independent, it clearly follows that the conditional distribution given A of the solutions of (2.7) are free of A, which concludes the proof of the theorem.

Thus if we write

$$f_i(\Lambda_i) = I_{[0, c_i]}(T_i(\Lambda_i)),$$

where $T_i(\Lambda_i)$ is one of the test criteria suggested earlier, Theorem 1 gives the multivariate analogue to Kimball's inequality.

REFERENCES

ANDERSON, T. W. (1958). *An Introduction to Multivariate Statistical Analysis*. Wiley, New York.

ANDERSON, T. W. and DAS GUPTA, S. (1964). Montonicity property of the power functions of some tests of the equality of two covariance matrices. *Ann. Math. Statist.* **35** 1059–1063.

DYKSTRA, R. L. and HEWETT, JOHN E. (1978). Positive dependence of the roots of a Wishart matrix. *Ann Statist.* **6** 235–238.

DYKSTRA, R. L., HEWETT, JOHN E. and THOMPSON, W. A. (1973). Almost independent events. *Ann Statist.* **1** 674–681.

KIMBALL, A. W. (1951). On dependent tests of significance in the analysis of variance. *Ann. Math. Statist.* **22** 600–602.

NETER, JOHN and WASSERMAN, WILLIAM (1974). *Applied Linear Statistical Models*. Richard D. Irwin, Inc., Homewood, Ill.

TIMM, NEIL H. (1975). *Multivariate Analysis with Applications in Education and Psychology*. Brooks Cole Publishing Co., Monterey, Calif.

DEPARTMENT OF STATISTICS
UNIVERSITY OF MISSOURI-COLUMBIA
222 MATH SCIENCES BLDG.
COLUMBIA, MISSOURI 65201