The Annals of Statistics
1976, Vol. 4, No. 4, 673-684

IMPORTANCE SAMPLING IN THE MONTE CARLO STUDY OF SEQUENTIAL TESTS[1]

BY D. SIEGMUND

Columbia University

Let $x_1, x_2, \cdots$ be independent random variables which under P_θ have probability density function of the form $P_\theta\{x_k \in dx\} = \exp(\theta x - \Psi(\theta))\, dH(x)$, where Ψ is normalized so that $\Psi(0) = \Psi'(0) = 0$. Let $a \leqq 0 < b$, $s_n = \sum_1^n x_k$, and $T = \inf\{n: s_n \notin (a, b)\}$. For $u < 0$, an unbiased Monte Carlo estimate of $P_u(s_T \geqq b)$ is the average of independent P_θ-realizations of $I_{\{s_T \geqq b\}} \exp\{(u - \theta)s_T - T(\Psi(u) - \Psi(\theta))\}$. It is shown that the choice $\theta = w$, where $w > 0$ is defined by $\Psi(w) = \Psi(u)$, is an asymptotically (as $b \to \infty$) optimal choice of θ in a sense to be defined. Implications of this result for Monte Carlo studies in sequential analysis are discussed.

1. Introduction and summary. The direct approach to Monte Carlo studies is to estimate probabilities by relative frequencies: to estimate $\alpha = P(A)$, the probability of an event A, one uses the average of n independent realizations of I_A, the indicator variable of A, which under P has the distribution $P\{I_A = 1\} = \alpha = 1 - P\{I_A = 0\}$. This estimator is unbiased and has variance equal to $1/n$ times

$$\operatorname{Var}_P(I_A) = \alpha(1 - \alpha)\,. \tag{1}$$

If α is small, rather large values of n are required to provide an accurate estimate.

Importance sampling (cf. Hammersley and Handscomb, 1964, pages 57–59) suggests that it may be helpful to write

$$P(A) = \int_A L\, dQ \tag{2}$$

for a suitable probability Q. Here L denotes the likelihood ratio of P relative to Q. ($L = p/q$ if P and Q have densities p and q.) Then one may estimate α by the average of n independent realizations of $I_A L$, generated to have the distribution induced by Q. This second estimator is also unbiased and has variance equal to $1/n$ times

$$\operatorname{Var}_Q(I_A L) = \int_A L^2\, dQ - \alpha^2\,, \tag{3}$$

which may be smaller than (1). For example, if $L \leqq 1$ on A, then

$$\int_A L^2\, dQ \leqq \int_A L\, dQ = P(A) = \alpha\,,$$

and (3) is not larger than (1). It is apparent from (3) that a suitable choice of Q, for the purpose of reducing the variance of the resulting estimator of α, is

Received April 1974; revised January 1976.

[1] Research supported in part by PHS Grant 5-R01-GM-16895-05 and ONR Grant N00014-75-C-0560.

AMS 1970 *subject classifications.* Primary 62L10, 65C05.

Key words and phrases. Sequential test, Monte Carlo, importance sampling.

one that makes L small and nearly constant on A. If A is the rejection region of a statistical test and P belongs to the null hypothesis, then to the extent that the test is approximately a likelihood ratio test a suitable Q may presumably be found among those probabilities permitted by the alternative hypothesis.

In this paper the technique of importance sampling is illustrated on the problem of estimating the error probabilities of the sequential probability ratio test. Since Wald's (1947) well-known approximations supplemented by more recent techniques (e.g., van Dobben de Bruyn, 1968; Siegmund, 1975a) provide reasonable numerical approximations to these probabilities under fairly general conditions, practical implications of these results are most important in more complicated problems in which alternative methods either do not exist or are to be checked for accuracy by simulation. Some examples are given in Section 3. The principal theoretical result is that in the case of a sequential probability ratio test the "natural" choice of Q is asymptotically optimal in a sense to be defined. It has the interesting interpretation that the analytic device employed by Esscher (1932), Wald (1947, page 48), Bahadur and Ranga Rao (1960), and Feller (1966, Chapters XI and XII) to estimate gambler's ruin and other large deviation probabilities is optimal when regarded as a Monte Carlo technique.

2. Notation and precise statement of main results. Let $x_1, x_2, \cdots$ be independent random variables with a common distribution function, such that

$$-\infty < Ex_k < 0 \qquad \text{and} \qquad P\{x_k > 0\} > 0 \,. \tag{4}$$

Let $s_n = \sum_1^n x_k$, and for $a \leqq 0 < b$ let

$$T = \inf\{n : n \geqq 1,\, s_n \notin (a, b)\} \,. \tag{5}$$

The subject of this paper is the Monte Carlo technique of "importance sampling" in attempting to estimate

$$\alpha = P\{s_T \geqq b\} \tag{6}$$

with an emphasis on asymptotic considerations as $b \to \infty$ (a arbitrary).

Perhaps the most important special case arises when the x's are log likelihood ratios. Then T is the stopping rule of Wald's sequential probability ratio test, and Wald's well-known arguments give an approximation to α under fairly general conditions (cf. Wald, 1947).

Assume that the probability P can be embedded in an exponential family $\{P_\theta\}$ as in Siegmund (1975b). The result of this embedding, which is fairly standard and not reproduced here, is that for θ in some interval with 0 as an interior point, under the probability P_θ the random variables $x_1, x_2, \cdots$ are independent with common probability density function of the form

$$P_\theta\{x_k \in dx\} = \exp(\theta x - \Psi(\theta))\, dH(x) \,. \tag{7}$$

The function Ψ is normalized so that

$$\Psi(0) = \Psi'(0) = 0 \,, \tag{8}$$

and then $dH(x) = P_0\{x_k \in dx\}$. For example, if originally the x's are normally distributed with mean μ and variance 1, then under P_θ they are normal with mean θ and variance 1, so that their probability density is given by (7) with $\Psi(\theta) = \frac{1}{2}\theta^2$ and $dH(x) = (2\pi)^{-\frac{1}{2}} \exp(-x^2/2)\, dx$. Also Ψ is a strictly convex function for which

$$\Psi'(\theta) = E_\theta(x_k)\,, \qquad \Psi''(\theta) = \operatorname{Var}_\theta x_k\,, \tag{9}$$

so by (8) and (9)

$$\operatorname{sgn} E_\theta(x_k) = \operatorname{sgn}\theta\,. \tag{10}$$

Let u denote that value of θ such that $P = P_*$. By (4) and (10) $u < 0$. By (8) and the convexity of Ψ there exists at most one value w, necessarily positive, for which

$$\Psi(w) = \Psi(u)\,. \tag{11}$$

Assume that such a value w exists.

It is easy to show (cf. Lemma 1 in Section 4) that for arbitrary θ

$$\alpha = P_u\{s_T \geqq b\} = \int_{(s_T \geqq b)} \exp[(u-\theta)s_T - T(\Psi(u) - \Psi(\theta))]\, dP_\theta\,, \tag{12}$$

which for $\theta = w$, by (11), becomes

$$\alpha = P_u\{s_T \geqq b\} = \int_{(s_T \geqq b)} \exp[-(w-u)s_T]\, dP_w\,. \tag{13}$$

The identity (13), from which follow at once the inequalities

$$P_u\{s_T \geqq b\} \leqq \exp[-(w-u)b]P_w\{s_T \geqq b\} \leqq \exp[-(w-u)b]\,, \tag{14}$$

forms the basis for Wald's analysis of the sequential probability ratio test. With the proper interpretation (cf. Lemma 1), equation (12) is a special case of (2) and (regardless of whether one makes this interpretation) suggests using the average of n independent P_θ-realizations of

$$I_{(s_T \geqq b)} \exp[(u-\theta)s_T - T(\Psi(u) - \Psi(\theta))] \tag{15}$$

to estimate α. For the important special case $\theta = w$ the variance of this estimator is $1/n$ times

$$\begin{aligned}
&\int_{(s_T \geqq b)} \exp[-2(w-u)s_T]\, dP_w - \alpha^2 \\
&\qquad \leqq \exp[-(w-u)b] \int_{(s_T \geqq b)} \exp[-(w-u)s_T]\, dP_w - \alpha^2 \\
&\qquad = \exp[-(w-u)b]\alpha - \alpha^2\,,
\end{aligned} \tag{16}$$

which even for moderate values of b is much smaller than the $\alpha(1-\alpha)$ of direct Monte Carlo. To the extent that (14) is almost an equality this variance is $O(\alpha^2)$ as $b \to \infty$ rather than $O(\alpha)$ as in straightforward Monte Carlo. A numerical example is given in Section 3.

The algebraic simplification in (12) which results from the choice $\theta = w$ makes possible some analysis and hence Wald's approximation to α. However, for Monte Carlo purposes one may well ask whether some other choice of θ would result in yet greater variance reduction. Choosing θ to minimize the variance (under P_θ) of the random variable (15) is equivalent to minimizing

$$(17) \qquad \mu_2(\theta) = \int_{(s_T \geqq b)} \exp[2(u-\theta)s_T - 2T(\Psi(u) - \Psi(\theta))\, dP_\theta\,.$$

The following theorem shows that in the family $\{P_\theta\}$ the choice $\theta = w$ minimizes μ_2 asymptotically as $b \to \infty$. (No restriction is placed on the behavior of a except that $a \leqq 0$.)

THEOREM 1. *Assume that the random variables x_k have a nonlattice distribution or a lattice distibution supported by $0, \pm h, \pm 2h, \cdots$ for some $h > 0$ and that $E_w(x_k) < \infty$. Then as $b \to \infty$, for all $\theta \neq w$ $\mu_2(w)/\mu_2(\theta)$ converges to 0 at an exponential rate. (In the lattice case $b \to \infty$ through multiples of h.)*

A proof is given in Section 4.

REMARKS. (i) The expected number of x's required to compute a single realization of (15) is a function of θ and rather than minimizing $\mu_2(\theta)$ one may prefer to find that value of θ which minimizes (cf. Hammersley and Handscomb, 1964, page 51)

$$(18) \qquad (E_\theta T)(\mu_2(\theta) - \alpha^2)\,.$$

However, $E_w(T) = O(b)$ as $b \to \infty$ (e.g., Chow, Robbins, and Siegmund, 1971, page 29), and since the convergence to 0 in Theorem 1 is exponentially fast the choice $\theta = w$ would still be asymptotically optimal under the criterion (18). From a practical point of view one will frequently want to estimate along with α the expectation of T. Hence the x's generated under P_w to estimate $P_u(s_T \geqq b)$ can also be used to estimate $E_w(T)$ (and by a similar argument, those x's generated under P_u to estimate $P_w(s_T \leqq a)$ can also serve to estimate $E_u(T)$). Thus the possibility that $E_w(T)$ exceeds $E_u(T)$ typically will not present serious objections to the use of importance sampling.

(ii) The representation (13) has proved very useful in developing analytic approximations to α (cf. Wald, 1947; Siegmund, 1974a; and in a similar although not identical context, Feller, 1966, page 393). Its usefulness has depended in large part on its simplicity compared to the more general (12). Theorem 1 shows that in the class of representations (12) the choice $\theta = w$ which yields (13) has an intrinsic optimality property for estimating α which explains its success in a more satisfactory manner than does the convenience of algebraic simplicity.

(iii) To understand better the preceding remark it is instructive to consider a simpler fixed sample size problem. For any fixed n, it is easy to see from (7) that

$$(19) \qquad P_u(s_n \geqq b) = \left(\int_{(s_n \geqq b)} \exp[(u-\theta)s_n]\, dP_\theta\right) \exp[-n(\Psi(u) - \Psi(\theta))]\,.$$

For obtaining asymptotic approximations for $P_u(s_n \geqq b)$ as $n \to \infty$, the representation (19) has proved useful, but in this case with the choice $\theta = 0$, giving

$$(20) \qquad P_u(s_n \geqq b) = \exp(-n\Psi(u)) \int_{(s_n \geqq b)} \exp(us_n)\, dP_0\,.$$

(cf. Bahadur and Ranga Rao, 1960, or in the present notation Siegmund, 1975b). From the point of view of variance reduction in Monte Carlo, one should choose θ in (19) to minimize

$$\int_{(s_n \geqq b)} \exp[2(u-\theta)s_n]\, dP_\theta \exp[-2n(\Psi(u)-\Psi(\theta))] \,. \tag{21}$$

By (7) and (8), this quantity equals

$$\int_{(s_n \geqq b)} \exp[(2u-\theta)s_n]\, dP_0 \exp[-n(2\Psi(u)-\Psi(\theta))] \,. \tag{22}$$

An application of the central limit theorem to the integral in (22), as in Bahadur and Ranga Rao (1960), shows that this integral is for large n a multiple of $n^{-\frac{1}{2}}$. Hence (22) is minimized asymptotically as $n \to \infty$ by minimizing the exponential factor, which by (8) and the strict convexity of Ψ is achieved by putting $\theta = 0$. Thus in this fixed sample size case, as in the sequential case, the convenient analytic choice of θ is the asymptotically optimal Monte Carlo choice.

3. Examples. The following numerical example compares direct simulation of α with the average of independent realizations of

$$I_{(s_T \geqq b)} \exp[-(w-u)s_T] \tag{23}$$

generated according to P_w. Under P_θ the random variables $x_1, x_2, \cdots$ are independent and normally distributed with mean θ and variance 1.

For this example $a = -b$ and results are given for different values of b and $u = E_u(x_1)$. The entries in Table 1 are not obtained from an actual simulation but are analytic approximations based on the results of Siegmund (1975a). Columns of Table 1 give the values of b, $-u$, α, σ = standard deviation of (23), $[\alpha(1-\alpha)]^{\frac{1}{2}}$, the relative efficiency $\alpha(1-\alpha)/\sigma^2$, and Wald's approximation to α.

TABLE 1

b	$-u = w$	α	σ	$[\alpha(1-\alpha)]^{\frac{1}{2}}$	R.E.	Wald
9	.5	6.92×10^{-5}	3.11×10^{-5}	8.32×10^{-3}	7.16×10^{4}	1.23×10^{-4}
9	.25	8.24×10^{-3}	2.07×10^{-3}	9.04×10^{-2}	1910	1.10×10^{-2}
9	.125	8.35×10^{-2}	2.73×10^{-2}	2.77×10^{-1}	102	9.53×10^{-2}
5	.5	3.76×10^{-3}	1.71×10^{-3}	6.12×10^{-2}	1280	6.66×10^{-3}
5	.25	5.78×10^{-2}	2.00×10^{-2}	2.33×10^{-1}	136	7.59×-10^{-2}
5	.125	1.99×10^{-1}	1.02×10^{-1}	3.99×10^{-1}	15	2.23×10^{-1}

Many statistical problems exhibit sufficient symmetry to permit combining a second variance reducing technique with importance sampling to produce yet more effective estimation of α. The following paragraph describes briefly one technique which is complementary to importance sampling. Its effect is greatest for u near 0, where importance sampling degenerates into direct Monte Carlo.

If $\Psi(\theta) = \Psi(-\theta)$ and the P_0 distribution of x_k is symmetric so that $P_u\{x_k \in dx\} = P_w\{-x_k \in dx\}$, and if $a = -b$, then $P_u\{s_T \geqq b\} = P_w\{s_T \leqq -b\}$. Under these conditions one may estimate α by an average of independent P_w realizations of the convex combination

$$cI_{(s_T \geqq b)} \exp(-2ws_T) + (1 - c)I_{(s_T \leqq -b)} \tag{24}$$

for $0 < c < 1$. Since the two terms appearing in (24) are negatively correlated, for a proper choice of c the variance of the combination estimator is smaller than the variances of the individual terms. It is easy to see that (24) has minimum variance for $c^* = 1/(1 + \alpha + \sigma^2/\alpha)$, where as above $\sigma^2 = \mathrm{Var}_w\,[I_{(s_T \geqq b)} \exp(-2ws_T)]$. For example, for $b = 5$ and $w = .125$, corresponding to the last row of Table 1, the optimal value c^* is .8; and with this choice the relative efficiency for the combination estimator compared to importance sampling used alone is 29, which gives an overall relative efficiency of 435 compared to direct Monte Carlo. In practice c^* must be estimated either by approximate calculations or empirically.

The following examples illustrate situations in which analytic methods are less satisfactory or do not exist, and then Monte Carlo methods play a more important role.

(i) Sequential t-test: For testing whether a normal mean μ is negative or positive when the variance σ^2 is unknown, Rushton (1950) suggested that one use a sequential probability ratio test based on the t-statistic for testing H_0: $\mu/\sigma = -\delta_0$ against H_1: $\mu/\sigma = \delta_1$, where δ_1 and δ_2 are two positive numbers. Wald's error probability approximations apply when μ/σ equals one of the two values—δ_0 or δ_1, but for other parameter points there is no known analytic method. While the optimality property of Theorem 1 may not hold for this problem, importance sampling of the kind discussed in this paper can be profitably used as a variance reducing technique in Monte Carlo studies.

(ii) Robbins and Siegmund (1974) (cf. also Flehinger and Louis, 1970) study the problem of testing which of the two normal means, μ_1 and μ_2, is larger in such a way that a minimum average number of observations is taken on the population giving the smaller mean response. They discuss a sequential probability ratio test stopping rule and various sampling rules. While the Wald error probability approximations apply in this problem, the expected sample size approximation does not except for essentially deterministic sampling rules. In the course of studying the expected sample size by Monte Carlo it is no additional work to obtain estimates of the error probabilities which are usually more accurate than Wald's approximations and whose accuracy would be increased still further by importance sampling. A version of this problem for binomial data is now under investigation. In this case Wald's approximations do not apply at all, except through the central limit theorem, and Monte Carlo methods are necessary.

(iii) In the previous two examples the stopping rule was of the form of a sequential probability ratio test but accurate computation of error probabilities was made more difficult because the stochastic process under consideration was not a random walk. By way of contrast, for the stopping rules of Schwarz (1962) and Chernoff (1961) approximations to the error probabilities have not been obtained even in the case of independent normal random variables. For closed stopping regions made up from straight live segments as in Anderson (1960) or a truncated sequential probability ratio test, exact results may be obtained for Brownian motion which yield approximations of the order of accuracy of Wald's for a discrete normal process. In these problems simulation may be helpful to evaluate probabilities or check approximations, and importance sampling as suggested in Section 2 gives some gain in efficiency over direct simulation although the magnitude of this again and the possibilities for further improvement vary from one situation to another.

(iv) The following example illustrates the dangers of the careless use of importance sampling. In the theory of open-ended tests Darling and Robbins (1968) study stopping rules of the form

$$N = \text{first} \quad n \geqq 1 \quad \text{such that} \quad s_n \geqq b_n \tag{25}$$
$$= \infty \quad \text{if} \quad s_n < b_n \quad \text{for all} \quad n ,$$

where

$$0 < b_n/n \to 0 \qquad n \to \infty , \tag{26}$$

is such that

$$P_0\{N < \infty\} < 1 . \tag{27}$$

By (10), (26), and the strong law of large numbers $P_\theta\{N < \infty\} = 1$ for all $\theta > 0$, and hence Lemma 1 with $\theta' = 0$, $\theta'' = \theta$, $A = \Omega$, and $\tau = N$ implies

$$P_0\{N < \infty\} = E_\theta[\exp(-\theta s_N + N\Psi(\theta))] \qquad \theta > 0 . \tag{28}$$

Darling and Robbins (1968) suggest that $P_0(N < \infty)$ may be estimated by Monte Carlo methods by averaging independent P_θ-realizations of

$$\exp(-\theta s_N + N\Psi(\theta)) \tag{29}$$

for suitable $\theta > 0$. The example is different in principle from those previously discussed, for direct simulation is impossible due to (27), and some other method must be found.

If N were defined by (25) with $b_n = b + cn$, then a shift in location of the x's would transform this problem into a limiting case of the problem of Section 2 with $a = -\infty$. It is easy to see from the proof of Theorem 1 that in this case an asymptotically optimal θ would exist. However, the theory of open-ended tests requires that $b_n = o(n)$ as $n \to \infty$, and with this condition, R. Berk (1969) has shown for normally distributed x's that

$$E_\theta[\exp(-r\theta s_N + rN\Psi(\theta)] = \infty \qquad \text{for all} \quad r > 1. \tag{30}$$

A simpler proof which is not restricted to normally distributed x's is given in Section 4.

4. Theoretical developments. Let $P_\theta^{(n)}$ denote the restriction of P_θ to the space of $x_1, \cdots, x_n$ $(n = 1, 2, \cdots)$. It follows from (7) and (8) that for each θ', θ''

$$dP_{\theta'}^{(n)} = \exp[(\theta' - \theta'')s_n - n(\Psi(\theta') - \Psi(\theta''))]\, dP_{\theta''}^{(n)} . \tag{31}$$

In particular by (11)

$$dP_u^{(n)} = \exp[-(w - u)s_n]\, dP_w^{(n)} . \tag{32}$$

For any stopping rule τ for the sequence $x_1, x_2, \cdots$ let $\mathscr{F}_\tau$ denote the class of all events A such that $A \cap \{\tau = n\}$ is defined in terms of $x_1, \cdots, x_n$ for every $n = 1, 2, \cdots$. A version of the fundamental identity of sequential analysis (cf. Chow, Robbins and Siegmund, 1971, page 33) is

LEMMA 1. *For each θ', θ'' and $A \in \mathscr{F}_\tau$*

$$P_{\theta'}(A\{\tau < \infty\}) = \int_{A(\tau<\infty)} \exp[(\theta' - \theta'')s_\tau - \tau(\Psi(\theta') - \Psi(\theta''))]\, dP_{\theta''} . \tag{33}$$

PROOF. Writing $\int_{A(\tau<\infty)} = \sum_1^\infty \int_{A(\tau=n)}$ and appealing to (31) and the definition of $\mathscr{F}_\tau$ yield (33).

REMARK. According to Wald (1947, page 157) $P_\theta\{T < \infty\} = 1$ for all θ. Hence for $\theta' = u$, $\theta'' = \theta$, $A = \{s_T \geqq b\}$, and $\tau = T$ the equation (33) becomes equation (12).

PROOF OF THEOREM 1. To simplify the exposition and eliminate some special cases, assume that $\Psi(\theta)$ is defined for all real $\theta \geqq u$. Then by (4)

$$\lim_{\theta\to\infty} \Psi(\theta) = +\infty . \tag{34}$$

Lemma 1 may be interpreted to say that if P_θ is considered as a measure on $\mathscr{F}_\tau \cap \{\tau < \infty\}$ (which by a slight abuse of notation will still be denoted P_θ), then for any two values θ', θ'', the measures $P_{\theta'}$ and $P_{\theta''}$ are mutually absolutely continuous and

$$dP_{\theta'}/dP_{\theta''} = \exp[(\theta' - \theta'')s_\tau - \tau(\Psi(\theta') - \Psi(\theta''))] . \tag{35}$$

Putting $\theta' = \theta$, $\theta'' = 0$, and $\tau = T$ in (35) allows (17) to be written

$$\mu_2(\theta) = \int_{(s_T \geqq b)} \exp[(2u - \theta)s_T - T(2\Psi(u) - \Psi(\theta))]\, dP_0 . \tag{36}$$

From (36) it is apparent that any choice $\theta_1 < 0$ can be improved on by any value $\theta_2 > 0$ such that $\Psi(\theta_2) \leqq \Psi(\theta_1)$. Also by (34) there exists $\bar{\theta} > 0$ such that

$$\Psi(\bar{\theta}) = 2\Psi(u) . \tag{37}$$

On $[u, \bar{\theta}]$ define $\xi = \xi(\theta) \geqq 0$ by

$$\Psi(\xi) = 2\Psi(u) - \Psi(\theta) . \tag{38}$$

It is easy to see that ξ is strictly increasing on $[u, 0]$, strictly decreasing on $[0, \bar{\theta}]$, and by (8) and (11)

$$\xi(0) = \bar{\theta} , \qquad \xi(\bar{\theta}) = 0 , \qquad \xi(w) = \xi(u) = w . \tag{39}$$

Putting $\theta' = 0$, $\theta'' = \xi$ and $\tau = T$ in (35) shows by (38) that (36) may be rewritten as

$$\mu_2(\theta) = \exp[(2u - \theta - \xi)b] \int_{(s_T \geq b)} \exp[(2u - \theta - \xi)(s_T - b)]\, dP_\xi \tag{40}$$

for $u \leqq \theta \leqq \bar{\theta}$. Let $\tau_b = \inf\{n : s_n \geqq b\}$. Then for any $\lambda > 0$, $\omega > 0$

$$\begin{aligned} &\int_{(s_T \geq b)} \exp[-\lambda(s_T - b)]\, dP_\omega \\ &\quad = E_\omega\{\exp[-\lambda(s_{\tau_b} - b)]\} - \int_{(s_T \leq a)} E_\omega(\exp[-\lambda(s_{\tau_b} - b)] \mid s_T)\, dP_\omega\,. \end{aligned} \tag{41}$$

It is an easy consequence of the renewal theorem (cf. Feller, 1966, page 356) that the expectation and conditional expectation on the right-hand side of (41) converge to a nonzero (finite) limit as $b \to \infty$, and for the conditional expectation convergence is uniform in s_T. Hence for $u < \theta < \bar{\theta}$ the integral on the right-hand side of (40) lies in (0, 1) and is bounded away from 0. (Typically it will converge, but since no restriction has been placed on the behavior of $a \leqq 0$, it may oscillate.)

It follows that if $\xi(\theta) + \theta$ has a unique maximum in $[0, \bar{\theta}]$, then the theorem is established for all $\theta < \bar{\theta}$.

From (39) it follows that $\xi(0) + 0 = \xi(\bar{\theta}) + \bar{\theta} = \bar{\theta}$ and $\xi(w) + w = 2w$. By strict convexity of Ψ, (15), and (16)

$$\tfrac{1}{2}\Psi(2w) = \tfrac{1}{2}(\Psi(0) + \Psi(2w)) > \Psi(w) = \Psi(u)\,,$$

so by (37) and the monotonicity of Ψ on $\{0, \infty)$

$$2w > \bar{\theta}\,. \tag{42}$$

Hence $\xi(\theta) + \theta$ must assume its maximum on $[0, \bar{\theta}]$ at an interior point θ^* at which

$$\xi'(\theta^*) + 1 = 0\,. \tag{43}$$

Differentiating (38) gives $\Psi'(\xi)\xi' = -\Psi'(\theta)$, so that at θ^* satisfying (43)

$$\Psi'(\xi(\theta^*)) = \Psi'(\theta^*)\,. \tag{44}$$

However, Ψ' is strictly increasing and hence (44) implies that $\xi(\theta^*) = \theta^*$, for which the unique solution is $\theta^* = w$.

Since by the preceding argument $\log \mu_2(w) \sim -2b(w - u)$, to complete the proof it suffices to show by direct computation that for all $\theta \geqq \bar{\theta}$ there exists $\varepsilon > 0$ such that

$$\mu_2(\theta) \geqq \exp[-2b(w - \varepsilon - u)] \tag{45}$$

for b sufficiently large. The following argument shows that in fact $\lim_{b\to\infty} \mu_2(\theta) = \infty$ for $\theta > \bar{\theta}$. A slightly simpler argument incorporating (42) shows that (45) holds for $\theta = \bar{\theta}$, which completes the proof.

Assume then that $\theta > \bar{\theta}$, so that

$$\Psi(\theta) - 2\Psi(u) > 0\,. \tag{46}$$

Then

$$\begin{aligned} \mu_2(\theta) &= \int_{(s_T \geq b)} \exp[(2u - \theta)s_T - T(2\Psi(u) - \Psi(\theta))]\, dP_0 \\ &\geqq P_0(s_T \geqq b, T \geqq b^{\frac{3}{2}}, x_T < b) \exp[2(2u - \theta)b \\ &\qquad + b^{\frac{3}{2}}(\Psi(\theta) - 2\Psi(u))]\,. \end{aligned} \tag{47}$$

By (46) and (47), to show that $\mu_2(\theta) \to +\infty$ as $b \to \infty$, it suffices to show that

$$bP_0(s_T \geqq b,\, T \geqq b^{\frac{3}{2}},\, x_T < b)$$

is bounded away from 0 as $b \to \infty$. But

$$\begin{aligned} P_0(s_T \geqq b,\, T \geqq b^{\frac{3}{2}},\, x_T < b) \geqq\ & P_0(s_T \geqq b) - P_0(s_T \geqq b,\, T < b^{\frac{3}{2}}) \\ & - P_0(T \leqq b^4,\, x_T \geqq b) - P_0(T > b^4)\,, \end{aligned} \tag{48}$$

and hence it suffices to show that $bP_0(s_T \geqq b)$ is bounded away from 0 and that the remaining three terms on the right-hand side of (48) are $o(P_0(s_T \geqq b))$ as $b \to \infty$. The following three lemmas complete the proof in the case that $|a|/b$ remains bounded as $b \to \infty$. The case $|a|/b \to \infty$ may be treated similarly. The case $\liminf |a|/b < \infty$ and $\limsup |a|/b = \infty$ may be reduced to the preceding by considering subsequences.

LEMMA 2. *Let $a = 0$ and define $\tau_- = \inf\{n : n \geqq 1, s_n \leqq 0\}$. Then $P_0\{s_T \geqq b\} \sim -E_0 s_{\tau-}/b$ as $b \to \infty$.*

(For a more precise result cf. Siegmund, 1975a.)

PROOF. By (9) and Wald's lemma

$$0 = E_0 s_T = P_0(s_T \geqq b)b + \int_{(s_T \geqq b)} (s_T - b)\, dP_0 + \int_{(s_T \leqq 0)} s_T\, dP_0\,. \tag{49}$$

Now

$$\int_{(s_T \leqq 0)} s_T\, dP_0 = E_0 s_{\tau-} - \int_{(s_T \geqq b)} E_0(s_{\tau-} \mid s_T)\, dP_0\,. \tag{50}$$

It follows from the renewal theorem that the conditional expectation on the right-hand side of (50) converges to a finite limit as $b \to \infty$ uniformly in s_T on $\{s_T \geqq b\}$. Hence

$$\int_{(s_T \leqq 0)} s_T\, dP_0 = E_0 s_{\tau-} + O(P_0(s_T \geqq b))\,. \tag{51}$$

Similar reasoning shows that

$$\int_{(s_T \geqq b)} (s_T - b)\, dP_0 = O(P_0(s_T \geqq b)) + o(1)\,. \tag{52}$$

The lemma follows upon substitution of (51) and (52) into (49).

In Lemmas 3 and 5 $[x]$ denotes the largest integer $\leqq x$.

LEMMA 3. *As $b \to \infty$*

$$P_0(s_T \geqq b,\, T < b^{\frac{3}{2}}) = O(\exp(-b^{\frac{1}{4}}))\,.$$

PROOF. By the Doob–Kolmogorov inequality for submartingales (cf. Chow, Robbins and Siegmund, 1971, page 24), for any $\theta > 0$

$$\begin{aligned} P_0(s_T \geqq b,\, T < b^{\frac{3}{2}}) &\leqq P_0(\max\nolimits_{1 \leqq k < b^{\frac{3}{2}}} s_k \geqq b) \\ &= P_0(\max\nolimits_{1 \leqq k < b^{\frac{3}{2}}} \exp(\theta s_k) \geqq \exp(\theta b)) \\ &\leqq \exp(-\theta b) E_0(\exp(\theta s_{[b^{\frac{3}{2}}]})) \\ &\leqq \exp(-\theta b + b^{\frac{3}{2}}\Psi(\theta))\,. \end{aligned}$$

Since $\Psi(\theta) \sim (E_0 x_1^{\,2})\theta^2/2$ as $\theta \to 0$, the lemma follows from the choice $\theta = b^{-\frac{3}{4}}$.

LEMMA 4. *$P_0(T \leqq b^4, x_T \geqq b) = O(e^{-b})$ as $b \to \infty$.*

PROOF. For arbitrary $\theta > 0$,

$$\begin{aligned} P_0(T \leqq b^4, x_T \geqq b) &\leqq P_0(\max_{k \leqq b^4} x_k \geqq b) \\ &\leqq b^4 P_0(x_1 \geqq b) = b^4 \int_{(x_1 \geqq b)} e^{-\theta x_1 + \Psi(\theta)}\, dP_\theta \\ &\leqq b^4 e^{-\theta b + \Psi(\theta)} \end{aligned}$$

from which the lemma follows at once.

LEMMA 5. *If $|a| = O(b)$ as $b \to \infty$, then*

$$P_0(T > b^4) = O(e^{-b})\,. \tag{53}$$

PROOF. Assume $|a|/b < c - 1$ for all large b. Then by independence of the x's and the definition of T

$$\begin{aligned} P_0(T > b^4) &\leqq P_0(\max_{1 \leqq k \leqq b^2} |s_{[kb^2]} - s_{[(k-1)b^2]}| < cb) \\ &\leqq \{P_0\{|s_{[b^2-1]}| < cb\}\}^{[b^2]}\,. \end{aligned} \tag{54}$$

By the central limit theorem the probability within braces on the right-hand side of (54) converges to a limit < 1 as $b \to \infty$, and the lemma follows.

REMARK. A more sophisticated argument shows that (53) remains true even if $|a|/b \to \infty$. However, in this case it is easier to replace Lemma 2 by $P_0(s_T \geqq b) \to 1$ and (53) by the weaker result that $P_0\{T > b^4\} \leqq P_0\{\tau_b > b^4\} \to 0$ as $b \to \infty$ in order to complete the proof of Theorem 1.

PROOF OF (30). Let $\theta > 0$ and $r > 1$ be arbitrary. Let $\varepsilon > 0$ be such that

$$(r-1)\Psi(\theta) - 3\varepsilon r\theta > 0\,. \tag{55}$$

Since $b_n = o(n)$, for all n sufficiently large $b_k < \varepsilon(k+1)$ for all $k \geqq n$. Then since $s_{T-1} < b_{T-1}$

$$\begin{aligned} E_\theta[\exp(-r\theta s_N + rN\Psi(\theta))] &\geqq \int_{\{N>n, x_N < \varepsilon N\}} \exp[-r\theta(b_{N-1} + \varepsilon N) + rN\Psi(\theta)]\, dP_\theta \\ &\geqq \int_{\{N>n, x_N < \varepsilon N\}} \exp[(r\Psi(\theta) - 2\varepsilon r\theta)N]\, dP_\theta \\ &\geqq \exp[(r\Psi(\theta) - 2\varepsilon r\theta)n] P_\theta\{N > n, x_N < \varepsilon N\}\,. \end{aligned}$$

Also since $\{N > n\}$ is independent of $x_{n+1}, x_{n+2}, \cdots$

$$\begin{aligned} P_\theta\{N > n, x_N < \varepsilon N\} &= P_\theta\{N > n\} - P_\theta\{N > n, x_N \geqq \varepsilon N\} \\ &\geqq P_\theta\{N > n\}(1 - \textstyle\sum_{k=n+1}^\infty P_\theta\{x_k > \varepsilon k\}) \\ &\geqq \tfrac{1}{2} P_\theta\{N > n\} \qquad \text{for } n \text{ sufficiently large.} \end{aligned}$$

Now by (31)

$$\begin{aligned} P_\theta\{N > n\} &= \int_{\{N>n\}} \exp(\theta s_n - n\Psi(\theta))\, dP_0 \\ &\geqq \int_{\{N>n, s_n \geqq -n\varepsilon\}} \exp(\theta s_n - n\Psi(\theta))\, dP_0 \\ &\geqq \exp(-n\varepsilon\theta - n\Psi(\theta)) P_0\{N > n, s_n \geqq -n\varepsilon\} \\ &\geqq \exp(-n\varepsilon\theta - n\Psi(\theta))[P_0\{N = \infty\} - P_0\{s_n < -n\varepsilon\}]\,. \end{aligned}$$

By (10) and the law of large numbers $P_0\{s_n < -n\varepsilon\} \to 0$; hence the preceding estimates in conjunction with (27) and (55) show that

$$E_\theta[\exp(-r\theta s_T + rT\Psi(\theta))] \geqq \tfrac{1}{2}\exp[\{(r-1)\Psi(\theta) - 3\varepsilon r\theta\}n](P_0\{N=\infty\} - o(1)) \to \infty \quad \text{as} \quad n \to \infty .$$

REFERENCES

ANDERSON, T. W. (1960). A modification of the sequential probability ratio test to reduce the sample size. *Ann. Math. Statist.* **31** 165–197.

BAHADUR, R. R. and RAO, R. RANGA (1960). On deviations of the sample mean. *Ann. Math. Statist.* **31** 1015–1027.

BERK, R. (1969). A note on a Monte Carlo estimator of Darling and Robbins. *Israel J. Math.* **7** 365–368.

CHERNOFF, H. (1960). Sequential tests for the mean of a normal distribution. *Proc. Fourth Berkeley Symp. Math. Statist. Prob.* **1** 79–91. Univ. of California Press.

CHOW, Y. S., ROBBINS, H. and SIEGMUND, D. (1971). *Great Expectations: The Theory of Optimal Stopping.* Houghton Mifflin, Boston.

DARLING, D. and ROBBINS, H. (1968). Some further remarks on inequalities for sample sums. *Proc. Nat. Acad. Sci. U.S.A.* **60** 1175–1182.

ESSCHER, F. (1932). On the probability function in the collective theory of risk. *Skand. Aktuarietidskr.* **15** 175–195.

FELLER, W. (1966). *An Introduction to Probability Theory and Its Applications* **2**. Wiley, New York.

FLEHINGER, B. and LOUIS, T. (1970). Sequential medical trials with data dependent treatment allocation. *Proc. Sixth Berkeley Symp. Math. Statist. Prob.* **4** 43–51.

HAMMERSLEY, J. and HANDSCOMB, D. (1964). *Monte Carlo Methods.* Methuen, London.

ROBBINS, H. and SIEGMUND, D. (1974). On sequential tests involving two populations. *J. Amer. Statist. Assoc.* **69** 132–139.

RUSHTON, S. (1950). On a sequential t-test. *Biometrika* **37** 326–333.

SCHWARZ, G. (1962). Asymptotic shapes of Bayes sequential testing regions. *Ann. Math. Statist.* **33** 224–236.

SIEGMUND, D. (1975a). A note on the error probabilities and average sample number of the sequential probability ratio test. *J. Roy. Statist. Soc. Ser.* **37** 394–401.

SIEGMUND, D. (1975b). Large deviation probabilities in the strong law of large numbers. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **31** 107–113.

WALD, A. (1947). *Sequential Analysis.* Wiley, New York.

DEPARTMENT OF MATHEMATICAL STATISTICS
COLUMBIA UNIVERSITY
NEW YORK, NEW YORK 10027