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AN APPLICATION OF A THEOREM OF ROBBINS AND SIEGMUND

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A stochastic approximation process for estimating an unknown parameter in nonlinear regression is discussed. The process was suggested by Albert and Gardner [Stochastic Approximation and Nonlinear Regression. Research Monograph No. 42. M.I.T. Press, Cambridge, Massachusetts]. An almost sure convergence of the process is proved. The proof is an application of a theorem of Robbins and Siegmund on the almost sure convergence of nonnegative almost supermatingales. The conditions given here are weaker than those given by Albert and Gardner.

1. Introduction. In their paper on the convergence of almost super martingales Robbins and Siegmund [3] have proved the following theorem:

THEOREM. *Let* $(\Omega, \mathscr{F}, P)$ *be a probability space and* $\mathscr{F}_1 \subseteqq \mathscr{F}_2 \subseteqq \cdots$ *be a sequence of sub* σ*-fields of* $\mathscr{F}$. *Let* U_n, β_n, ξ_n *and* ζ_n, $n = 1, 2, \cdots$, *be nonnegative* $\mathscr{F}_n$*-measurable random variables such that*

$$E(U_{n+1} | \mathscr{F}_n) \leqq (1 + \beta_n)U_n + \xi_n - \zeta_n\,, \qquad n = 1, 2, \cdots. \tag{1}$$

Then on the set $\{\sum_n \beta_n < \infty, \sum_n \xi_n < \infty\}$ U_n *converges* a.s. *to a random variable and* $\sum_n \zeta_n < \infty$ a.s.

As Robbins and Siegmund have demonstrated in [3], this theorem can be used as a strong tool for proofs of convergence of various processes.

In this note this theorem is applied to obtain a proof of a.s. convergence of a stochastic approximation process for estimating an unknown parameter θ in a nonlinear regression setup. This process was first discussed by Albert and Gardner in [1].

Consider the following situation. Let $F_n(x)$, $n = 1, 2, \cdots$ be a sequence of real valued functions of a real variable. The functions F_n are known. One wishes to estimate a real parameter θ, the information about which is obtained by means of observable random variables $Y_n = F_n(\theta) + Z_n$.

Let t_1 be a random variable. For $n \geqq 1$ define

$$t_{n+1} = t_n - a_n(t_1, \cdots, t_n)[F_n(t_n) - Y_n]\,, \tag{2}$$

where a_n is a real valued function which is measurable with respect to the σ-field generated by $(t_1, Y_1, \cdots, Y_{n-1})$. Albert and Gardner have proved that under some conditions $t_n \to \theta$ a.s. (For details see [1] Theorem 2.1 page 11.) They

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require fairly strong differentiability conditions on the F_n's. In the next section an a.s. convergence of the process (2) is proved under much more plausible conditions.

2. A convergence theorem. In what follows, θ denotes some fixed (unknown) real number. Assume:

(3) For every $n \geqq 1$, $F_n(x)$ is a monotone function of the real variable x.

(4) For every $0 < \varepsilon < 1$ there exists a sequence of nonnegative number $\{b_n(\varepsilon)\}$ such that for every $n \geqq 1$

$$\inf_{\varepsilon < |x-\theta| < \varepsilon^{-1}} |F_n(x) - F_n(\theta)| \geqq b_n(\varepsilon) .$$

(5) For every $n \geqq 1$ there exists a number $0 < A_n < \infty$ such that

$$|F_n(x) - F_n(\theta)| \leqq A_n|x - \theta| \qquad \text{for all } x .$$

Let t_1 and Z_n, $n \geqq 1$ be random variables, and let $Y_n = F_n(\theta) + Z_n$. Let $\mathscr{F}_1$ be the σ-field generated by t_1 and $\mathscr{F}_n$ be the σ-field generated by $(t_1, Y_1, \cdots, Y_{n-1})$ for $n \geqq 2$. Denote $V_n = E(Z_n^2 | \mathscr{F}_n)$.

THEOREM. *With the above notation, suppose that conditions* (3), (4) *and* (5) *hold and that* $E(Z_n | \mathscr{F}_n) = 0$. *Suppose also that* a_n, $n \geqq 1$, *is an* $\mathscr{F}_n$*-measurable random variable with*

$$\text{(6)} \qquad \operatorname{sgn}(a_n) = 1 \quad \text{if } F_n \text{ is nondecreasing}$$
$$= -1 \quad \text{if } F_n \text{ is nonincreasing}$$

and that

$$\text{(7)} \qquad \begin{aligned} &\text{(i)} \quad \textstyle\sum a_n^2 A_n^2 < \infty \\ &\text{(ii)} \quad \textstyle\sum a_n^2 V_n < \infty \quad \text{and} \\ &\text{(iii)} \quad \textstyle\sum |a_n| b_n(\varepsilon) = \infty \quad \text{for every } \varepsilon > 0 , \quad \text{a.s.,} \end{aligned}$$

then $t_n \to \theta$ a.s., *where* t_n *is given by* (2).

PROOF. Let $U_n = (t_n - \theta)^2$. Then,

$$\text{(8)} \qquad U_{n+1} = U_n - 2a_n(t_n - \theta)(F_n(t_n) - F_n(\theta)) + 2a_n(t_n - \theta)Z_n + a_n^2(F_n(t_n) - F_n(\theta) - Z_n)^2 .$$

Taking expectations on both sides of (8) and using the convexity inequality $(a + b)^2 \leqq 2(a^2 + b^2)$, one obtains

$$E(U_{n+1}|F_n) \leqq U_n - 2a_n(t_n - \theta)(F_n(t_n) - F_n(\theta)) + 2a_n^2(F_n(t_n) - F_n(\theta))^2 + 2a_n^2 V_n .$$

By (3), (5) and (6) it follows that

$$\begin{aligned} E(U_{n+1}|\mathscr{F}_n) &\leqq U_n - 2|a_n||t_n - \theta||F_n(t_n) - F_n(\theta)| + 2a_n^2 A_n^2 U_n + 2a_n^2 V_n \\ &= (1 + 2a_n^2 A_n^2)U_n + 2a_n^2 V_n - 2|a_n||t_n - \theta||F_n(t_n) - F_n(\theta)| . \end{aligned}$$

If we set $\beta_n = 2a_n^2 A_n^2$, $\xi_n = 2a_n^2 V_n$ and $\zeta_n = 2|a_n||t_n - \theta||F_n(t_n) - F_n(\theta)|$, conditions (7) (i) and (ii) imply that $\lim_n U_n$ exists a.s. and

$$\sum |a_n||t_n - \theta||F_n(t_n) - F_n(\theta)| < \infty \qquad \text{a.s.} \tag{9}$$

Let X be the a.s. limit of U_n. It remains to show that $X = 0$ a.s. Suppose this is not the case. Then there exists a set S with $P(S) > 0$ such that $X(\omega) > 0$ for every $\omega \in S$. Let $\omega \in S$. Then there exists $\varepsilon > 0$ and N such that for $n \geqq N$, $\varepsilon < |t_n(\omega) - \theta| < \varepsilon^{-1}$. Hence by (4),

$$\sum_n |a_n(\omega)||t_n(\omega) - \theta||F_n(t_n) - F_n(\theta)| \geqq \varepsilon \sum_{n \geqq N} |a_n(\omega)| b_n(\varepsilon) = \infty$$

by (7)(iii), which contradicts (9).

In some applications it may occur that the functions F_n are random. This situation was encountered by the author while constructing an adaptive procedure for determining an optimal inspection policy in a stochastically failing system. The details are discussed elsewhere (see [2]). If instead of $F_n(x)$ one has $F_n(x, \omega)$ which is (jointly) measurable with respect to $(\mathscr{B} \times \mathscr{F}_n)$ where $\mathscr{B}$ is the Borel σ-field on the real line, then it is easy to see that the above theorem remains true if the following modifications are made:

(a) Conditions (3) and (6) hold for every *fixed* ω.

(b) Conditions (4), (5) with A_n being $\mathscr{F}_n$-measurable functions and (7) hold with probability one.

REMARKS. 1. As Albert and Gardner point out it seems advisable to use truncated procedure whenever it is known that θ lies in some finite interval (a, b). That is, let $f(x)$ be a real valued function. The truncation of $f(x)$ in an interval (a, b) is defined by

$$\begin{aligned} [f(x)]_a^b &= b && \text{if } f(x) \geqq b \\ &= f(x) && \text{if } a < f(x) < b \\ &= a && \text{if } f(x) \leqq a\,. \end{aligned}$$

A truncated process analogous to (2) is defined by

$$t_{n+1}^* = [t_n^* - a_n(F_n(t_n^*) - Y_n)]_a^b\,. \tag{10}$$

Since

$$|t_{n+1}^* - \theta| \leqq |(t_n^* - \theta) - a_n(F_n(t_n^*) - Y_n)|\,,$$

it is clear that the proof of convergence of the untruncated process (2) holds for the truncated process (10) as well.

2. As one can readily check, all our conditions except condition (7)(i) follow from Albert and Gardner's conditions of their Theorem 2.1. If we assume differentiability of the functions F_n, with $\sup_x |\dot{F}_n(x)| < \infty$, where $\dot{F}_n$ denotes the derivative of F_n, then our condition (5) is satisfied with $A_n = \sup_x |\dot{F}_n(x)|$ and Albert and Gardner's condition (2) is implied by our condition (7)(i).

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