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SCHUR FUNCTIONS IN STATISTICS
I. THE PRESERVATION THEOREM[1]

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This is Part I of a two-part paper; the purpose of this two-part paper is (a) to develop new concepts and techniques in the theory of majorization and Schur functions, and (b) to obtain fruitful applications in probability and statistics. The main theorem of Part I states that if $f(x_1, \cdots, x_n)$ is Schur-concave, and if $\phi(\lambda, x)$ is totally positive of order 2 and satisfies the semigroup property for $\lambda_1 > 0$, $\lambda_2 > 0$: $\phi(\lambda_1 + \lambda_2, y) = \int \phi(\lambda_1, x)\phi(\lambda_2, y - x)\, d\mu(x)$, where μ is Lebesgue measure on $[0, \infty)$ or counting measure on $\{0, 1, 2, \cdots\}$, then $h(\lambda_1, \cdots, \lambda_n) \equiv \int \cdots \int \prod_1^n \phi(\lambda_i, x_i) f(x_1, \cdots, x_n)\, d\mu(x_1) \cdots d\mu(x_n)$ is also Schur-concave. This theorem is then applied to obtain renewal theory results, moment inequalities, and shock model properties.

1. Introduction and summary. The purpose of this two-part paper is (a) to develop new concepts and techniques in the theory of majorization and Schur functions, and (b) to obtain fruitful applications in probability and statistics. More specifically, we (a) derive a basic theorem concerning the preservation of Schur functions under certain integral transformations (Part I), (b) introduce a stochastic version of majorization and develop its properties (Part II), and (c) obtain a number of applications of the preservation theorem and introduce the new notion of stochastic majorization to multivariate distributions.

1.2. *Some definitions and the preservation theorem.* We give definitions of majorization and Schur functions because, unfortunately, some previous definitions of majorization suffer from the lack of distinction between a vector and the vector resulting from a decreasing rearrangement of its coordinates.

Given a vector $\mathbf{x} = (x_1, \cdots, x_n)$, let $x_{[1]} \geqq x_{[2]} \geqq \cdots \geqq x_{[n]}$ denote a nonincreasing rearrangement of $x_1, \cdots, x_n$. A vector $\mathbf{x}$ is said to *majorize* a vector $\mathbf{x}'$ if

$$\sum_{i=1}^{j} x_{[i]} \geqq \sum_{i=1}^{j} x'_{[i]}, \quad j = 1, \cdots, n-1,$$

and

$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} x'_{[i]};$$

in symbols, $\mathbf{x} \geqq^{\mathrm{m}} \mathbf{x}'$. Notice that whenever $(\pi_1, \cdots, \pi_n)$ is a permutation of $(1, \cdots, n)$ and $\mathbf{x}' = (x_{\pi_1}, \cdots, x_{\pi_n})$, we have $\mathbf{x} \geqq^{\mathrm{m}} \mathbf{x}'$ and $\mathbf{x}' \geqq^{\mathrm{m}} \mathbf{x}$. If a function

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f satisfies the property that $f(\mathbf{x}) \geqq (\leqq) f(\mathbf{x}')$ whenever $\mathbf{x} \geqq^{\mathrm{m}} \mathbf{x}'$, then f is called a *Schur-convex* (*Schur-concave*) function. Functions which are either Schur-convex or Schur-concave are called *Schur functions* (or alternatively, are said to possess the *Schur property*). Note that a Schur function is necessarily permutation-invariant; that is, $f(\mathbf{x}) = f(\mathbf{x}')$ whenever $x' = (x_{\pi_1}, \cdots, x_{\pi_n})$ and $(\pi_1, \cdots, \pi_n)$ is a permutation of $(1, \cdots, n)$. A Schur function need not be measurable. However, throughout this paper when we say that f is a Schur function we will always mean that f is a Borel measurable Schur function.

A function $\phi(\lambda, x)$ on R_2 is *totally positive of order* 2 (TP_2) if (a) $\phi(\lambda, x) \geqq 0$, and (b) $\lambda_1 < \lambda_2$, $x_1 < x_2$ imply that

$$\begin{vmatrix} \phi(\lambda_1, x_1) & \phi(\lambda_1, x_2) \\ \phi(\lambda_2, x_1) & \phi(\lambda_2, x_2) \end{vmatrix} \geqq 0 .$$

For instance, see Karlin (1968), Chapter 1.

A function $\phi(\lambda, x)$, defined on $(0, \infty) \times [0, \infty)$, is said to satisfy the *semigroup property in* λ if

$$\phi(\lambda_1 + \lambda_2, x) = \int_0^\infty \phi(\lambda_1, x - y)\phi(\lambda_2, y)\, d\mu(y) \tag{1.1}$$

where throughout this paper μ *will denote either Lebesgue measure on* $[0, \infty)$ *or counting measure on the nonnegative integers.*

We may now state the preservation theorem whose proof appears in Section 2.

THEOREM 1.1. *Let the function* $\phi(\lambda, x)$ *defined on* $(0, \infty) \times [0, \infty)$ *be* TP_2 *and satisfy the semigroup property. Let* $f(\mathbf{x})$ *be Schur-convex (Schur-concave). Let*

$$h(\boldsymbol{\lambda}) = \int_0^\infty \cdots \int_0^\infty f(\mathbf{x}) \prod_{i=1}^n \phi(\lambda_i, x_i)\, d\mu(x_1) \cdots d\mu(x_n) \tag{1.2}$$

where the integral is assumed to exist. Then h is a Schur-convex (Schur-concave) function.

There is a long history of preservation theorems in the literature. Typical among these are the following:

THEOREM (Karlin, 1968, page 130). *Let* $f(x + y)$ *be* TP_2 *for* $x > 0$, $y > 0$ *and let* $\phi(\lambda, x)$ *be* TP_2 *for* $\lambda > 0$, $x > 0$ *and satisfy the semigroup property in* λ. *Define*

$$c(t) = \int_0^\infty \phi(t, x) f(x)\, d\mu(x) \qquad t > 0 ,$$

where μ *is either Lebesgue measure or counting measure. Then* $c(t + s)$ *is* TP_2 *for* $t, s > 0$.

(Actually Karlin gives a more general version involving higher order total positivity.)

THEOREM (Marshall–Olkin, 1974). *Let* $f(\mathbf{x})$ *and* $\phi(\mathbf{y})$ *be Schur-concave functions, and let* $h(\boldsymbol{\lambda}) = \int f(\mathbf{x})\phi(\boldsymbol{\lambda} - \mathbf{x}) \prod_1^n d\mu(x_i)$ *be well defined. Then h is Schur-concave.*

Preservation theorems have generally enabled one to understand the property preserved (such as TP_2, Schur-concavity, etc.) and to generate other functions with the same property. Our preservation theorem generalizes the Karlin theorem since $\prod_1^n f(x_i)$ is Schur-convex in $\mathbf{x}$ whenever $f(x+y)$ is TP_2. The Marshall–Olkin theorem and our preservation theorem overlap when $\phi(\boldsymbol{\lambda}, \mathbf{x})$ is of the form $\prod_1^n \psi(\lambda_i - x_i)$ and, in general, neither theorem can be derived from the other.

1.3. *Importance of the preservation theorem.* Theorem 1.1 shows that the integral transform $h(\boldsymbol{\lambda})$ given in (1.2) inherits the Schur property from the function $f(\mathbf{x})$ being transformed. The deceptively simple preservation Theorem 1.1 will be shown (especially in Part II) to have many applications to statistics. In one statistical context, we identify the $\phi(\lambda, x)$ of Theorem 1.1 as the density function of a random variable X, with parameter λ, and $f(\mathbf{x})$ as the indicator function of some set in R_n. Equation (1.2) then states that, under certain conditions, the probability of an observation lying in this set is a Schur function of the parametric vector $\boldsymbol{\lambda}$. A typical application is the following result of Wong and Yue (1973).

LEMMA 1.2. *Let* $\mathbf{Z}(\mathbf{p}) = (Z_1(\mathbf{p}), \cdots, Z_n(\mathbf{p}))$ *be a multinomial random vector with parameters* $(N, \mathbf{p})$. *Let* $W(\mathbf{p}) = \{\#Z_i(\mathbf{p}) = 0\}$ *be the number of empty cells. Define* $f_k(\mathbf{p}) = P(W(\mathbf{p}) \geqq k)$. *Then, for* $k = 0, 1, \cdots, n, f_k(\mathbf{p})$ *is a Schur function of* $\mathbf{p}$, *that is,* $\mathbf{p} \geqq^{\mathrm{m}} \mathbf{p}'$ *implies*

$$P[W(\mathbf{p}) \geqq k] \geqq P[W(\mathbf{p}') \geqq k] .$$

A simple proof of this result and various generalizations to other multivariate distributions are obtained as consequences of Theorem 1.1. Theorem 1.1 has many applications to statistics. In Section 3 we briefly indicate some of these applications to total positivity, shock models, renewal theory, and moment inequalities.

2. Preservation of the Schur property. In this section we prove the Schur property preservation result stated in Theorem 1.1 above. We first prove the result for the bivariate case in Lemma 2.1. We recall that μ stands for Lebesgue measure on $[0, \infty)$ or counting measure on the nonnegative integers.

LEMMA 2.1. *Let* $\phi(\lambda, x)$ *be* TP_2, $\phi(\lambda, x) = 0$ *for* $x < 0$, *and* $\phi(\lambda, x)$ *satisfy the semigroup property. Let* $f(x_1, x_2)$ *be Schur-convex (Schur-concave). Let*

$$h(\lambda_1, \lambda_2) = \int_0^\infty \int_0^\infty f(x_1, x_2)\phi(\lambda_1, x_1)\phi(\lambda_2, x_2)\, d\mu(x_1)\, d\mu(x_2) , \tag{2.1}$$

where the integral is assumed to exist. Then $h(\lambda_1, \lambda_2)$ *is Schur-convex (Schur-concave).*

PROOF. We shall prove the result for f Schur-concave. The result for f Schur-convex will follow by considering $-f$.

Let $(\lambda_1, \lambda_2) \geqq^{\mathrm{m}} (\lambda_1', \lambda_2')$. Without loss of generality, take $\lambda_1 > \lambda_2$, $\lambda_1' > \lambda_2'$. Then

$$\begin{aligned} h(\lambda_1, \lambda_2) &- h(\lambda_1', \lambda_2') \\ &= \int\int [\phi(\lambda_1, x_1)\phi(\lambda_2, x_2) - \phi(\lambda_1', x_1)\phi(\lambda_2', x_2)]f(x_1, x_2)\, d\mu(x_1)\, d\mu(x_2) \\ &= \int \phi(\lambda_1 - \lambda_1', y) \int\int [\phi(\lambda_1', x_1 - y)\phi(\lambda_2, x_2) \\ &\qquad - \phi(\lambda_1', x_1)\phi(\lambda_2, x_2 - y)]f(x_1, x_2)\, d\mu(x_1)\, d\mu(x_2)\, d\mu(y) \end{aligned}$$

[using the semigroup property and the fact that $\lambda_1 - \lambda_1' = \lambda_2' - \lambda_2$]

$$= \int \phi(\lambda_1 - \lambda_1', y) \iint_{x_1 \geqq x_2} [f(x_1 + y, x_2) - f(x_1, x_2 + y)]$$
$$\times [\phi(\lambda_1', x_1)\phi(\lambda_2, x_2) - \phi(\lambda_1', x_2)\phi(\lambda_2, x_1)]\, d\mu(x_1)\, d\mu(x_2)\, d\mu(y)$$

[by a change of variables and the permutation-invariant property $f(x_1, x_2) = f(x_2, x_1)$. The special nature of the measure μ is also used in this step].

Now, $f(x_1 + y, x_2) - f(x_1, x_2 + y) \leqq 0$ since f is Schur-concave. Also, $\phi(\lambda_1', x_1)\phi(\lambda_2, x_2) - \phi(\lambda_1', x_2)\phi(\lambda_2, x_1) \geqq 0$ since ϕ is TP_2 and $\lambda_1' \geqq \lambda_2$ and $x_1 \geqq x_2$. Thus $h(\lambda_1, \lambda_2) - h(\lambda_1', \lambda_2') \leqq 0$, i.e., h is Schur-concave. □

The proof of Theorem 1.1 is now an easy consequence.

PROOF OF THEOREM 1.1. We shall prove the result for f Schur-concave. The result for f Schur-convex will follow by considering $-f$.

Let $\boldsymbol{\lambda} \geqq^{\mathrm{m}} \boldsymbol{\lambda}'$. By the result of Hardy, Littlewood, and Pólya (1952, page 47), there exists a finite sequence of vectors $\boldsymbol{\lambda}_0 \geqq^{\mathrm{m}} \boldsymbol{\lambda}_1 \geqq^{\mathrm{m}} \cdots \geqq^{\mathrm{m}} \boldsymbol{\lambda}_k$ such that $\boldsymbol{\lambda}_0 = \boldsymbol{\lambda}$, $\boldsymbol{\lambda}_k = \boldsymbol{\lambda}'$, and for each $i (i = 0, 1, \cdots, k-1)$, $\boldsymbol{\lambda}_i$ and $\boldsymbol{\lambda}_{i+1}$ differ in two coordinates only. We may therefore, without loss of generality, assume that $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$ differ in two coordinates only, i.e., $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n)$, $\boldsymbol{\lambda}' = (\lambda_1', \lambda_2', \lambda_3, \cdots, \lambda_n)$, and $(\lambda_1, \lambda_2) \geqq^{\mathrm{m}} (\lambda_1', \lambda_2')$. It follows by Lemma 2.1 that for fixed $x_3, \cdots, x_n$,

$$\prod_3^n \phi(\lambda_i, x_i) \iint \phi(\lambda_1, x_1)\phi(\lambda_2, x_2) f(\mathbf{x})\, d\mu(x_1)\, d\mu(x_2)$$
$$\leqq \prod_3^n \phi(\lambda_i, x_i) \iint \phi(\lambda_1', x_1)\phi(\lambda_2', x_2) f(\mathbf{x})\, d\mu(x_1)\, d\mu(x_2)\,.$$

Integrating each side of the inequality with respect to $d\mu(x_3) \cdots d\mu(x_n)$, we conclude that $h(\boldsymbol{\lambda}) \leqq h(\boldsymbol{\lambda}')$. It follows that $h(\boldsymbol{\lambda}) \leqq h(\boldsymbol{\lambda}')$, i.e., $h(\boldsymbol{\lambda})$ is Schur-concave. □

REMARKS.

1. The requirement $\phi(\lambda, x)$ be TP_2 cannot in general be dropped from the hypotheses of Theorem 1.1. Counterexamples are easy to construct; one such can be extracted from the counterexample in Section 4 of Part II.

2. Let Λ be a semigroup contained in $(0, \infty)$. For instance, Λ could be the set of positive integers. A function $\phi(\lambda, x)$ defined on $\Lambda \times [0, \infty)$ is said to satisfy the semigroup property on Λ if (1.1) holds for $\lambda_1, \lambda_2 \in \Lambda$. A function $h(\lambda_1, \cdots, \lambda_n)$ defined on $\Lambda \times \cdots \times \Lambda$ is said to be Schur-convex if $h(\boldsymbol{\lambda}) \geqq h(\boldsymbol{\lambda}')$ whenever $\boldsymbol{\lambda}, \boldsymbol{\lambda}' \in \Lambda \times \cdots \times \Lambda$ and $\boldsymbol{\lambda} \geqq^{\mathrm{m}} \boldsymbol{\lambda}'$. Theorem 1.1 still holds if the first sentence in it is changed to "Let $\phi(\lambda, x)$ defined on $\Lambda \times [0, \infty)$ be TP_2 and satisfy the semigroup property on Λ." Corollary 3.3 uses this slight extension of Theorem 1.1.

3. Applications. We now give a few illustrative applications of Theorem 1.1 in the areas of total positivity, shock models, renewal theory and moment inequalities.

Consider the following shock model. A device is subject to n types of shocks. Let the probability of surviving k_1 shocks of type $1, \cdots, k_n$ shocks of type n be

$\bar{P}_{k_1,\cdots,k_n}$. Assume that shocks of type i occur according to a Poisson process having rate λ_i, $i = 1, \cdots, n$, with the n processes mutually independent. Then the probability $\bar{H}_t(\boldsymbol{\lambda})$ of surviving until time t is given by

$$\bar{H}_t(\boldsymbol{\lambda}) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} \prod_{i=1}^{n} [e^{-\lambda_i t}(\lambda_i t)^{k_i}/k_i!]\bar{P}_{k_1\cdots k_n} . \tag{3.1}$$

An application of Theorem 1.1 yields the following result:

COROLLARY 3.1. *Let* $\bar{P}_{k_1,\cdots,k_n}$ *be Schur-concave (Schur-convex) in* $\mathbf{k}$. *Then for fixed* $t > 0$, $\bar{H}_t(\boldsymbol{\lambda})$ *given in* (3.1) *is Schur-concave (Schur-convex) in* $\boldsymbol{\lambda}$.

For an example of a survival probability satisfying the conditions of Corollary 3.1, consider

$$\bar{P}_{k_1,\cdots,k_n} = \prod_{i=1}^{n} \bar{P}_{k_i}^* ,$$

where $\bar{P}_k^*$ is log concave in k (i.e., the independence case). It follows that $\bar{P}_{k_1,\cdots,k_n}$ is Schur-concave, and thus $\bar{H}_t(\boldsymbol{\lambda})$ is Schur-concave in $\boldsymbol{\lambda}$. Since

$$\bar{H}_t(\boldsymbol{\lambda}) = \prod_{i=1}^{n} \bar{H}_t^*(\lambda_i) ,$$

where

$$\bar{H}_t^*(\boldsymbol{\lambda}) = \sum_{k=0}^{\infty} \bar{P}_k^* e^{-\lambda t}(\lambda t)^k/k! ,$$

it follows that $\bar{H}_t^*(\lambda)$ is log concave in λ. This conclusion corresponds to (3.2) of Theorem 3.1 of Esary, Marshall and Proschan (1973).

COROLLARY 3.2. *Let* X_{ij}, $i = 1, \cdots, n$; $j = 1, 2, \cdots,$ *be independently and identically distributed according to a log-concave density* g *with support* $[0, \infty)$. *Let* $f(u_1, \cdots, u_n)$ *be Schur-concave. Define* $h(\mathbf{k}) = Ef(\sum_{j_1=1}^{k_1} X_{1j_1}, \cdots, \sum_{j_n=1}^{k_n} X_{nj_n})$. *Then* $h(\mathbf{k})$ *is Schur-concave.*

This result follows from Theorem 1.1 by expressing $h(k)$ as shown below:

$$h(\mathbf{k}) = \int \cdots \int \prod_{i=1}^{n} g^{(k_i)}(u)f(u_1, \cdots, u_n)\, du_1 \cdots du_n ,$$

where $g^{(k)}(u)$, the k-fold convolution of $g(u)$, satisfies the semigroup property: $g^{(r+s)}(u) = \int g^{(r)}(u-x)g^{(s)}(x)\, dx$ and is TP_2 in $k = 1, 2, \cdots$ and $u \geqq 0$ (Karlin and Proschan, 1960, Theorem 1).

An interesting special case of Corollary 3.2 may be stated as follows:

COROLLARY 3.2a. *Let* X_{ij}, $i = 1, \cdots, n$; $j = 1, 2, \cdots,$ *be independently and identically distributed according to a log-concave density* g *with support* $[0, \infty)$. *Let* S *be a set in* E^n *such that* $\mathbf{x} \in S$ *and* $\mathbf{x} \geqq^{\mathrm{m}} \mathbf{x}'$ *imply* $x' \in S$. *Then* $\mathbf{k} \geqq^{\mathrm{m}} \mathbf{k}'$ *implies*

$$P[(\sum_1^{k_1} X_{1i}, \cdots, \sum_1^{k_n} X_{ni}) \in S] \leqq P[(\sum_1^{k_1'} X_{1i}, \cdots, \sum_1^{k_n'} X_{ni}) \in S] . \tag{3.2}$$

Corollary 3.2a follows from Corollary 3.2 by choosing $f(\mathbf{x}) = 1$ if $\mathbf{x} \in S$ and $= 0$ otherwise, and noticing that f is Schur-concave.

In our next applications, we obtain inequalities for moments of a class of multivariate distributions.

COROLLARY 3.3. *Let* $g(\mathbf{x})$ *be a Schur-concave density, with* $g(\mathbf{x}) = 0$ *if any of* $x_1, \cdots, x_n$ *is negative.*

For $0 < \alpha_1 < \infty, \cdots, 0 < \alpha_n < \infty$, *let*

$$M(\boldsymbol{\alpha}) = \int \cdots \int \frac{x_1^{\alpha_1 - 1}}{\Gamma(\alpha_1)} \cdots \frac{x_n^{\alpha_n - 1}}{\Gamma(\alpha_n)} g(\mathbf{x})\, dx_1 \cdots dx_n \tag{3.3}$$

be a corresponding normalized multivariate moment. Then $M(\boldsymbol{\alpha})$ *is Schur-concave, where it is finite.*

PROOF. In Theorem 1.1, choose $\phi(\alpha, x) = e^{-x}x^{\alpha-1}/\Gamma(\alpha)$ for $x \geqq 0$, $\alpha > 0$, and $= 0$ elsewhere. Then $\phi(\alpha, x)$ is TP_2 and satisfies the semigroup property, as is easily verified. Let $f(\mathbf{x}) = g(x)\exp(\sum_1^n x_i)$. Then $f(\mathbf{x})$ is Schur-concave. The derived conclusion follows immediately from Theorem 1.1.

Corollary 3.3 generalizes Theorem 1 of Karlin, Proschan and Barlow (1961) for the univariate case when the density is log-concave.

REMARK 3.4. Corollary 3.3 yields Schur-concavity for the normalized moment $M(\boldsymbol{\alpha})$ under the assumption of Schur-concavity of the density. The result derives additional interest when we note that the ordinary (nonnormalized) multivariate moment

$$m(\boldsymbol{\alpha}) = \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^n t_i^{\alpha_i} f(\mathbf{t})\, dt_1 \cdots dt_n \tag{3.4}$$

is Schur-convex for any density $f(\mathbf{t})$ on $[0, \infty) \times \cdots \times [0, \infty)$ which is invariant under permutation. See Tong (1976).

A closely related result for multivariate Laplace transforms may be similarly obtained.

COROLLARY 3.5. *Let f be a nonnegative, integrable function on* $[0, \infty) \times \cdots \times [0, \infty)$ *which is invariant under permutation. Let*

$$f^*(\mathbf{s}) = \int_0^\infty \cdots \int_0^\infty e^{-\sum_i^n s_i t_i} f(\mathbf{t})\, dt_1 \cdots dt_n \tag{3.5}$$

be defined and finite for $s_i^0 \leqq s_i < \infty$, $i = 1, \cdots, n$. *Then f^* is Schur-convex on* $[s_1^0, \infty) \times \cdots \times [s_n^0, \infty)$.

Finally, Schur properties for binomial moments may be obtained as in Corollary 3.3, Remark 3.4 and Corollary 3.5. This is a consequence of the fact that $\phi(i, n) = \binom{n-1}{i-1}$ and $\phi^*(n, i) = \binom{n}{i}$ are TP_2. We leave the details to the reader.

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