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A SPECIAL PROPERTY OF LINEAR ESTIMATES OF THE NORMAL MEAN

BY GLEN MEEDEN

Iowa State University

Let X be a normal random variable with mean θ and variance 1 and consider the problem of estimating θ with squared error loss. If $\delta(x) = ax + b$ is a linear estimate with $0 \leqq a \leqq 1$ then it is well known that $\lambda\delta$ is an admissible proper Bayes estimate for $\lambda \in (0, 1)$. That is, all contractions of δ are proper Bayes estimates. In this note we show that no other estimates have this property.

THEOREM. *Let X be a normal random variable with mean θ and variance one. Let δ be a generalized Bayes estimator for estimating $\gamma(\theta) = \theta$ with squared error loss. Suppose there exists a constant $M > 0$ such that*

$$\delta(x) \leqq x + M \qquad \text{for} \quad x \geqq 0$$

and

$$\delta(x) \geqq x - M \qquad \text{for} \quad x < 0 .$$

If $\lambda\delta$ is a proper Bayes estimator for all $\lambda \in (0, 1)$ then $\delta(x) = ax + b$ for all x for some constants a and b where $0 \leqq a \leqq 1$.

PROOF. Consider the special case when $\delta(x) = x + \alpha(x)$ where α is a bounded function. Since δ is a generalized Bayes estimator we have by Theorem 3.2.1 of Strawderman and Cohen (1971) that

$$(1) \qquad \exp[\textstyle\int_0^x \delta(y)\,dy] = \exp[(\frac{1}{2})x^2]\exp[\int_0^x \alpha(y)\,dy]$$

is a moment generating function for some distribution function H. Since $\lambda\delta$ is a Bayes estimator we have that $\exp[\int_0^x \delta(y)\,dy]^\lambda$ is a moment generating function for all $\lambda \in (0, 1)$ and H must be an infinitely divisible distribution. The form of the moment generating function for an infinitely divisible distribution is well-known and we will show that the moment generating function in (1) cannot be of that form except in the case $\alpha(x) \equiv c$.

If $\lambda\delta$ is Bayes for $\lambda \in (0, 1)$ then we have by Gnedenko and Kolmogorov (1954) that there exist a real constant γ and a nondecreasing bounded function G such that

$$(2) \qquad \textstyle\int_0^x \delta(y)\,dy = \gamma x + \int_{-\infty}^{+\infty} \phi_x(u)\,dG(u) \qquad \text{for} \quad -\infty < x < +\infty$$

where

$$\phi_x(u) = \left[e^{xu} - 1 - \frac{xu}{1 + u^2}\right]\frac{1 + u^2}{u^2} \qquad \text{for} \quad u \neq 0$$
$$= x^2/2 \qquad \text{for} \quad u = 0 .$$

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If G is the function which puts mass d at zero and is constant elsewhere then (2) yields

$$\delta(x) = dx + \gamma .$$

The only choice of d which does not violate the condition that $|\delta(x) - x|$ is bounded is $d = 1$. With $d = 1$ we have that $\lambda(x + \gamma)$ is Bayes for $\lambda \in (0, 1)$. To complete the proof we will show that no other choice of G satisfies (2).

Suppose G does satisfy (2) but G does not concentrate its mass at zero. For definiteness assume there exist $0 < a < b$ such that $G(b) - G(a) > 0$. We first obtain a lower bound for the right hand side of (2) for large values of x. The integral in the right hand side can be broken up into three parts over the intervals $(-\infty, -1]$, $(-1, 0]$ and $(0, +\infty)$ respectively. For $x > 0$ the integral over $(-\infty, -1]$ is bounded below since $\phi_x(u) \geqq -2$ for $u \leqq -1$ and G is bounded. For large values of x the integral over $(-1, 0]$ is bounded below since for each fixed $u \in (-1, 0]$, $(d/dx)\phi_x(u) > 0$ for x sufficiently large independent of u. Finally since for $x > 0$ and $u > 0$ $\phi_x(u) \geqq 0$ we have

$$\begin{aligned} \textstyle\int_0^\infty \phi_x(u)\, dG(u) &\geqq \textstyle\int_a^b \phi_x(u)\, dG(u) \\ &\geqq \alpha e^{ax} + \beta_1 x + \beta_2 \end{aligned}$$

for $x > 0$ where $\alpha > 0$, β_1 and β_2 are some constants. Summarizing we have that

$$\gamma x + \textstyle\int_{-\infty}^{+\infty} \phi_x(u)\, dG(u) \geqq \alpha e^{ax} + \beta_1 x + \beta_2' \tag{3}$$

for x sufficiently large.

On the other hand we have from (1) and the fact that for $x > 0$ $\alpha(x)$ is bounded above, that

$$\textstyle\int_0^x \delta(y)\, dy \leqq x^2/2 + kx \tag{4}$$

for $x > 0$.

Note that (2) and (3) contradict (4) for large values of x and hence G must be constant on $(0, +\infty)$. If G is not constant on $(-\infty, 0)$ we get a similar contradiction by letting x approach $-\infty$. Hence G is constant everywhere except at zero and the theorem is proved in the special case. Since equations (2) and (3) do not depend on the form of δ in the special case just considered and (4) is true in general; the result follows.

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DEPARTMENT OF STATISTICS
SNEDECOR HALL
IOWA STATE UNIVERSITY
AMES, IOWA 50010