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A CHARACTERIZATION OF THE ASYMPTOTIC NORMALITY OF LINEAR COMBINATIONS OF ORDER STATISTICS FROM THE UNIFORM DISTRIBUTION

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A necessary and sufficient condition for the asymptotic normality of linear combinations of order statistics from the uniform distribution over [0, 1] is derived. The condition implies, that the variances of all weighted extremes are asymptotically zero compared with the total variance of the sum, which conversely, in the case of nonnegative constants, is sufficient for the asymptotic normality.

1. Introduction. Let $U_{1n} \leqq U_{2n} \leqq \cdots \leqq U_{nn}$ be the order statistics of n independent random variables $U_1, \cdots, U_n$ with uniform distribution on [0, 1]. For a given array of constants a_{in} $(i = 1, \cdots, n;\ n = 1, 2, \cdots)$ let

(1) $$T_n = \sum_{i=1}^n a_{in}(U_{in} - i/(n+1)) .$$

Since $EU_{in} = i/(n+1)$, we have $ET_n = 0$, and

(2) $$\sigma_n^2 \equiv \operatorname{Var} T_n = \sum_{j,k=1}^n a_{jn} a_{kn} \operatorname{Cov}(U_{jn}, U_{kn})$$

where

(3) $$\operatorname{Cov}(U_{jn}, U_{kn}) = (n+2)^{-1}\left(\min\left(\frac{j}{n+1}, \frac{k}{n+1}\right) - \frac{j}{n+1}\frac{k}{n+1}\right) .$$

In this paper a necessary and sufficient condition for the asymptotic normality of T_n/σ_n is derived. This problem is closely related to the case of more general types of underlying distribution functions, since statistics of the form (1) are often regarded as approximating sums in this case.

For some historical remarks on the subject, important results in the general case and for applications we refer to the papers of Chernoff, Gastwirth and Johns [2] and Stigler [7]. Later results are obtained by e.g., Shorack [3], [4], [5], [6] and Stigler [8], [9].

2. The result. Let $\rightarrow_d$ denote convergence in distribution, and $N(0, 1)$ the standard normal distribution. Then the result may be stated as follows:

THEOREM. $T_n/\sigma_n \rightarrow_d N(0, 1)$ *if and only if*

(4) $$(n\sigma_n)^{-1} \max_{i=1,\cdots,n} \left|\sum_{j=i}^n a_{jn}\right| \rightarrow 0 \qquad \textit{as} \quad n \rightarrow \infty$$

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PROOF. Let $X_1, X_2, \cdots$ be independent rv with standard exponential distribution, and

$$Z_i = \sum_{j=1}^{i} X_j \qquad i = 1, 2, \cdots. \tag{5}$$

Then $(U_{1n}, U_{2n}, \cdots, U_{nn})$ and $(Z_1/Z_{n+1}, Z_2/Z_{n+1}, \cdots, Z_n/Z_{n+1})$ are identically distributed (see, e.g., Breiman [1], page 285). Now let

$$b_{in} = \sum_{j=i}^{n} a_{jn} \qquad i = 1, 2, \cdots, n,\ b_{n+1,n} = 0 \tag{6}$$

and

$$\bar{b}_n = (n+1)^{-1} \sum_{i=1}^{n+1} b_{in} = (n+1)^{-1} \sum_{i=1}^{n} i a_{in}. \tag{7}$$

Then, if we define $U_{0n} \equiv 0$, partial summation yields

$$\sum_{i=1}^{n} a_{in} U_{in} = \sum_{i=1}^{n} b_{in}(U_{in} - U_{i-1,n}),$$

and hence T_n and

$$S_n \equiv (Z_{n+1})^{-1} \sum_{i=1}^{n} b_{in} X_i - \sum_{i=1}^{n} b_{in}/(n+1) \tag{8}$$

are identically distributed. By (5), (6) and (7), S_n may be written as

$$S_n = (Z_{n+1}/(n+1))^{-1} \tilde{S}_n$$

where

$$\tilde{S}_n = \sum_{i=1}^{n+1} (b_{in} - \bar{b}_n) X_i/(n+1). \tag{9}$$

Thus, T_n/σ_n has the same distribution as

$$S_n/\sigma_n = (Z_{n+1}/(n+1))^{-1} (\tilde{\sigma}_n/\sigma_n) \tilde{S}_n/\tilde{\sigma}_n \tag{10}$$

where

$$\tilde{\sigma}_n^2 = \operatorname{Var} \tilde{S}_n.$$

By (7), $\tilde{S}_n$ has the mean zero.

Comparing σ_n with $\tilde{\sigma}_n$, we find

$$\begin{aligned}
\tilde{\sigma}_n^2 &= (n+1)^{-2} \sum_{i=1}^{n+1} (b_{in} - \bar{b}_n)^2 \\
&= (n+1)^{-2} \sum_{i=1}^{n} \sum_{j,k=i}^{n} a_{jn} a_{kn} - (n+1)^{-1} \left(\sum_{i=1}^{n} a_{in} i/(n+1)\right)^2 \\
&= (n+1)^{-1} \sum_{j,k=1}^{n} a_{jn} a_{kn} \left(\min\left(\frac{j}{n+1}, \frac{k}{n+1}\right) - \frac{j}{n+1} \frac{k}{n+1}\right),
\end{aligned}$$

or by (2) and (3),

$$\tilde{\sigma}_n^2 = ((n+2)/(n+1)) \sigma_n^2. \tag{11}$$

Thus, since $Z_{n+1}/(n+1) \to 1$ almost surely, $T_n/\sigma_n \to_d N(0, 1)$ if and only if $\tilde{S}_n/\tilde{\sigma}_n \to_d N(0, 1)$. The latter convergence is equivalent to the condition

$$((n+1)\tilde{\sigma}_n)^{-1} \max_{i=1,\cdots,n+1} |b_{in} - \bar{b}_n| \to 0 \qquad \text{as} \quad n \to \infty \tag{12}$$

(see Lemma 1 of Chernoff, Castwirth and Johns [2]). Since $b_{n+1,n} = 0$, $\bar{b}_n = (n+1)^{-1} \sum_{i=1}^{n} b_{in}$ and by (11), this is equivalent to

$$(n\sigma_n)^{-1} \max_{i=1,\cdots,n} |b_{in}| \to 0 \qquad \text{as} \quad n \to \infty$$

which by (6) is the condition (4).

Remark. Condition (4) implies, that for any fixed m

$$(13) \qquad \lim_{n\to\infty} \sigma_n^{-2} \operatorname{Var}(a_{mn} U_{mn}) = \lim_{n\to\infty} \sigma_n^{-2} \operatorname{Var}(a_{n+1-m,n} U_{n+1-m,n}) = 0 ,$$

that is, the variances of the weighted extrems are asymptotically zero compared with the total variance. The converse is generally not true, since for example $T_n = U_{jn} - U_{j-1,n} - 1/(n+1)$ has the same distribution as $U_{1n} - 1/(n+1)$ for all $2 \leqq j \leqq n$ (especially take $j = [n/2]$); thus the distribution of T_n/σ_n does not converge to $N(0, 1)$ and hence (4) does not hold.

But it can be shown that (4) and (13) are equivalent, if all constants a_{in} are nonnegative (choose a sequence (c_n) of positive integers with $\lim_{n\to\infty} c_n = \infty$ and $\lim_{n\to\infty} v_n = 0$ where $v_n = \sigma_n^{-2} \operatorname{Var}(\sum_{i=1}^{c_n} a_{in} U_{in})$, then find upper bounds for $((n\sigma_n)^{-1} \sum_{i=1}^{c_n} a_{in})^2$ and $((n\sigma_n)^{-1} \sum_{i=c_n+1}^{[n/2]} a_{in})^2$ in terms of v_n and

$$\sigma_n^{-2} \operatorname{Var}(\sum_{i=c_n+1}^{[n/2]} a_{in} U_{in}) ,$$

respectively; finally, use the symmetry of the problem.) Thus (13) may be regarded as a simple stochastic version of (4) in this case.

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