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# THE SMALL SAMPLE DISTRIBUTION OF A MANN-WHITNEY TYPE STATISTIC FOR CIRCULAR DATA

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The union-intersection method of test construction can be used to derive from the Mann-Whitney rank test a test statistic for testing whether two samples of observations from circular distributions come from the same population. The null distribution of this test statistic will be investigated here. The approach followed is to use the principle of inclusion-exclusion to obtain an expression for the number of partitions of a positive integer which satisfy certain conditions. This enables the probabilities for values of the circular Mann-Whitney test statistic to be expressed explicitly in terms of the probabilities for values of the usual Mann-Whitney test statistic. Recurrence formulae enabling computation of the distribution for small sample sizes are given. There is a clear relationship between our work and results obtained by Steck for the Kolmogorov and Smirnov statistics. These results can also be derived from the present approach.

**1. Introduction.** Some discussion of a Mann-Whitney type test for circular data and its distribution can be found in Batschelet (1965). Various properties of two different Mann-Whitney type test statistics have been discussed by the author (Eplett (1976)) including an expression for the asymptotic null distributions. The relationships derived in that paper between these Mann-Whitney type statistics and the Smirnov statistics are central to the present work where the concern is with small sample sizes.

The results of Section 3 were suggested by the work of Steck (1969, 1971) and Mohanty (1971) in connection with the Kolmogorov and Smirnov test statistics. One of the results proved here may be regarded as a generalization of this work, but is different to the result obtained by Kreweras (1965) from which the distributions of the Smirnov statistics may also be obtained (Mohanty (1977)). The generalization arises from using the general form of the principle of inclusion-exclusion as given, for instance, on page 90 of Berge (1971). The main result of Section 3, Theorem 2, is an application of this generalization towards obtaining the probabilities for the values of the Mann-Whitney type statistic. Theorem 2 is also of purely combinatorial interest as it concerns the number of partitions of a positive integer which satisfy certain conditions.

Section 2 contains preliminaries required for the rest of the paper together with a result giving the range of possible values for the Mann-Whitney type statistic. Section 4 discusses recurrence formulae for the null distribution of the test statistic.

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**2. The Mann-Whitney type test.** Let $X_1 < \cdots < X_m$ be an ordered sample of $m$ independent, identically distributed random variables with a continuous distribution function $F$ and $Y_1 < \cdots < Y_n$ be an ordered sample of $n$ independent, identically distributed random variables with a continuous distribution function $G$ (if the random variables are defined on the circle, then the ordering is by angular displacement relative to some fixed point on the circumference of the circle). We shall use $R_i$ to denote the rank of $X_i$ in the ordered combined sample of $N(= m + n)$ random variables. Then the usual Mann-Whitney test statistic is

$$W_N(R) = \Sigma_{i=1}^m R_i \qquad \text{for} \quad R = (R_1, \cdots, R_m).$$

Suppose that $\mathfrak{R} = \{(r_1, \cdots, r_m): r_i \text{ integers}, 1 \leqslant r_1 < \cdots < r_m \leqslant N\}$ and that $\mathcal{G}$ is the group of transformations of $\mathfrak{R}$ onto itself generated by the two transformations

$$g_i: (r_1, \cdots, r_m) \to (N + 1 - r_m, \cdots, N + 1 - r_1)$$

and

$$g_r: (r_1, \cdots, r_m) \to (r_1 - 1, \cdots, r_m - 1) \quad \text{if} \quad r_1 > 1,$$
$$(r_1, \cdots, r_m) \to (r_2 - 1, \cdots, r_m - 1, N) \quad \text{if} \quad r_1 = 1.$$

Then we shall investigate the distribution of the statistic

$$\xi_N(R) = \max_{g \in \mathcal{G}}\{W_N(g(R))\}, \tag{1}$$

under the assumption that $F(\cdot) = G(\cdot)$.

If $F_m(z)$ and $G_n(z)$ are used to denote the empirical distribution functions for the $X$'s and $Y$'s, then the usual Smirnov statistics are defined by

$$D^+(m, n) = \sup_z\{F_m(z) - G_n(z)\}$$
$$D^-(m, n) = \sup_z\{G_n(z) - F_m(z)\}$$
$$D(m, n) = \max\{D^+(m, n), D^-(m, n)\}.$$

One easily verifies (cf. Eplett (1976)) that

$$\xi_N = \max\{W_N(R) + mnD^+(m, n), m(N + 1) - W_N(R) + mnD^-(m, n)\}, \tag{2}$$

where $\xi_N$ in this case is a linear transformation of the statistic used in that paper.

The largest value which $\xi_N$ may assume is clearly the same as the largest value which $W_N$ may assume. The least value which $\xi_N$ may assume seems less obvious and is derived using elementary number theory.

PROPOSITION 1. *The largest value attained by $\xi_N$ is $m(N + n + 1)/2$ and the smallest value attained by $\xi_N$ is $(N(m + 1) + m - d)/2$ where $d = (m, n)$ is the* g.c.d. *of $m$ and $n$.*

PROOF. It is required to show that

$$\min_{R \in \mathfrak{R}}\max_{g \in \mathcal{G}}\{W_N(g(R))\} = (N(m + 1) + m - d)/2. \tag{3}$$

Without loss of generality the problem may be reduced to finding the least possible value of $\xi_N$ for those $R \in \mathfrak{R}$ for which $W_N(R) \geqslant W_N(g(R))$ for all $g \in \mathcal{G}$. Form (2) this last condition is equivalent to

$$mnD^+(m, n) \leqslant 0 \quad \text{and} \quad mnD^-(m, n) \leqslant 2\Sigma_{i=1}^m R_i - m(N+1),$$

which, using Theorems 2.1 and 2.2 of Steck (1969) to express $mnD^+(m, n)$ and $mnD^-(m, n)$ in terms of the $R_i$, becomes

$$(4) \qquad Ni \leqslant mR_i \leqslant 2\Sigma_{i=1}^m R_i - N(m+1) + Ni \qquad i = 1, \cdots, m.$$

This suggests considering $R_i = < Ni/m >$, $i = 1, \cdots, m$, where $< x >$ denotes the smallest integer greater than or equal to $x(-[-x]$ in standard notation). Then $\Sigma_{i=1}^m R_i = N + \Sigma_{i=1}^{m-1} < Ni/m >$. An argument often used in studying the Dedekind sum is employed in order to obtain an expression for the last term on the right-hand side.

Put $((x)) = < x > - x$, so that $((x))$ is periodic with period 1, and consider $\Sigma_{i=1}^{m-1}((Ni/m))$. Then, since $(m, n) = d$ implies that $(m, N) = d$, it follows that in the set of residue classes $\{Ni\colon i = 1, \cdots, m-1\}$ each nonzero multiple of $d$ appears $d$ times and 0 appears $d - 1$ times (page 32 of Le Veque (1965)). Thus

$$\Sigma_{i=1}^{m-1}((Ni/m)) = d\Sigma_{1 \leqslant k \leqslant m-1, d|k}((k/m)) = (m-d)/2.$$

In consequence,

$$\Sigma_{i=1}^{m-1} < Ni/m > = (N(m-1))/2 + (m-d)/2$$

and hence

$$\Sigma_{i=1}^m R_i = (N(m+1) + m - d)/2.$$

The upper bound on $mR_i$ from (4) then becomes $(m-d) + Ni$ for $i = 1, \cdots, m$ and since $R_i = < Ni/m >$ does satisfy this upper bound, it gives the minimum value for $\xi_N$ over $\mathfrak{R}$ (and is unique up to orbits of $\mathcal{G}$). This result is intuitively acceptable, since the smallest value of $\xi_N$ occurs when the ranks of the $X$'s in the combined sample are evenly spaced.

From (2) it follows that

$$(5) \qquad P(\xi_N < e) = \Sigma_t P(W_N(R) = t, mnD^+(m, n) < e - t, mnD^-(m, n) < e + t - m(N+1)),$$

where the range of summation is given by $m(N+1) - e < t < e$. Using the theorems from Steck mentioned previously, (5) may be written as

$$(6) \quad P(\xi_N < e) = \Sigma_t P(W_N(R) = t; (iN - (e-t))/m < R_i < (e + t - (m + 1 - i)N)/m, i = 1, \cdots, m).$$

This last expression suggests determining $W(b, c, t)$ which equals the number of ways the event $\{\Sigma_{i=1}^m R_i = t; b_i < R_i < c_i, i = 1, \cdots, m\}$ can occur in the ordered combined sample, where $b = (b_1, \cdots, b_m)$ and $c = (c_1, \cdots, c_m)$ are two increasing sequences of integers such that $i - 1 \leqslant b_i < c_i \leqslant n + i + 1$. The numbers $W(b, c, t)$ count the number of partitions of $t$ subject to the restrictions we have imposed.

**3. A generating function for $W(b, c, t)$.** If $T_i = R_i - i$, $u_i = b_i - i + 1$, $v_i = c_i - i - 1$, $i = 1, \cdots, m$ and $w = t - (m(m+1))/2$, then

$$\{\Sigma_{i=1}^m R_i = t;\ b_i < R_i < c_i,\ i = 1, \cdots, m\}$$

if and only if

$$\{\Sigma_{i=1}^m T_i = w;\ u_i \leqslant T_i \leqslant v_i,\ i = 1, \cdots, m\}$$

and $0 \leqslant T_1 \leqslant \cdots \equiv T_m \leqslant n$. The number of vectors of integers which satisfy the second of these conditions will for convenience be denoted by $N_m(u, v, w)$ (where $u = (u_1, \cdots u_m)$ and $v = (v_1, \cdots, v_m)$).

The main result is stated in terms of the generating functions

$$N_m^*(u, v, z) = \Sigma_{w=0}^\infty N_m(u, v, w) z^w$$

and

$$\pi^*(m, n, z) = \Sigma_{w=0}^\infty \pi(m, n, w) z^w$$

where $\pi(m, n, w)$ equals the number of ways in which the event $\{\Sigma_{i=1}^m R_i = w,\ 1 \leqslant R_1 < \cdots < R_m \leqslant N\}$ can occur and is just $\binom{N}{m} P(W_N(R) = w)$. Both $N_m^*$ and $\pi^*$ are polynomials since only a finite number of terms in the summation are nonzero.

THEOREM 2. *If the components of the $m \times m$ matrix $(d(i,j))$ are given by*

$$d(i,j) = z^{(b_j - j)(j-i+1)} \pi^*(j - i + 1, c_i - b_j - 2, z), \tag{7}$$

*then*

$$N_m^*(u, v, z) = \det\{(d(i,j))\}. \tag{8}$$

The generating function for the numbers $W(b, c, t)$ is then obtained from

$$W^*(b, c, z) = \Sigma_t W(b, c, t) z^t = z^{(m(m+1))/2} N_m^*(u, v, z). \tag{9}$$

The significance of Theorem 2 is that using (9) together with (6), the probabilities for $\xi_N$ can be explicitly expressed in terms of the probabilities for $W_N$. The actual expression is rather complicated but seems to be a promising step towards obtaining useful results about the distribution of $\xi_N$ using material about the usual Mann-Whitney statistic $W_N$.

Let us make a few preliminary remarks about the form of the matrix defined by (7). If $i - j > 1$ or $c_i - b_j \leqslant 1$, then $d(i,j) = 0$. If $i = j + 1$, then $d(i,j) = 1$ since we take $\pi^*(0, n, z) = 1$ for $n \geqslant 0$. Thus the matrix $(d(i,j))$ has the same form as those obtained by Steck in connection with the Kolmogorov and Smirnov statistics. Since the $d(i,j)$ are members of the polynomial ring over $Z$, caution must be exercised in applying the standard results about determinants of matrices whose entries are elements of a field.

The principle of inclusion-exclusion is used to derive Lemma 3 from which Theorem 2 may then be deduced. As usual, $Z$ denotes the set of integers. Suppose that $X^*$ is a function mapping the finite subsets of $Z^m$ into a commutative ring such that if $\mathbb{S}_1$, $\mathbb{S}_2$ are two such disjoint subsets, $X^*(\mathbb{S}_1 \cup \mathbb{S}_2) = X^*(\mathbb{S}_1) + X^*(\mathbb{S}_2)$. The function $X^*$ is said to be multiplicative if there exist functions $(X^*)_1, \cdots, (X^*)_m$ such rhat if $l_1, \cdots, l_k > 0$, $l_1 + \cdots + l_k = m$ and $Z(i) \subset Z^{l_i}$, then $X^*(Z(1) \times \cdots \times Z(k)) = \pi_{i=1}^k (X^*)_{l_i}(Z(i))$, and we shall write $(X^*)_{l_i}(Z(i))$ as

$X^*(Z(i))$. The number of elements in a finite subset $\mathbb{S}$ is denoted by $|\mathbb{S}|$.

Let $\chi_m(u, v) = \{(x_1, \cdots, x_m)$: $x_i$ integers, $0 \leqslant x_1 \leqslant \cdots \leqslant x_m \leqslant n$, $u_i \leqslant x_i \leqslant v_i$, $i = 1, \cdots, m\}$. The sets $\mathcal{L}(i,j)$ are defined for $1 \leqslant i \leqslant j \leqslant m$ by $\mathcal{L}(i,j) = \{(x_i, \cdots, x_j)$: $x_k$ integers, $v_i \geqslant x_i > \cdots > x_j \geqslant u_j\}$ (which will be empty if $v_i - u_j \leqslant j - i - 1$). By convention take $X^*(\mathcal{L}(i, i-1)) = 1$ and otherwise $X^*(\mathcal{L}(i,j)) = 0$ for $1 \leqslant j + 1 < i \leqslant m$.

LEMMA 3. *If the components of the $m \times m$ matrix $(d(i,j))$ are given by*

$$d(i,j) = X^*(\mathcal{L}(i,j)), \tag{10}$$

*then, if $X^*$ is multiplicative,*

$$X^*(\chi_m(u, v)) = \det\{(d(i,j))\}. \tag{11}$$

PROOF. The matrix $(d(i,j))$ is of the same special form as that of Theorem 2. To each subset of elements on the subdiagonal $\{d(i+1, i) = 1: i = 1, \cdots, m-1\}$ there corresponds at most one nonzero term in the determinant containing these and only these elements on the subdiagonal as a factor. By considering the structure of these terms we find that

$$\det\{(d(i,j))\} = \Sigma_{k=1}^{m}(-1)^{m-k}\Sigma_{0=i_0<i_1<\cdots<i_k=m}\pi_{\nu=1}^{k}d(i_{\nu-1}+1, i_\nu),$$

and in view of (10) and $X^*$ being multiplicative we have

$$\det\{(d(i,j))\} = \Sigma_{k=1}^{m}(-1)^{m-k}\Sigma_{0=i_0<i_1<\cdots<i_k=m}X^*(\mathcal{T}(i_0, \cdots, i_k)), \tag{12}$$

where

$$\mathcal{T}(i_0, \cdots, i_k) = \{(x_1, \cdots, x_m) \in Z^m: u_i \leqslant x_i \leqslant v_i, i = 1, \cdots, m;$$
$$v_{i_{\nu-1}+1} \geqslant x_{i_{\nu-1}+1} > \cdots > x_{i_\nu} \geqslant u_{i_\nu}, \nu = 1, \cdots, k\}.$$

The principle of inclusion-exclusion yields that the right-hand side of (12) equals $X^*(\chi_m(u, v))$ and the lemma is proved.

Suppose that $f(x_1, \cdots, x_m) = g(x_1) + \cdots + g(x_m)$, where $g$ is a function mapping $Z$ into itself. Then if $X^*(\mathbb{S})$ is defined as the generating function for the values of $f$ obtained when $f$ acts on $\mathbb{S}$, that is $X^*(\mathbb{S}) = \Sigma_{x\in\mathbb{S}}z^{f(x)}$, it follows that $X^*$ is multiplicative and Lemma 3 applies.

To prove the theorem, apply Lemma 3 to the case where $X^*$ is the generating function for $f(x_1, \cdots, x_m) = x_1 + \cdots + x_m$ over subsets $\mathbb{S}$ of $Z^m$. Notice that for integers $x_k$ and $i \leqslant j$,

$$\Sigma_{k=i}^{j}x_k = Q, \qquad u_j \leqslant x_j > \cdots > x_i \leqslant v_i$$

if and only if

$$\Sigma_{k=i}^{j}z_k = Q - (j - i + 1)(u_j - 1),$$
$$1 \leqslant z_j < \cdots < z_i \leqslant v_i - u_j + 1,$$

where $z_k = x_k - u_j + 1$, $k = i, \cdots, j$. Then

$$X^*(\mathcal{L}(i,j)) = \Sigma_k\pi(j - i + 1, v_i - u_j - j + i, k - (j - i + 1)(u_j - 1))z^k.$$

But the nature of $\pi$ and $u_j$ ensures that

$$z^{(j-i+1)(u_j-1)}\pi^*(j-i+1, v_i-u_j-j+i, z)$$
$$= \Sigma_k \pi(j-i+1, v_i-u_j-j+i, k-(j-i+1)(u_j-1))z^k,$$

and so substituting for $u_i$ and $v_i$ in terms of $b_i$ and $c_i$, Theroem 2 follows.

Suppose we apply Lemma 3 when $X^*$ is defined by $X^*(\mathbb{S}) = |\mathbb{S}|$ for finite subsets $\mathbb{S}$ of $Z^m$. Then

$$X^*(\mathfrak{L}(i,j)) = |\mathfrak{L}(i,j)|$$
$$= \binom{v_i - u_j + 1}{j-i+1}_+,$$

where

$$\binom{t}{s}_+ = \binom{t}{s} \quad \text{if} \quad 1 \leqslant s \leqslant t$$
$$= 0 \quad \text{if} \quad t < s.$$

Substituting for $u_i$ and $v_i$ in terms of $b_i$ and $c_i$, (11) gives

$$|\chi_m(u, v)| = \det\left\{\left(\binom{c_i - b_j + j - i - 1}{j-i+1}_+\right)\right\},$$

which is the expression obtained as Theorem 4.1 of Steck (1969).

Restricting $X^*$ to subsets of integers is for convenience and such restrictions may be removed without affecting the validity of the proof of Lemma 3. For instance, suppose that $X^*$ is the distribution of $(U_1, \cdots, U_m)$, where the $U_i$ are independent random variables uniformly disributed over [0, 1] and we require an expression for $P(u_i \leqslant U^{(i)} \leqslant v_i, i = 1, \cdots, m)$ where $U^{(i)}$ is the $i$th order statistic of the sample and $u_i$, $v_i$ are now real numbers lying between 0 and 1. The arguments of Lemma 3 still hold and

$$X^*(\mathfrak{L}(i,j)) = P(v_i \geqslant U_i > \cdots > U_j \geqslant u_j)$$
$$= \{(v_i - u_j)_+^{j-i+1}\}/(j-i+1)!,$$

where $(x)_+ = \max(x, 0)$. This provides another proof of Theorem 2 of Steck (1971).

Using the principle of inclusion-exclusion offers a unified approach to these results and gives every term in the determinant a probabilistic (or combinatorial) meaning.

Returning to Theorem 2, we use (8) to express the result in the form

$$(13) \qquad N_m^*(u, v, z) = \Sigma_{k=1}^m (-1)^{m-k} \Sigma_{0=i_0<i_1<\cdots<i_k=m} z^{\Sigma_{\nu=1}^k (b_{i_\nu} - i_\nu)(i_\nu - i_{\nu-1})}$$
$$\times \pi_{\nu=1}^k \pi^*(i_\nu - i_{\nu-1}, c_{i_{\nu-1}+1} - b_{i_\nu} - 2, z),$$

from which we use (9) to obtain an expression for the numbers $W(b, c, t)$ in terms of the $\pi$'s as

(14) $$W(b, c, t) = \Sigma_{k=1}^{m}(-1)^{m-k}\Sigma_{0=i_0<i_1<\cdots<i_k=m}\Sigma_{0=j_0<j_1<\cdots<j_k=\gamma}$$
$$\pi_{\nu=1}^{k}\pi(i_\nu - i_{\nu-1}, c_{i_{\nu-1}+1} - b_{i_\nu} - 2, j_\nu - j_{\nu-1}),$$

where

$$\gamma = t - (m(m+1))/2 - \Sigma_{\nu=1}^{k}(b_{i_\nu} - i_\nu)(i_\nu - i_{\nu-1}).$$

In order to apply the results to obtaining the distribution of $\xi_N$, the probabilities can be obtained from $W(b, c, t)$ by dividing by $\binom{N}{m}$ and substituting in (6). The values of $b_i$ and $c_i$ depend on $t$ and $e$ and are given by

$$b_i = \max([(iN - e + t)/m], i - 1),$$
$$c_i = \min(< (e + t - (m + 1 - i)N)/m >, n + i + 1),$$
$$i = 1, \cdots, m.$$

The values of $\pi(m, n, w)$ can be obtained from tables of the exact probabilities for the Mann-Whitney statistic (remembering to convert from probabilities to integers) or directly from the recurrence formula $\pi(m, n, w) = \pi(m-1, n, w-N) + \pi(m, n-1, w)$. Hence (14) provides a way of evaluating the probabilities for $\xi_N$—although this appears to be a rather cumbersome procedure.

**4. Recurrence relationships.** A recurrence formula for $N_m^*$ may be obtained by expanding the determinant in Theorem 2 by the $m$th column to obtain

(15) $$N_m^*(u, v, z) = \Sigma_{k=1}^{m}(-1)^{k+1}N_{m-k}^*(u, v, z)\pi^*$$
$$\cdot (k, v_{m-k+1} - u_m - k + 1, z)z^{k(u_m-1)},$$

or equivalently

(16) $$N_m(u, v, w) = \Sigma_{k=1}^{m}(-1)^{k+1}\Sigma_{i=1}^{w}N_{m-k}(u, v, w - i)\pi$$
$$\cdot (k, v_{m-k+1} - u_m - k + 1, i - k(u_m - 1)),$$

where in $N_{m-k}^*(u, v, z)$ and $N_{m-k}(u, v, w)$, $u$ and $v$ are reduced to their first $m - k$ components and

$$N_0(u, v, w) = 0 \quad \text{if } w > 0$$
$$= 1 \quad \text{if } w = 0,$$

so that $N_0^*(u, v, z) = 1$. Since the $\pi$'s can be obtained along the lines described in Section 3, (16) provides a way of generating values of $N_m(u, v, w)$.

There is another recurrence formula for $W(b, c, t)$ which does not involve the $\pi$'s. Write $W(b, c, t)$ in more detail as $W(m, n, b, c, t)$. Then, if $S_i = R_i - R_1$, $i = 2, \cdots, m$, $W(m, n, b, c, t)$ equals the sum over $R_1$ of the number of vectors $(S_2, \cdots, S_m)$ for which

(a) $S_i$ are integers, $1 \leqslant S_2 < \cdots < S_m \leqslant N - R_1$;
(b) $b_i - R_1 < S_i < c_i - R_1$, $i = 2, \cdots, m$;
(c) $\Sigma_{i=2}^{m}S_i = t - mR_1$,

which is just $W(m - 1, n - R_1 + 1, b'(R_1), c'(R_1), t - mR_1)$ where $b'(R_1) = (b_2 - R_1, \cdots, b_m - R_1)$ and $c'(R_1) = (c_2 - R_1, \cdots, c_m - R_1)$. Then

$$W(m, n, b, c, t) = \Sigma_{R_1=b_1+1}^{c_1-1} W(m - 1, n - R_1 + 1, b'(R_1), c'(R_1), t - mR_1),$$

which enables us to generate the numbers $W(m, n, b, c, t)$ from the values of $W(1, n, b, c, t)$, which are obvious. An observation which might help to simplify computation is that for any real vectors $b$, $c$, one may always replace $b$, $c$ by

$$b_i = \max([b_i], i - 1)$$
$$c_i = \min(< c_i >, n + i + 1) \qquad i = 1, \cdots, m,$$

so that $W$ does not depend on $n$.

Finally, let us note that our results may be related to the null distribution of another Mann-Whitney type test statistic. Suppose that $\bar{\mathcal{G}}$ is the subgroup of $\mathcal{G}$ generated by $g_r$ and put

$$\bar{\xi}_N = \max_{g \in \bar{\mathcal{G}}}\{W_N(g(R))\},$$

then discussion of the null distribution of $\bar{\xi}_N$ follows the same lines as that for $\xi_N$ and in particular $W(b, c, t)$ can be used by setting $c_i = n + i + 1$, $i = 1, \cdots, m$. The statistic $\bar{\xi}_N$ can also be used for testing against the general alternative hypothesis that $F(\cdot) \neq G(\cdot)$.

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