The Annals of Statistics
1977, Vol. 5, No. 3, 580-582

A CHARACTERISTIC PROPERTY OF THE EXPONENTIAL DISTRIBUTION

BY M. AHSANULLAH

Empresa Brasileira de pesquisa agropecuária
Convênio IICA/Embrapa, Brazil

Let X be a nonnegative random variable with probability distribution function F. Suppose $X_{i,n}$ $(i = 1, \cdots, n)$ is the ith smallest order statistics in a random sample of size n from F. A necessary and sufficient condition for F to be exponential is given which involves the identical distribution of the random variables X and $(n-i)$ $(X_{i+1,n} - X_{i,n})$ for some i and n, $(1 \leqq i < n)$.

1. Introduction. Let X be a random variable (rv) whose probability density function f is given, for some $\theta > 0$, by

$$f_\theta(x) = \Theta^{-1} \exp(-x/\theta), \quad x > 0, \qquad (1.1)$$
$$= 0, \quad \text{otherwise.}$$

Suppose $X_1, X_2, \cdots, X_n$ is a random sample of size n from a population with density f and let $X_{1,n} < X_{2,n} < \cdots < X_{n,n}$, be the associated order statistics.

Kotz (1974) and Galambos (1975) discussed extensively the characterization of exponential distribution by order statistics. Desu (1971) showed that the exponential distribution is the only one with the property that for all K, K times the minimum of the random sample of size K from the distribution has the same distribution as a single observation from the distribution. Arnold (1971) proved that the characterization is preserved if in Desu's theorem "for all K" is replaced by two integers K_1 and K_2 relatively prime and distinct from 1.

Puri and Rubin (1970) proved that if X_1 and X_2 are independent copies of a rv X with density f then X and $|X_1 - X_2|$ have the same distribution if and only if f is as given in (1.1). Barlow and Proschan (1966) considered many interesting properties concerning order statistics and their spacings from certain restricted families of positive random variables.

In this paper we will give a characterization of the exponential distribution that requires X and $(n-i)(X_{i+1,n} - X_{i,n})$ to be identically distributed for some i and n, $1 \leqq i < n$.

2. Notation and result. Let F be the distribution function of a nonnegative rv and let $\bar{F} = 1 - F$, for $x \geqq 0$. We will call F "new better than used" (NBU), if $\bar{F}(x+y) \leqq \bar{F}(x)\bar{F}(y)$, $x, y \geqq 0$, and F is "new worse than used" (NWU) if

Received December 1974; revised August 1976.
AMS 1970 *subject classifications*. Primary 62E10; Secondary 62G30.
Key words and phrases. Exponential distribution, characterization, identical distribution, order statistics.

$\bar{F}(x+y) \geqq \bar{F}(x)\bar{F}(y)$, $x, y \geqq 0$. We will say that F belongs to the class C, if F is either NBU or NWU.

THEOREM. *Let X be a nonnegative* rv *having an absolutely continuous* (with respect to Lebesgue measure) *distribution function F that is strictly increasing on* $[0, \infty)$. *Then the following properties are equivalent*:

(a) *X has an exponential distribution with density as given in* (1.1),

(b) *for some i and n, $1 \leqq i < n$, the statistics $(n-i)(X_{i+1,n} - X_{i,n})$ and X are identically distributed and F belongs to class C.*

PROOF. It is well known [see, e.g., Galambos (1975)] that (a) $\Rightarrow$ (b), so we prove only that (b) $\Rightarrow$ (a).

From the density of $Y_i = X_{i+1,n} - X_{i,n}$ which is given, e.g., by Pyke (1965), it follows that $Z = (n-i)\, Y_i$ has the density

$$f_Z(z) = L(n, i) \int_0^\infty (F(u))^{i-1}(1 - F(u + z(n-i)^{-1}))^{n-i-1} \times f(u)f(u + z(n-i)^{-1})\, du/(n-i)\,, \tag{2.1}$$

where $L(n, i) = n!/((i-1)!\,(n-i-1)!)$.

By using the hypothesis $f_Z = f$ and writing $(n-i)/L(n, i) = \int_0^\infty (F(u))^{i-1}(1 - F(u))^{n-i}f(u)\, du$, it follows that

$$0 = \int_0^\infty (F(u))^{i-1}f(u)g(u, z)\, du\,, \qquad \text{for all} \quad z\,, \tag{2.2}$$

where

$$g(u, z) = f(z)(1 - F(u))^{n-i} - (1 - F(u + z(n-i)^{-1}))^{n-i-1}f(u + z(n-i)^{-1})\,.$$

Integrating (2.2) with respect to z from 0 to z_1 and interchanging the order of integration (which is permitted here), we get

$$0 = \int_0^\infty (F(u))^{i-1}(1 - F(u))^{n-i}f(u)G(u, z_1)\, du\,, \qquad \text{for all} \quad z_1\,, \tag{2.3}$$

where

$$G(u, z_1) = ((1 - F(u + z_1(n-i)^{-1}))/(1 - F(u)))^{n-i} - (1 - F(z_1))\,.$$

Now if F is NBU, then for any integer $k > 0$, $\bar{F}(x/k) \geqq (\bar{F}(x))^{1/k}$, so $G(0, z_1) \geqq 0$. Thus if (2.3) holds, it must be $G(0, z_1) \equiv 0$. Similarly, if F is NWU, then $G(0, z_1) \leqq 0$ and hence for (2.3) to be true $G(0, z_1) \equiv 0$. Writing $G(0, z_1)$ in terms of F, we get,

$$1 - F(z_1) = (1 - F(z_1(n-i)^{-1}))^{n-i}\,, \qquad \text{for all} \quad z_1\,. \tag{2.4}$$

Substituting $\bar{F}(z_1) = 1 - F(z_1)$, and $n - i = k$, we get from (2.4),

$$\bar{F}(z_1/k) = (\bar{F}(z_1))^{1/k}\,, \qquad \text{for all} \quad z_1 > 0\,, \quad \text{and some integer} \quad k > 0\,. \tag{2.5}$$

The solution of (2.5), Aczél (1966), is for $k > 1$,

$$\bar{F}(z_1) = 1 - F(z_1) = e^{-\lambda_1 z_1}\,, \qquad \text{for some} \quad \lambda_1 > 0 \quad \text{and all} \quad z_1 > 0\,. \tag{2.6}$$

If $k = 1$, then $G(u, z_1) = (\bar{F}(u + z_1))(\bar{F}(u))^{-1} - \bar{F}(z_1)$ and (2.3) gives $0 = \int_0^\infty (F(u))^{n-2} f(u)\bar{F}(u)[(\bar{F}(u + z_1))(\bar{F}(u))^{-1} - \bar{F}(z_1)]\, du$, for all z_1, and with $F \in C$. This means $(\bar{F}(u + z_1))(\bar{F}(u))^{-1} = \bar{F}(z_1)$, so again we get (2.6).

Acknowledgment. I am indebted to the referee for helpful comments which simplified the proof of the theorem.

REFERENCES

[1] Aczél, J. (1966). *Lectures on Functional Equations and Their Applications*. Academic Press, New York.

[2] Arnold, B. C. (1971). Two characterizations of the exponential distribution using order statistics. Unpublished.

[3] Barlow, R. E. and Proschan, F. (1966). Inequalities for linear combinations of order statistics from restricted families. *Ann. Math. Statist.* **37** 1574–1592.

[4] Desu, M. M. (1971). A characterization of the exponential distribution by order statistics. *Ann. Math. Statist.* **42** 837–838.

[5] Galambos, J. (1975). Characterization of probability distributions by properties of order statistics I. In *Statistical Distribution in Scientific Work* 3 71–88 (G. P. Patil et al., eds.). Reidel, Dordrecht.

[6] Kotz, S. (1974). Characterizations of statistical distribution: a supplement to recent surveys. *Rev. Inst. Internat. Statist.* **42** 39–65.

[7] Puri, P. S. and Rubin, H. (1970). A characterization based on absolute difference to two i.i.d. random variables. *Ann. Math. Statist.* **41** 2113–2122.

[8] Pyke, R. (1965). Spacings. *J. Roy. Statist. Soc. Ser. B* **27** 395–436.

60 East Park Drive
Ottawa, Ontario K1B 4A1
Canada