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INEQUALITIES FOR SEMIREGULAR GROUP DIVISIBLE DESIGNS

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Let s_{ju} be the number of varieties in common to the jth and uth blocks of a symmetric semiregular group divisible design. Connor (1952) and Saraf (1961) have given inequalities for s_{ju}. Both these inequalities lead to the same stronger inequality $\lambda_1 \leqq s_{ju} \leqq 2\lambda_2 - \lambda_1 - 1$. Both the upper and lower bounds are attained by a series of designs derived from lattices.

1. Introduction. Clatworthy (1973, pages 8-9) gives two inequalities for the number of treatments common to two blocks of a semiregular group divisible design. These are due to Connor (1952) and Saraf (1971). Following Connor we let s_{ju} denote the number of treatments that appear in both the jth and uth blocks. Connor's inequality for symmetric SRGD designs is

(1) $$\lambda_1 \leqq s_{ju} \leqq -\lambda_1 + 2\lambda_2 k^2/(k - \lambda_1 + v\lambda_2) .$$

Saraf's inequality holds for any SRGD design; it is

(2) $$k - (r - \lambda_1) \leqq s_{ju} \leqq r - \lambda_1 - k + 2k[c(\lambda_1 - \lambda_2 + 1) + k\lambda_2]/(n + v\lambda_2) ,$$

where $c = k/m$ (c is an integer).

For symmetric designs this inequality becomes

(3) $$\lambda_1 \leqq s_{ju} \leqq -\lambda_1 + 2k[c(\lambda_1 - \lambda_2 + 1) + k\lambda_2]/(n + v\lambda_2) .$$

We shall denote the upper bounds of (1) and (3) by C and S respectively; $[C]$ and $[S]$ will denote the largest integers contained in C and S. Clatworthy points out that when $\lambda_2 - \lambda_1 = 1$ we have $C = S$, and states that "Saraf claims superiority for his inequalities when $\lambda_2 - \lambda_1 > 1$." Clatworthy's choice of the word "claims" was judicious. It will be shown that, whereas $S < C$ whenever $\lambda_2 - \lambda_1 > 1$ the difference never becomes so large as to make $[S] < [C]$. Since s_{ju} is an integer, the integers $[C]$ and $[S]$ are more appropriate upper bounds. We shall obtain the sharper inequality

(4) $$\lambda_1 \leqq s_{ju} \leqq 2\lambda_2 - \lambda_1 - 1$$

by showing that $2\lambda_2 - \lambda_1 - 1 = [S] = [C]$. An example will be given of a series of symmetric SRGD designs for which both the upper and lower bounds of (4) are attained.

2. Properties of SRGD designs. Some relations between the parameters of the designs which will be needed later are developed in this section. The reader is referred to Connor (1952) and Bose and Connor (1952) for further details.

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For any GD design we have $r(k-1) = (n-1)\lambda_1 + n(m-1)\lambda_2$; for semiregular designs we have $rk = \lambda_2 v$. Subtracting we have $r - \lambda_1 = n(\lambda_2 - \lambda_1)$.

We write $\lambda_2 - \lambda_1 = d$, and confine ourselves to symmetric designs. Putting $r = k$ in the previous paragraph we obtain

$$k^2 = v\lambda_2 ; \qquad k = \lambda_1 + nd .$$

Substituting mc for k and mn for v gives $ck = n\lambda_2$. It follows that $\lambda_1 = k(c-1)/(n-1)$, and

$$k - \lambda_1 = nd = k(n-c)/(n-1) .$$

3. Derivation of the inequality. We have

$$C = -\lambda_1 + 2\lambda_2 k^2/(k - \lambda_1 + k^2) = -\lambda_1 + 2\lambda_2 - C^* ,$$

where $C^* = 2\lambda_2(k - \lambda_1)/(k^2 + k - \lambda_1)$. We now show that $C^* < 1$; C^* is clearly positive.

From the results of the previous section we may write

$$C^* = 2kc(n-c)/[n(n-1)k + n(n-c)] .$$

However, $0 < c < n$ and so $4c(n-c) \leqq n^2$, so that

$$C^* \leqq \frac{nk}{2[(n-1)k + (n-c)]} < \frac{nk}{2(n-1)k} \leqq 1 .$$

It follows that $[C] = -\lambda_1 + 2\lambda_2 - 1$. Proceeding in the same way we have

$$S = -\lambda_1 + 2\lambda_2 - S^* ,$$

where $S^* = 2kcd/(k^2 + n) = 2k^2c(n-c)/[n(n-1)(k^2+n)] < 1$. It follows that $S^*/C^* \geqq 1$ with equality if and only if $d = 1$.

We have thus shown that

$$2\lambda_2 - \lambda_1 - 1 = [S] = [C] < S \leqq C$$

and that

$$\lambda_1 \leqq s_{ju} \leqq 2\lambda_2 - \lambda_1 - 1 .$$

The upper bound may be written as $\lambda_1 + 2d - 1$, which is less than k. This proves that a symmetric SRGD design cannot contain any repeated blocks.

4. Agrawal's identity. Agrawal (1964) has proved the following inequality for the general SRGD design:

$$\max\,[0, 2k - v, -r + \lambda_1 + k] \leqq s_{ju} \leqq \min\left[\frac{2\{r(k-1) + \lambda_1\}}{b} - k + r - \lambda_1\right].$$

He showed that the upper bound of his inequality is always less than Saraf's upper bound. We recall that for symmetric designs $\lambda_1 \geqq 2r - b = 2k - v$, and

Agrawal's inequality reduces to

$$\lambda_1 \leqq s_{ju} \leqq \frac{2\{k(k-1)+\lambda_1\}}{v} - \lambda_1 .$$

Denote this upper bound by A, and the largest integer contained in A by $[A]$. It follows at once that $A = 2\lambda_2 - \lambda_1 - 2(k-\lambda_1)/v$, so that $[A] = 2\lambda_2 - \lambda_1 - 1$, which is in accord with our equation (4) of the first section. As a practical matter all three inequalities are equivalent.

5. An example. The balanced lattice design, based on mutually orthogonal latin squares, has parameters $v = p^2$, $b = p^2 + p$, $r = p + 1$, $k = p$, $\lambda = 1$. Such a design exists when p is a prime or a power of a prime. The lattice is resolvable. If one replication is omitted the remaining blocks form a symmetric SRGD design with

$$v = b = p^2 , \qquad r = k = p , \qquad m = n = p , \qquad \lambda_1 = 0 , \qquad \lambda_2 = 1 .$$

Blocks in the same replication are disjoint; blocks in different replications have one common treatment. The complementary design has

$$v = b = p^2 , \qquad r = k = p(p-1) , \qquad m = n = p ,$$
$$\lambda_1 = p(p-2) , \qquad \lambda_2 = (p-1)^2 .$$

The blocks fall into groups corresponding to replications in the original. Blocks in the same group have $p(p-2) = \lambda_1$ treatments in common; blocks in different groups have $s_{ju} = (p-1)^2 = 2\lambda_2 - \lambda_1 - 1$, and both bounds in the inequality are attained.

6. Clatworthy's listing is confined to designs with $r \leqq 10$ and $k \leqq 10$. The only symmetric SRGD designs listed have either (i) $\lambda_1 = 0$ or (ii) $d = 1$. Designs that satisfy neither of these restrictions can be found by applying the following theorem. The proof of the theorem is straightforward and is not included.

THEOREM. *Let* $\mathbf{N}$ *be the incidence matrix of a symmetric* SRGD *design with parameters* $v, k, \cdots, \lambda_2$ *where* $v = 2k$ *and* $k = 2\lambda_2$, *and let* $\mathbf{N}^c$ *be the incidence matrix of the complementary design. Let* $\mathbf{N}^* = \begin{pmatrix} \mathbf{N} & \mathbf{N} \\ \mathbf{N} & \mathbf{N}^c \end{pmatrix}$.

Then $\mathbf{N}^*$ is the incidence matrix of a symmetric SRGD design with parameters $v^* = 2v$, $k^* = 2k$, $m^* = 2m$, $n^* = n$, $\lambda_1^* = 2\lambda_1$ and $\lambda_2^* = 2\lambda_2$.

If $\mathbf{N}$ is the incidence matrix of either of Clatworthy's solutions for SR68 with $v = 12$, $k = 6$, $m = 3$, $n = 4$, $\lambda_1 = 2$, $\lambda_2 = 3$, $\mathbf{N}^*$ is the incidence matrix of a design with $v = 24$, $k = 12$, $m = 6$, $n = 4$, $\lambda_1 = 4$, $\lambda_2 = 6$.

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