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WEAK CONVERGENCE OF PROGRESSIVELY CENSORED LIKELIHOOD RATIO STATISTICS AND ITS ROLE IN ASYMPTOTIC THEORY OF LIFE TESTING[1]

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Progressive censoring schemes are often adopted in clinical trials and life testing problems with a view to monitoring the experiment from the start with the objective of a possible early termination of the experiment depending on the cumulative evidence at its various stages. Along with a basic martingale property, a Wiener process approximation for progressively censored likelihood ratio statistics is established here and the same is incorporated in the formulation of some asymptotic sequential tests for the life testing problem.

1. Introduction. For n ($\geqq 1$) items under a life test, the smallest observation comes first, the second smallest second, and so on, until the largest observation emerges last. Thus, the entire span of experimentation may be quite time consuming (as well as costly), and hence, the experiment is often carried out either for a specified length of time (*truncation*) or until a prespecified proportion of the subjects respond (*censoring*). In most practical problems, a single point of truncation or censoring may lead to considerable loss of efficiency; a too early termination usually leads to an increased risk of making incorrect statistical decisions, while unnecessary prolongation may involve greater amount of cost and time as well as more sacrifice of lives of the experimental units without any significant increase in the sensitivity of the experiment. Also, for the censoring scheme, the actual period of experimentation is a random variable and may run into conflict with the other limitations on time as set by other practical considerations on the study. For these reasons, in clinical trials and life testing problems, *progressively censoring schemes* (PCS) are often adopted to monitoring the experiment from the start with the objective of an early termination whenever feasible, and this usually leads to a substantial amount of savings of time, cost and lives of the experimental units. By constitution, the statistical procedure based on a continuous updating of the data in a PCS is essentially sequential in nature and this introduces additional complications in its formulation.

The current paper is devoted to a basic invariance principle for *progressively censored likelihood ratio statistics* (PCLRS) for a broad class of survival distributions.

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Along with the preliminary notions, the main theorem is formulated in Section 2. Section 3 is devoted to the proof of the main theorem. Section 4 deals with a class of asymptotic (sequential) tests for the life testing problem based on the basic invariance principle for the PCLRS. A potential reader interested only in the applications may look carefully in Section 2 and then proceed to Section 4.

2. Preliminary notions and the main theorem. Let $\{X_i, i \geqq 1\}$ be a sequence of independent and identically distributed random variables (i.i.d. rv) with a continuous probability density function (pdf) $f_\theta(x)$ and distribution function (df) $F_\theta(x)$, $x \in R$, the real line $(-\infty, \infty)$, $\theta \in \Theta \subset R$. In a life testing problem, the X_i are nonnegative, so that $F_\theta(0) = 0$ for all $\theta \in \Theta$. We desire to test the null hypothesis

$$(2.1) \qquad H_0: \theta = \theta_0 \quad \text{(specified), against one or two-sided alternatives.}$$

For $n\,(\geqq 1)$ items under a life test, the observable random variables are $X_{n,1} < \cdots < X_{n,n}$, the order statistics corresponding to $X_1, \cdots, X_n$; by virtue of the assumed continuity of F, ties among the X_i (and hence, $X_{n,i}$) can be neglected in probability. In a censored plan, for some fixed r $(1 \leqq r \leqq n)$, the experiment is terminated at $X_{n,r}$ and the test for H_0 in (2.1) is then based on $(X_{n,1}, \cdots, X_{n,r})$. In a truncated scheme, the experiment is continued for a predetermined length of time T $(0 < T < \infty)$, and if r^* $(0 \leqq r^* \leqq n)$ of the observations lie in the interval $[0, T]$, the test is based on them; if $r^* = 0$, one works with the probability $\{1 - F_\theta(T)\}^n$, while for $r^* \geqq 1$, with $X_{n,1}, \cdots, X_{n,r^*}$. In a PCS, we allow the possibility of terminating the experiment prior to $X_{n,r}$ through monitoring from the beginning. Here, if for some k $(\leqq r)$, $(X_{n,1}, \cdots, X_{n,k})$ advocate a clear statistical decision in favor of either of the hypotheses, the experiment is stopped following $X_{n,k}$. Thus, both the *stopping number* k and the *stopping time* $X_{n,k}$ are stochastic variables.

We denote by

$$(2.2) \qquad Z_k = X_{n,k}\,, \quad \mathbf{Z}^{(k)} = (Z_1, \cdots, Z_k)\,, \quad 1 \leqq k \leqq n\,; \qquad Z_0 = \mathbf{Z}^{(0)} = 0\,.$$

Then, for every k: $1 \leqq k \leqq n$, the (joint) pdf of $\mathbf{Z}^{(k)}$ is

$$(2.3) \qquad \begin{aligned} p_\theta(\mathbf{z}^{(k)}, n) &= n^{[k]}\{\textstyle\prod_{i=1}^k f_\theta(z_i)\}\{1 - F_\theta(z_k)\}^{n-k}\,; \\ n^{[k]} &= n \cdots (n-k+1)\,, \end{aligned}$$

and (2.3) is defined on the domain $A_k^{(n)} = \{\mathbf{z}^{(k)}: 0 < z_1 < \cdots < z_k < \infty\}$. In the sequel we relate the censoring number r to n by letting $r/n \to p$ as $n \to \infty$, where $p \in (0, 1]$. Also, we assume that Θ is an open interval $(\subset R)$, $f_\theta(x) > 0$ for every $x \in R^+$, $\theta \in \Theta$, $f_\theta(x)$ is a continuously (twice) differentiable function of θ and for every $\theta \in \Theta$,

$$(2.4) \qquad |(\partial^i/\partial\theta^i) f_\theta(x)| \leqq U_i(x) \quad \text{where} \quad \textstyle\int_{-\infty}^\infty U_i(x)\,dx < \infty\,, \qquad i = 1, 2\,;$$

$$(2.5) \qquad 0 < J(\theta) = \textstyle\int_0^\infty \dot{f}_\theta^2(x)\,dF_\theta(x) < \infty\,,$$

where for a nonnegative $h_\theta(x)$, $\dot{h}_\theta(x) = (\partial/\partial\theta) \log h_\theta(x)$ and $\ddot{h}_\theta(x) = (\partial/\partial\theta)\dot{h}_\theta(x)$.

Let $r_\theta(x) = f_\theta(x)/\{1 - F_\theta(x)\}$, $x \in R^+$ be the *hazard function* and assume that $\forall\, \theta \in \Theta$,

$$|(\partial^i/\partial\theta^i)r_\theta(x)| \leqq U_i^*(x) \quad \text{where} \quad \int_{-\infty}^{\infty} U_i^*(x)\,dx < \infty\,, \qquad i = 1, 2\,. \tag{2.6}$$

Note that by assumption $f_\theta(x) > 0$ for all $x \in R^+$, $\theta \in \Theta$, so that

$$r_\theta(x) > 0 \quad \text{whenever} \quad 0 < F_\theta(x) < 1\,. \tag{2.7}$$

We assume that for all $\theta \in \Theta$,

$$\int_0^\infty \dot{r}_\theta^4(x)\,dF_\theta(x) < \infty \quad \text{and} \quad \int_0^\infty \ddot{r}_\theta^2(x)\,dF_\theta(x) < \infty\,. \tag{2.8}$$

Then, if we denote by $\tilde{r}_\theta(x) = \{r_\theta(x)\}^{-1}(\partial^2/\partial\theta^2)r_\theta(x) = \dot{r}_\theta^2(x) + \ddot{r}_\theta(x)$, we have

$$\int_0^\infty \tilde{r}_\theta^2(x)\,dF_\theta(x) < \infty\,, \qquad \text{for all} \quad \theta \in \Theta\,. \tag{2.9}$$

Further, let us denote by $G_\theta(x) = 1 - F_\theta(x)$ and define the logarithmic derivatives $\dot{G}_\theta(x)$ and $\ddot{G}_\theta(x)$ as before. Note that $\dot{G}_\theta(x) = \dot{f}_\theta(x) - \dot{r}_\theta(x)$ and $\ddot{G}_\theta(x) = \ddot{f}_\theta(x) - \ddot{r}_\theta(x)$, so that by (2.5) and (2.8),

$$|E_\theta \ddot{G}_\theta(X)| < \infty \qquad \text{for all} \quad \theta \in \Theta\,. \tag{2.10}$$

Finally, for every $\alpha \in (0, 1)$, let

$$J_\alpha(\theta) = \int_0^{F_\theta^{-1}(\alpha)} \dot{f}_\theta^2(x)\,dF_\theta(x) + (1-\alpha)^{-1}\Big\{\int_{F_\theta^{-1}(\alpha)}^\infty \dot{f}_\theta(x)\,dF_\theta(x)\Big\}^2\,, \qquad \theta \in \Theta\,, \tag{2.11}$$

so that for every $\theta \in \Theta$, $J_\alpha(\theta)$ is $\nearrow$ in $\alpha \in (0, 1)$ and $\lim_{\alpha\to 1} J_\alpha(\theta) = J(\theta)$.

We are primarily concerned with an invariance principle for the partial sequence

$$\lambda_{n,k} = \dot{p}_\theta(\mathbf{Z}^{(k)}, n)\,, \qquad k = 1, \cdots, n \quad \text{and} \quad \lambda_{n,0} = 0\,. \tag{2.12}$$

To formulate this, we define

$$J_{n,0}(\theta) = 0 \quad \text{and} \quad J_{n,k}(\theta) = E_\theta\{\lambda_{n,k}^2\}\,, \quad \text{for} \quad k = 1, \cdots, n\,. \tag{2.13}$$

Note that $J_{n,k}(\theta)$ is nondecreasing in k ($1 \leqq k \leqq n$) for every $\theta \in \Theta$. For every (n, r) ($n \geqq r \geqq 1$), consider a stochastic process $W_{n,r} = W_{n,r}^\theta = \{W_{n,r}^\theta(t), 0 \leqq t \leqq 1\}$ by introducing a sequence of integer-valued, nondecreasing and right-continuous functions $\{k_n(t), 0 \leqq t \leqq 1\}$ where

$$k_n(t) = \max\{k : J_{n,k}(\theta) \leqq tJ_{n,r}(\theta)\}\,, \qquad 0 \leqq t \leqq 1 \tag{2.14}$$

and then letting

$$W_{n,r}(t) = \lambda_{n,k_n(t)}/J_{n,r}^{\frac{1}{2}}(\theta)\,, \qquad 0 \leqq t \leqq 1\,. \tag{2.15}$$

Then, for every $n \geqq r \geqq 1$, $W_{n,r}$ belongs to the $D[0, 1]$ space, endowed with the Skorokhod J_1 topology. Then, the main theorem of the paper is the following.

THEOREM 1. *Under the assumptions made before, whenever $r/n \to p(\in (0, 1])$ and $J_p(\theta) > 0$,*

$$W_{n,r} \to_{\mathscr{D}} W\,, \quad \textit{in the Skorokhod} \quad J_1 \quad \textit{topology on} \quad D[0, 1]\,, \tag{2.16}$$

where $W = \{W(t),\ 0 \leqq t \leqq 1\}$ *is a standard Wiener process on the unit interval* $[0, 1]$.

The proof of the theorem is considered in Section 3. Also, under some additional regularity conditions, the result is extended in Section 3 for contiguous alternatives (viz., Theorem 2). The latter result is useful for the study of the asymptotic power properties of the tests considered in Section 4.

3. Proof of the main theorem. First, we consider the following three lemmas which are needed in the main proof. Let $\mathscr{B}_{n,k}$ be the sigma-field generated by $\mathbf{Z}^{(k)}$, for $k = 1, \cdots, n$ and $\mathscr{B}_{n,0}$ be the trivial sigma-field. Then, for every $n\ (\geqq 1)$, $\mathscr{B}_{n,k}$ is nondecreasing in $k: 0 \leqq k \leqq n$. Also, define the $\lambda_{n,k}$ as in (2.12).

LEMMA 3.1. *For every* $n\ (\geqq 1)$, $\{\lambda_{n,k}, \mathscr{B}_{n,k}, 0 \leqq k \leqq n\}$ *is a martingale.*

PROOF. Given $\mathscr{B}_{n,k-1}$, the conditional pdf of Z_k (defined for $Z_k \geqq z_{k-1}$) is given by

$$q_\theta(Z_k \mid \mathscr{B}_{n,k-1}) = (n-k+1) f_\theta(Z_k)\{1 - F_\theta(Z_k)\}^{n-k}/\{1 - F_\theta(z_{k-1})\}^{n-k+1}, \tag{3.1}$$

so that for every $k: 1 \leqq k \leqq n$,

$$p_\theta(\mathbf{Z}^{(k)}, n) = p_\theta(\mathbf{Z}^{(k-1)}, n) q_\theta(Z_k \mid \mathscr{B}_{n,k-1}) = \prod_{i=1}^k q_\theta(Z_i \mid \mathscr{B}_{n,i-1}); \tag{3.2}$$

$$\lambda_{n,k} = \sum_{i=1}^k \lambda_{n,i}^*; \qquad \lambda_{n,k}^* = \dot{q}_\theta(Z_k \mid \mathscr{B}_{n,k-1}), \qquad k = 1, \cdots, n. \tag{3.3}$$

Now, assumptions (2.4) through (2.8) insure the integrability of $\lambda_{n,k}^*$ and the differentiability under the integral sign, so that for every $1 \leqq k \leqq n$,

$$\begin{aligned} E\{\lambda_{n,k}^* \mid \mathscr{B}_{n,k-1}\} &= \textstyle\int_{z_{k-1}}^\infty \{(\partial/\partial\theta) \log q_\theta(z \mid \mathscr{B}_{n,k-1})\} q_\theta(z \mid \mathscr{B}_{n,k-1})\, dz \\ &= (\partial/\partial\theta) \textstyle\int_{z_{k-1}}^\infty q_\theta(z \mid \mathscr{B}_{n,k-1})\, dz = 0. \end{aligned} \tag{3.4}$$

Hence, the lemma follows from (3.3) and (3.4). □

LEMMA 3.2 *Under the regularity conditions of Section* 2, $r/n \to \alpha(\in (0, 1])$ *insures that*

$$n^{-1} J_{n,r}(\theta) \to J_\alpha(\theta) \qquad \text{for all} \quad \theta \in \Theta. \tag{3.5}$$

PROOF. For $r = n$, we note that $J_{n,n}(\theta) = -E_\theta \ddot{p}_\theta(\mathbf{Z}^{(n)}, n) = \sum_{i=1}^n E_\theta\{-\ddot{f}_\theta(X_i)\} = nJ(\theta)$. So, we need to prove the result for $\alpha < 1$, where for large n, we may take $r \leqq n-1$. Note that by (2.3), for $r < n$,

$$J_{n,r}(\theta) = -E_\theta \ddot{p}_\theta(\mathbf{Z}^{(r)}, n) = \sum_{i=1}^r E_\theta\{-\ddot{f}_\theta(Z_i)\} + (n-r) E_\theta\{-\ddot{G}_\theta(Z_r)\}. \tag{3.6}$$

Hence, for $r \leqq n-1$,

$$\begin{aligned} n^{-1} \textstyle\sum_{i=1}^r E_\theta\{-\ddot{f}_\theta(Z_i)\} &= n^{-1} \textstyle\sum_{i=1}^r E_\theta(E\{-\ddot{f}_\theta(Z_i) \mid Z_{r+1}\}) \\ &= \frac{(n-1)!}{(r-1)!(n-r-1)!} \textstyle\int_0^\infty [\int_0^z \{-\ddot{f}_\theta(x)\}\, dF_\theta(x)] \\ &\quad \times \{F_\theta(z)^{r-1}\}\{1 - F_\theta(z)\}^{n-r-1}\, dF_\theta(z) \\ &= E_\theta\{\gamma(X_{n-1,r})\}; \quad \gamma(x) = \textstyle\int_0^x \{-\ddot{f}_\theta(y)\}\, dF_\theta(y), \end{aligned} \tag{3.7}$$

where $X_{n-1,r}$ is the rth order statistics of a sample of size $n-1$ from the df F_θ. Thus, by the moment convergence of sample quantiles [viz., Sen (1959), Sarkadi (1974)] and (2.5), the rhs of (3.7) converges (as $n \to \infty$) to

$$-\int_0^{F_\theta^{-1}(\alpha)} \ddot{f}_\theta(x)\,dF_\theta(x) = -\int_0^{F_\theta^{-1}(\alpha)} \{(\partial^2/\partial\theta^2) f_\theta(x)\}\,dx + \int_0^{F_\theta^{-1}(\alpha)} \dot{f}_\theta^2(x)\,dF_\theta(x)\,. \tag{3.8}$$

The same moment convergence result of the sample quantiles implies that as $n \to \infty$,

$$\begin{aligned} n^{-1}(n-r)E_\theta\{-\ddot{G}_\theta(Z_r)\} &\to -(1-\alpha)\ddot{G}_\theta(F_\theta^{-1}(\alpha)) \\ &= -\int_{F_\theta^{-1}(\alpha)}^\infty \ddot{f}_\theta(x)\,dF_\theta(x) + (1-\alpha)^{-1}\{\int_{F_\theta^{-1}(\alpha)}^\infty \dot{f}_\theta(x)\,dx\}^2\,. \end{aligned} \tag{3.9}$$

Then, (3.5) follows from (3.7), (3.8), (3.9) and (2.11). □

Let $\{Y_i, i \geqq 1\}$ be a sequence of i.i.d. nonnegative rv's with a continuous pdf $h(y)$, $y \in R^+$, where we assume that

$$0 < h(0) = \lim_{y\downarrow 0} h(y) < \infty\,. \tag{3.10}$$

Also, let $g = \{g(y), y \in R^+\}$ be such that in some neighbourhood of the origin, $g(y)$ has a continuous first derivative $g'(y)$ and $\lim_{y\downarrow 0} g'(y) = g'(0)$ exists and further for some $d > 0$, $E|g(Y)|^d < \infty$. Finally, let $Y_{n,1} = \min(Y_1, \cdots, Y_n)$, $n \geqq 1$.

LEMMA 3.3. *Under the assumptions made above, $E\{|ng(Y_{n,1})|^k\}$ exists for every k: $0 \leqq k \leqq nd$, and for every (fixed) a ($\geqq 0$),*

$$\lim_{n\to\infty} E\{[n|g(Y_{n,1}) - g(0)|]^a\} = \underline{|a+1}\{|g'(0)|/h(0)\}^a\,. \tag{3.11}$$

The proof follows along the lines of the proof of Theorem 3.1 of Sen (1961), and hence is omitted.

Let us now proceed on the proof of Theorem 1. We define

$$\sigma_{n,k}^2 = E\{\lambda_{n,k}^{*2} \mid \mathscr{B}_{n,k-1}\}\,, \quad k = 1, \cdots, n\,; \qquad \sigma_{n,0}^2 = 0 \quad \text{and} \tag{3.12}$$

$$V_{n,k} = \sum_{s=0}^k \sigma_{n,s}^2\,, \quad k = 0, \cdots, n\,.$$

Then

$$E_\theta \sigma_{n,k}^2 = J_{n,k}^*(\theta) = E_\theta \lambda_{n,k}^{*2} \quad \text{and} \tag{3.13}$$

$$E_\theta V_{n,k} = J_{n,k}(\theta) = \sum_{s=0}^k J_{n,s}^*(\theta)\,, \quad k = 0, \cdots, n\,.$$

To prove (2.16), we verify the conditions of Corollary (3.8) of McLeish (1974); by our Lemma 3.1, we need to prove only that as $n \to \infty$,

$$V_{n,k_n(t)}/J_{n,r}(\theta) \to_p t\,, \qquad \text{for every} \quad t \in [0, 1]\,, \tag{3.14}$$

$$\sum_{k=1}^r E\{\lambda_{n,k}^{*2} I(|\lambda_{n,k}^*| > \varepsilon J_{n,r}^{\frac12}(\theta))\}/J_{n,r}(\theta) \to 0\,, \qquad \text{for every} \quad \varepsilon > 0\,, \tag{3.15}$$

where $r/n, \to p$, $k_n(t)$ is defined by (2.14), $I(A)$ stands for the indicator function of the set A and the conditional Lindeberg condition in McLeish (1974) is replaced here by (3.15) for its relative simplicity of verification. Since

$\sigma^2_{n,k} = E\{-\ddot{q}_\theta(Z_k \mid \mathscr{B}_{n,k-1}) \mid \mathscr{B}_{n,k-1}\}$, we have for every $k: 1 \leqq k \leqq r\,(\leqq n)$,

$$\begin{aligned} n^{-1}V_{n,k} &= n^{-1} \textstyle\sum_{s=1}^k E(-\ddot{f}_\theta(Z_s) \mid \mathscr{B}_{n,s-1}) \\ &\quad - n^{-1} \textstyle\sum_{s=1}^k (n-s)E(-\ddot{G}_\theta(Z_s) \mid \mathscr{B}_{n,s-1}) \\ &\quad + n^{-1} \textstyle\sum_{s=1}^k (n-s+1)\ddot{G}_\theta(Z_{s-1})\,; \qquad Z_0 = 0\,, \end{aligned} \tag{3.16}$$

which can be rewritten as

$$\begin{aligned} &-n^{-1} \textstyle\sum_{s=1}^k \ddot{f}_\theta(Z_s) + n^{-1} \sum_{s=1}^k \{\ddot{f}_\theta(Z_s) - E(\ddot{f}_\theta(Z_s) \mid \mathscr{B}_{n,s-1})\} \\ &\quad - n^{-1}(n-k)\ddot{G}_\theta(Z_k) \\ &\quad + n^{-1} \textstyle\sum_{s=1}^k (n-s)\{\ddot{G}_\theta(Z_s) - E(\ddot{G}_\theta(Z_s) \mid \mathscr{B}_{n,s-1})\}\,. \end{aligned} \tag{3.17}$$

Thus, by virtue of (2.14) and Lemma 3.2, to prove (3.14) it suffices to show that for every $\alpha: 0 < \alpha < 1$ and $k/n \to \alpha$, as $n \to \infty$,

$$-n^{-1} \textstyle\sum_{s=1}^k \ddot{f}_\theta(Z_s) \to_p \int_0^{F_\theta^{-1}(\alpha)} \{-\ddot{f}_\theta(x)\}\, dF_\theta(x)\,, \tag{3.18}$$

$$n^{-1} \textstyle\sum_{s=1}^k \{\ddot{f}_\theta(Z_s) - E(\ddot{f}_\theta(Z_s) \mid \mathscr{B}_{n,s-1})\} \to_p 0\,, \tag{3.19}$$

$$n^{-1} \textstyle\sum_{s=1}^k (n-s)\{\ddot{G}_\theta(Z_s) - E(\ddot{G}_\theta(Z_s) \mid \mathscr{B}_{n,s-1})\} \to_p 0\,, \tag{3.20}$$

$$n^{-1}(n-k)\ddot{G}_\theta(Z_k) \to_p (1-\alpha)\ddot{G}_\theta(F_\theta^{-1}(\alpha))\,. \tag{3.21}$$

Now, (3.21) follows from the stochastic convergence of Z_k to $F_\theta^{-1}(\alpha)$ and the continuity of $\ddot{G}_\theta(x)$ (in x). Also, on denoting by $F_n(x)$ the empirical df based on $X_1, \cdots, X_n$, we may rewrite (3.18) as

$$\textstyle\int_0^{F_n^{-1}(k/n)} \{-\ddot{f}_\theta(x)\}\, dF_n(x); \; F_n^{-1}(t) = \inf\{x: F_n(x) \geqq t\}\,. \tag{3.22}$$

Since $F_n^{-1}(k/n)$ converges in probability to $F_\theta^{-1}(\alpha)$ and by the law of large numbers,

$$\begin{aligned} \textstyle\int_0^{F_\theta^{-1}(\alpha)} \ddot{f}_\theta(x)\, dF_n(x) &= n^{-1} \textstyle\sum_{i=1}^n \ddot{f}_\theta(X_i) I(X_i \leqq F_\theta^{-1}(\alpha)) \\ &\to_p \textstyle\int_0^{F_\theta^{-1}(\alpha)} \ddot{f}_\theta(x)\, dF_\theta(x)\,, \end{aligned} \tag{3.23}$$

the proof of (3.18) follows readily from (3.22), (3.23) and some standard steps. Hence, we need to prove (3.19) and (3.20). Note that, by constitution, the summands in (3.19) or (3.20) are mutually orthogonal, so it suffices to show that

$$n^{-2} \textstyle\sum_{s=1}^k E\{\ddot{f}_\theta(Z_s) - E(\ddot{f}_\theta(Z_s) \mid \mathscr{B}_{n,s-1})\}^2 \to 0 \quad \text{as} \quad n \to \infty\,, \tag{3.24}$$

$$n^{-2} \textstyle\sum_{s=1}^k (n-s)^2 E\{\ddot{G}_\theta(Z_s) - E(\ddot{G}_\theta(Z_s) \mid \mathscr{B}_{n,s-1})\}^2 \to 0 \quad \text{as} \quad n \to \infty\,. \tag{3.25}$$

We prove only (3.25); the proof of (3.24) follows on parallel lines.

Let now $\tilde{X}_1, \cdots, \tilde{X}_{n-s+1}$ be i.i.d. rv's with a df $\tilde{F}(x) = \{F_\theta(x) - F_\theta(Z_{s-1})\}/\{1 - F_\theta(Z_{s-1})\}$ for $x \geqq Z_{s-1}$ and 0 otherwise. Then, $\tilde{f}(x) = (d/dx)\tilde{F}(x) = f_\theta(x)/\{1 - F_\theta(Z_{s-1})\}$, $x \geqq Z_{s-1}$, and hence, by (2.7), $\tilde{f}(Z_{s-1}) = r_\theta(Z_{s-1}) > 0$ a.s. Also, given $\mathscr{B}_{n,s-1}$, Z_s has the same (conditional) df as of $\min(\tilde{X}_1, \cdots, \tilde{X}_{n-s+1})$ and further,

$$\begin{aligned} &(n-s)^2 E(\ddot{G}_\theta(Z_s) - E(\ddot{G}_\theta(Z_s) \mid \mathscr{B}_{n,s-1})\}^2 \\ &\qquad \leqq (n-s)^2 E\{[\ddot{G}_\theta(Z_s) - \ddot{G}_\theta(Z_{s-1})]^2 \mid \mathscr{B}_{n,s-1}\}\,, \end{aligned} \tag{3.26}$$

and, finally,

$$(3.27)\qquad (d/dx)\ddot{G}_\theta(x) = r_\theta(x)\{\dot{r}_\theta^2(x) + \ddot{r}_\theta(x)\}\,, \qquad \text{for every} \quad x \in R^+\,.$$

Consequently, by (2.8), Lemma 3.3 and (3.27), we obtain by standard arguments that the rhs of (3.26) exists (a.s.) for every $n \geqq r \geqq 1$, and as $n \to \infty$, it is convergent equivalent (a.s.) to

$$(3.28)\qquad 2\{\dot{r}_\theta^2(Z_{s-1}) + \ddot{r}_\theta(Z_{s-1})\}^2 = 2\tilde{r}_\theta^2(Z_{s-1})\,.$$

Also, we note that

$$(3.29)\qquad \begin{aligned} n^{-1}\textstyle\sum_{s=1}^{k} E_\theta\{\tilde{r}_\theta^2(Z_{s-1})\} &= n^{-1}\textstyle\sum_{s=1}^{k-1} E_\theta\{\tilde{r}_\theta(Z_s)\} \leqq n^{-1}\textstyle\sum_{s=1}^{n} E_\theta\{\tilde{r}_\theta(Z_s)\} \\ &= \textstyle\int_0^\infty \tilde{r}_\theta^2(x)\, dF_\theta(x) < \infty\,, \quad \text{by (2.9)}. \end{aligned}$$

From (3.26) through (3.29), we conclude that (3.25) holds. Hence, the proof of (3.14) is complete.

To prove (3.15), by virtue of Lemma 3.2, it suffices to show that under the hypothesis of the theorem, for some $m > 2$,

$$(3.30)\qquad \limsup\,\{n^{-1}\textstyle\sum_{k=1}^{r} E_\theta|\lambda_{n,k}^*|^m\} < \infty\,.$$

Note that by definition of the $\lambda_{n,k}^*$, for $m > 2$,

$$(3.31)\qquad |\lambda_{n,k}^*|^m \leqq 2^{m-1}\{|\dot{r}_\theta(Z_k)|^m + |(n-k+1)\{\dot{G}_\theta(Z_k) - \dot{G}_\theta(Z_{k-1})\}|^m\}\,,$$

where for every $n \geqq r \geqq 1$, $m \leqq 4$,

$$(3.32)\qquad n^{-1}\textstyle\sum_{k=1}^{r} E_\theta|\dot{r}_\theta(Z_k)|^m \leqq n^{-1}\textstyle\sum_{i=1}^{n} E_\theta|\dot{r}_\theta(Z_i)|^m = \textstyle\int_0^\infty |\dot{r}_\theta(x)|^m\, dF_\theta(x) < \infty\,,$$

by (2.8). Similarly, on writing

$$(3.33)\qquad E_\theta|\dot{G}_\theta(Z_k) - \dot{G}_\theta(Z_{k-1})|^m = E(E\{|\dot{G}_\theta(Z_k) - \dot{G}_\theta(Z_{k-1})|^m \mid \mathscr{B}_{n,k-1}\})\,,$$

and following steps similar to those in (3.26) through (3.29), it follows that as $n \to \infty$,

$$(3.34)\qquad \begin{aligned} &n^{-1}\textstyle\sum_{k=1}^{r} E_\theta\{|(n-k+1)(\dot{G}_\theta(Z_k) - \dot{G}_\theta(Z_{k-1}))|^m\} \\ &\qquad \leqq n^{-1}\textstyle\sum_{k=1}^{n} E_\theta\{|(n-k+1)(\dot{G}_\theta(Z_k) - \dot{G}_\theta(Z_{k-1}))|^m\} \\ &\qquad \to \lfloor m+1 \textstyle\int_0^\infty |\dot{r}_\theta(x)|^m\, dF_\theta(x) < \infty\,, \end{aligned}$$

for every $m \leqq 4$. Hence, (3.30) follows from (3.31), (3.32) and (3.34). The proof of Theorem 1 is complete.

We would like to extend Theorem 1 to contiguous alternatives. For this, we consider a sequence of values of θ, specified by

$$(3.35)\qquad \theta_n = \theta_0 + n^{-\frac{1}{2}}u\,, \qquad \text{where} \quad u \quad \text{is real and finite,} \quad \theta_0 \in \Theta\,.$$

We are then interested in the limiting behaviour of the process $W_{n,r}^{\theta_0}$ when the true θ is specified by θ_n in (3.35). For this purpose, we make a comparatively more restrictive regularity assumption: for $h_\theta(x) = -\ddot{f}_\theta(x)$ or $-\ddot{G}_\theta(x)$,

$$(3.36)\qquad \lim_{\delta\to 0} E_{\theta_0}\{\sup_{|\theta-\theta_0|<\delta} |h_\theta(X) - h_{\theta_0}(X)|\} = 0\,.$$

Now, (3.36) implies that

(3.37) $$\lim_{\delta\to 0}\{\sup_{\theta:|\theta-\theta_0|<\delta} E_{\theta_0}[h_\theta(X)]\} = E_{\theta_0}[h_{\theta_0}(X)] .$$

THEOREM 2. *Under* (3.35), (3.36) *and the regularity conditions of Theorem* 1, *as* $n \to \infty$,

$$\{W^{\theta_0}_{n,r}(t) - ut, 0 \leqq t \leqq 1\} \to_{\mathscr{D}} W , \quad \textit{in the } J_1 \textit{ topology on } D[0, 1] .$$

PROOF. For testing $H_0: \theta = \theta_0$ vs. $H_n: \theta = \theta_n$, the usual log-likelihood ratio statistic based on $Z_1, \cdots, Z_r$ is

(3.38) $$\lambda^0_{n,r} = \log p_{\theta_n}(\mathbf{Z}^{(r)}, n) - \log p_{\theta_0}(\mathbf{Z}^{(r)}, n) .$$

By the usual expansion, one obtains that

(3.39) $$\lambda^0_{n,r} = n^{-\frac{1}{2}}J^{\frac{1}{2}}_{n,r}(\theta_0)uW^{\theta_0}_{n,r}(1) + (\tfrac{1}{2}u^2)n^{-1}\ddot{p}_{\theta_n'}(\mathbf{Z}^{(r)}, n) ; \quad \theta_n' = \theta_0 + hun^{-\frac{1}{2}} ,$$

where $0 < h < 1$. Now, by (2.3),

(3.40) $$\begin{aligned} -n^{-1}\ddot{p}_\theta(\mathbf{Z}^{(r)}, n) &= n^{-1} \textstyle\sum_{i=1}^r \{-\ddot{f}_\theta(Z_i)\} - \{(n-r)/n\}\ddot{G}_\theta(Z_r) \\ &= \textstyle\int_0^{Z_r} \{-\ddot{f}_\theta(x)\}\, dF_n(x) + \dfrac{n-r}{n}\{-\ddot{G}_\theta(Z_r)\} . \end{aligned}$$

Thus, proceeding as in (3.18) through (3.29) and using (3.36)–(3.37), it follows that the second term on the rhs of (3.39) converges in probability to $-\frac{1}{2}u^2J_p^{\frac{1}{2}}(\theta_0)$, so that under H_0,

(3.41) $$\lambda^0_{n,r} \to_p J_p^{\frac{1}{2}}(\theta_0)\{uW^{\theta_0}_{n,r}(1) - \tfrac{1}{2}u^2\} , \quad \text{as} \quad n \to \infty .$$

Also, by Theorem 1, $W^{\theta_0}_{n,r}(1)$ is asymptotically normally distributed with 0 mean and unit variance, so that by (3.41), under H_0, $\lambda^0_{n,r}$ is asymptotically normally distributed with mean $-\frac{1}{2}u^2J_p^{\frac{1}{2}}(\theta_0)$ and variance $u^2J_p(\theta_0)$. Hence, by the Le Cam first lemma (viz., Hájek and Šidak (1967), page 204, Corollary), we conclude that $\{p_{\theta_n}(\mathbf{Z}^{(r)}, n)\}$ is contiguous to $\{p_{\theta_0}(\mathbf{Z}^{(r)}, n)\}$, and this implies that the sequence of densities $\{p_{\theta_n}(\mathbf{Z}^{(k)}, n), 1 \leqq k \leqq r\}$ is contiguous to $\{p_{\theta_0}(\mathbf{Z}^{(k)}, n), 1 \leqq k \leqq r\}$.

For a process $x \in D[0, 1]$, we define for every $\delta: 0 < \delta < 1$,

(3.42) $$\omega_\delta'(x) = \sup\{\min(|x(t) - x(s)|, |x(u) - x(t)|) : 0 \leqq s < t < u \leqq 1, \; u - s < \delta\} .$$

Then, by the tightness property of $W^{\theta_0}_{n,r}$ (under H_0), established in Theorem 1,

(3.43) $$\lim_{\delta\to 0}\lim_n \sup P\{\omega_\delta'(W^{\theta_0}_{n,r}) > \varepsilon \mid \theta_0\} = 0 \qquad \text{for every} \quad \varepsilon > 0 .$$

By virtue of (3.43) and the contiguity of $\{p_{\theta_n}\}$ with respect to $\{p_{\theta_0}\}$, we conclude that

(3.44) $$\lim_{\delta\to 0}\lim_n \sup p\{\omega_\delta'(W^{\theta_0}_{n,r}) > \varepsilon \mid \theta_n\} = 0 \qquad \text{for every} \quad \varepsilon > 0 ,$$

that is, under $H_n: \theta = \theta_n$ in (3.35), the process $\{W^{\theta_0}_{n,r}\}$ is also tight. Thus, to prove Theorem 2, it suffices to show that the finite dimensional distributions of $\{W^{\theta_0}_{n,r}(t) - ut, 0 \leqq t \leqq 1\}$ converge to those of W when $\{H_n\}$ holds. For this, note that under the hypothesis of the theorem, whenever $k/n \to \alpha: 0 < \alpha \leqq 1$,

$$(3.45) \qquad n^{-\frac{1}{2}}\{\dot{p}_{\theta_n}(\mathbf{Z}^{(k)}, n) - \dot{p}_{\theta_0}(\mathbf{Z}^{(k)}, n)\} \to_p -uJ_\alpha(\theta_0), \quad \text{as} \quad n \to \infty.$$

Thus, for every (fixed) m ($\geqq 1$) and $t_1, \cdots, t_m$ (all belonging to $[0, 1]$), on defining the $k_n(t_j)$ as in (2.14) and noting that $J_{n,k_n(t_j)}(\theta_n)/J_{n,r}(\theta_n) \to t_j$, $j = 1, \cdots, m$, we obtain from (3.45), Lemma 3.2 and Theorem 1 that under $\{H_n\}$, as $n \to \infty$,

$$(3.46) \qquad \{W_{n,r}^{\theta_0}(t_j) - ut_j, j = 1, \cdots, m\} \to_{\mathscr{D}} \{W(t_j), j = 1, \cdots, m\}.$$

Hence, the proof of the theorem is complete.

4. Asymptotic theory of progressively censored life testing. For simple exponential distributions, Epstein and Sobel (1954, 1955) considered some (sequential as well as nonsequential) life testing procedures which rest on a simple Poisson process characterization of the associated likelihood ratio process on which the celebrated results of Dvoretzky, Kiefer and Wolfowitz (1953) directly apply. However, the theory does not hold when $f_\theta(x)$ is not a simple exponential pdf. Our Theorems 1 and 2 provide us with the necessary tools for constructing a class of asymptotic tests for a broad class of survival distributions, and we present the same as follows.

First, consider the one-sided alternative $H_1^+: \theta > \theta_0$. Given (n, r) and a desired level of significance α ($0 < \alpha < 1$), we conceive of a positive constant $C_{n,\alpha}^{(r)}$ and continue monitoring of the experiment so long as $\{k \leqq r$ and $\dot{p}_{\theta_0}(\mathbf{Z}^{(k)}, n) < C_{n,\alpha}^{(r)}\}$. If, for the first time, for $k = N(\leqq r)$, $\dot{p}_{\theta_0}(\mathbf{Z}^{(N)}, n) \geqq C_{n,\alpha}^{(r)}$, we terminate experimentation along with the rejection of H_0 in (2.1). If no such $N(\leqq r)$ exists, the experiment is terminated when $X_{n,r}$ has been observed and H_0 is accepted. For testing H_0 vs. $H_1^-: \theta < \theta_0$, we simply change $\dot{p}_{\theta_0}(\mathbf{Z}^{(k)}, n)$ to $(-1)\dot{p}_{\theta_0}(\mathbf{Z}^{(k)}, n)$, $k \geqq 0$. From Theorem 1, it follows that asymptotically (as $n \to \infty$, with $r/n \to p \in (0, 1]$), $n^{-\frac{1}{2}}C_{n,\alpha}^{(r)} \to \tau_{\alpha/2}J_p^{\frac{1}{2}}(\theta_0)$, where $1 - \Phi(\tau_\beta) = \beta, 0 < \beta < 1$ and Φ is the standard normal df: For two-sided alternatives $H^*: \theta \neq \theta_0$, we replace $\dot{p}_{\theta_0}(\mathbf{Z}^{(k)}, n)$ by $|\dot{p}_\theta(\mathbf{Z}^{(k)}, n)|$, $k \geqq 0$ and proceed similarly. Again, by Theorem 1, asymptotically, $n^{-\frac{1}{2}}C_{n,\alpha}^{(r)} \to \omega_\alpha J_p^{\frac{1}{2}}(\theta_0)$ where ω_α is the upper $100\alpha\%$ point of the df $\Phi^*(x) = \sum_{k=-\infty}^{\infty}(-1)^k\{\Phi((2k+1)x) - \Phi((2k-1)x)\}$, $x \geqq 0$. From Theorem 2, it follows that for local alternatives in (3.35), the power function of these tests can be expressed in terms of appropriate boundary crossing probabilities of a drifted Brownian motion on the unit interval $[0, 1]$. Chatterjee and Sen (1973) have considered a class of progressively censored sequential tests based on linear rank statistics. Though the two problems are different (and so are the basic invariance principles), the properties of the tests are similar. In view of this, we refer to [2] for further motivation and other properties of this type of tests.

Secondly, as in Sen and Ghosh (1974) dealing with sequential rank test for location, we may consider another type of asymptotic tests as follows. Suppose that

$$(4.1) \qquad H_0: \theta = \theta_0 \quad \text{vs.} \quad H_\Delta: \theta = \theta_1 = \theta_0 + \Delta, \qquad \Delta\,(> 0) \ \text{is small and specified.}$$

Then, we introduce

$$T_{n,k} = \Delta \dot{p}_{\theta^*}(\mathbf{Z}^{(k)}, n)/J_p^{\frac{1}{2}}(\theta^*) \; ; \qquad \theta^* = \theta_0 + \tfrac{1}{2}\Delta,\ k \geqq 0\,, \tag{4.2}$$

and corresponding to the desired strength $(\alpha, \beta): 0 < \alpha, \beta \leqq \frac{1}{2}$, we choose two numbers $(b_n, a_n): -\infty < b_n < 0 < a_n < \infty$ and continue monitoring experimentation as long as $\{k < r$ and $b_n < T_{n,k} < a_n\}$. If, for the first time, for $k = N(< r)$, $T_{n,N}$ is $\leqq b_n$ (or $\geqq a_n$), we accept H_0 (or H_Δ); if no such $N(< r)$ exists, the experiment is stopped when $X_{n,r}$ has been observed, and H_0 or H_Δ is accepted according as $T_{n,r}$ is $\leqq$ or > 0. Here also, Theorems 1 and 2 provide us with the asymptotic (as $\Delta \to 0$ with $n \to \infty$ and $r/n \to p \in (0, 1]$) expressions for a_n and b_n, where we need to use the basic theorms of Anderson (1960) on the boundary crossing probabilites of the drifted Wiener processes on finite intervals. Basically, the properties of this test are similar to those of the sequential rank tests considered by Sen and Ghosh (1974), and in view of this, we shall not enter into a detailed discussion on these.

We conclude this section with the final remark that even for the single point truncated or censored scheme, Theorems 1 and 2 provide us with the necessary tools for constructing a (nonsequential) test based on $\lambda_{n,r}$ (or λ_{n,r^*}), making use of its asymptotic normality property. For the truncation scheme, $r^*/n \to F_{\theta_0}(T) = p^*$, say, so that as an immediate corollary to Theorem 1, we conclude that under H_0, $\lambda_{n,r^*}/J_{p^*}^{\frac{1}{2}}(\theta_0)$ is asymptotically normally distributed with 0 mean and unit variance, while Theorem 2 provides the parallel result for the contiguous alternative hypotheses.

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