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APPLICATIONS OF PRODUCTS TO THE GENERALIZED COMPOUND SYMMETRY PROBLEM

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Arnold (1973) studied testing problems where the covariance matrix is assumed to have the generalized correlation structure under both hypotheses. That paper showed how to transform such problems to "products" of unpatterned problems. This paper extends those results to testing problems where the covariance matrix is assumed to have Geisser's (1963) generalization of the pattern of compound symmetry. We prove theorems indicating how to transform such problems to products of unpatterned problems. These results are then applied to three problems: 1. a general problem where both the mean vectors and covariance matrix are patterned (this problem is general enough to include both the multivariate analysis of variance (MANOVA) and classification problems.); 2. the MANOVA problem when only the covariance matrix is patterned; 3. a problem arising only when the covariance matrix is patterned. In this paper we only show how to transform such problems to products of unpatterned problems that have been studied, since in Arnold (1973) it was showing how to convert results about known problems to results about their product.

1. Introduction. In Arnold (1973) problems involving certain patterned covariance matrices were transformed to products of unrestricted problems. In this paper these results are extended to show how to transform problems involving Geisser's (1963) generalization of Votaw's (1948) model of compound symmetry. In Section 2 we prove the basic theorems telling how to transform such problems to products of problems where nothing is patterned. In Section 3, these results are applied to three problems. Arnold (1973) showed that results about component problems could be used to generate results about their product. So, in this paper, a problem is considered "solved" when we have shown it to be a product of problems that have been studied.

1.1. *Definitions.* We write

$$A = (A_{ij}), \quad (A_{ij} : p_i \times q_j), \qquad i = 1, \cdots, n, j = 1, \cdots, m,$$
$$B = (B_i), \quad (B_i : p_i \times r), \qquad i = 1, \cdots, n,$$

to mean that A_{ij} is a $p_i \times q_j$ matrix, and B_i is a $p_i \times r$ matrix, and

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nm} \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 \\ \vdots \\ B_n \end{pmatrix}.$$

If A is a $k \times m$ matrix and $B = (b_{ij})$, $(b_{ij} : 1 \times 1)$ is an $n \times p$ matrix, then the

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Kronecker product of A and B, $A * B$ is the $nk \times mp$ matrix

$$C = (C_{ij}), \quad (C_{ij}: k \times m), \qquad i = 1, \cdots, n, j = 1, \cdots, p,$$

where $C_{ij} = b_{ij}A$. Clearly $(A * B)' = A' * B'$; $(A * B)(C * D) = AC * BD$ (if all multiplications are defined); $(A * B)^{-1} = A^{-1} * B^{-1}$ (if A and B are invertible). The following matrices will be used frequently in this paper. Let $I(j)$ be the $j \times j$ identity matrix; let $E(i, j)$ be the $i \times j$ matrix with 1 in every position; let $F(i, j)$ be the $i \times j$ matrix with $(ij)^{\frac{1}{2}}$ in the first row, first column and 0 in every other position. Let $U(j)$ be a $j \times j$ orthogonal matrix with first column equal to $j^{\frac{1}{2}}(1, \cdots, 1)'$. Then $U(i)'E(i, j)U(j) = F(i, j)$.

To state the generalized compound symmetry model, let $X(i, 1), \cdots, X(i, k_i)$ be $p_i \times 1$ vectors, $i = 1, \cdots, m$. Let $X_i' = (X(i, 1)', \cdots, X(i, k_i)')$ and $X' = (X_1', \cdots, X_m')$. If X has a $t\,(= \sum_{i=1}^m p_i k_i)$ dimensional normal distribution with mean vector μ and covariance matrix Σ, then the $X(i, 1), \cdots, X(i, k_i)$ are interchangeable for each $i = 1, \cdots, m$ if and only if there exist $p_i \times 1$ vectors δ_i, $p_i \times p_i$ matrices A_i and $p_i \times p_j$ matrices B_{ij} such that

$$\Sigma = (\Sigma_{ij}), \quad (\Sigma_{ij}: k_i p_i \times k_j p_j), \qquad i, j = 1, \cdots, m, \tag{1.1}$$
$$u = (\mu_i), \quad (\mu_i: k_i p_i \times 1), \qquad i = 1, \cdots, m,$$

where

$$\Sigma_{ii} = (A_i - B_{ii}) * I(k_i) + B_{ii} * E(k_i, k_i), \tag{1.2}$$
$$\Sigma_{ij} = B_{ij} * E(k_i, k_j), \qquad i \neq j,$$

and

$$\mu_i = \delta_i * E(k_i, 1). \tag{1.3}$$

If there exists A_i, and B_{ij} satisfying (1.2) we say that Σ has pattern E. If there exists such δ_i satisfying (1.3), we say that μ has *pattern D*. Pattern E is Geisser's (1963) generalization of Votaw's (1948) model of compound symmetry.

In this paper we show how to transform problems involving patterns D and E to products of unpatterned problems. In this paragraph we therefore define such a product. A testing problem P consists of the following three elements: an observed random variable X having density from a general class $D(\theta)$ (for example $N(\mu, \Sigma)$); a null set Ω; and an alternative set Θ. We use the following shorthand for P:

$$P: X \sim D(\theta),$$
$$H: \theta \in \Omega,$$
$$A: \theta \in \Theta.$$

We make one convention for this notation. All random variables are independent unless otherwise specified. Let P_1 and P_2 be the problems

$$P_1: X_1 \sim D_1(\theta_1), \qquad P_2: X_2 \sim D_2(\theta_2),$$
$$H_1: \theta_1 \in \Omega_1, \qquad H_2: \theta_2 \in \Omega_2,$$
$$A_1: \theta_1 \in \Theta_1, \qquad A_2: \theta_2 \in \Theta_2.$$

Then the *product* P of P_1 and P_2 (written $P = P_1 \times P_2$) is the problem

$$\begin{aligned} P&: X_1 \sim D_1(\theta_1)\,, && X_2 \sim D_2(\theta) \\ H_0&: \theta_1 \in \Omega_1\,, && \theta_2 \in \Omega_2\,, \\ A_0&: \theta_1 \in \Theta_1\,, && \theta_2 \in \Theta_2\,, \end{aligned}$$

That is, the product P is just the problem of testing P_1 and P_2 simultaneously and independently. In addition, we define one of the component problems to be *trivial* if it has the same null set and alternative set. Theorems A and B of Arnold (1973) show how to transform results about the component problems to results about their product. The problems studied in this paper transform to a product of more than 2 problems. Such a product can be defined recursively by

$$P_1 \times \cdots \times P_k = (P_1 \times \cdots \times P_{k-1}) \times P_k\,.$$

It is clear that Theorems A and B can be immediately extended to products of k problems. For this reason we show how to transform the testing problems involving pattern covariance matrices to products of unpatterned problems that have been studied. Since results about the components will then generate results about their product, in this paper we do no more than show how to transform the problems involving patterned covariance matrices to products of unpatterned problems that have been studied.

2. Basic theorems.

THEOREM 1. *If Σ has pattern E as defined in* (1.1) *and* (1.2), *there exists an orthogonal matrix C independent of A_i and B_{ij} such that*

$$C'\Sigma C = \begin{pmatrix} \Xi_0 & 0 & \cdots & 0 \\ 0 & \Xi_1 * I(k_1 - 1) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \Xi_m * I(k_m - 1) \end{pmatrix} = \Xi \tag{2.1}$$

where

$$\begin{aligned} &\Xi_i = A_i - B_{ii} \quad \textit{and} \quad \Xi_0 \quad \textit{is} \quad q \times q \quad (q = \textstyle\sum_{i=1}^m p_i)\,, \\ &\Xi_0 = (D_{ij})\,, \quad (D_{ij} : p_i \times p_j)\,, \qquad\qquad i, j = 1, \cdots, m\,, \\ &D_{ii} = A_i + (k-1)B_{ii}\,, \qquad D_{ij} = (k_i k_j)^{\frac{1}{2}} B_{ij}\,, \qquad i \neq j\,. \end{aligned} \tag{2.2}$$

PROOF. Define

$$\begin{aligned} &\Gamma = (\Gamma_{ij})\,, \quad (\Gamma_{ij} : p_i k_i \times p_j k_j)\,, && i, j = 1, \cdots, m \\ &\Gamma_{ii} = I(p_i) * U(k_i)\,, \qquad \Gamma_{ij} = 0\,, && i \neq j\,. \end{aligned}$$

Clearly Γ is orthogonal. Let $V = \Gamma'\Sigma\Gamma$. Then

$$V = (V_{ij})\,, \quad (V_{ij} : k_i p_i \times k_j p_j)\,, \qquad i, j = 1, \cdots, m\,,$$

$$\begin{aligned} V_{ii} &= (I(p_i) * U(k_i))'[(A_i - B_{ii}) * I(k_i) + B_{ii} * E(k_i, k_i)](I(p_i) * U(k_i)) \\ &= (A_i - B_{ii}) * I(k_i) + B_{ii} * F(k_i, k_i) \\ &= \begin{pmatrix} A_i + (k-1)B_{ii} & 0 \\ 0 & (A_i - B_{ii}) * I(k_i - 1) \end{pmatrix} \end{aligned}$$

and

$$V_{ij} = (I(p_i) * U(k_i))'[B_{ij} * E(k_i, k_j)](I(p_j) * U(k_j)) = B_{ij} * F(k_i, k_j)$$
$$= \begin{pmatrix} (k_i k_j)^{\frac{1}{2}} B_{ij} & 0 \\ 0 & 0 \end{pmatrix}.$$

Upon writing V out, it is clear that there is a permutation matrix P such that $P'VP = P'\Gamma'\Sigma\Gamma P = \Xi$ defined in (2.1) and (2.2). Let $C = \Gamma P$. □

COROLLARY. *If Σ has pattern E, then $\Sigma > 0$ if and only if $\Xi_i > 0$, $i = 0, 1, \cdots, m$. Therefore, although Σ was patterned, the Ξ_i are unrestricted.*

Now we are ready to transform random variables with patterned covariances to random variables with unrestricted covariance matrices. So let X and S be independent, $X(t \times r) \sim N(\mu, \Sigma)$ and $S \sim W(n, \Sigma)$. That is, X is a $t \times r$ random matrix whose columns are independently normally distributed with common covariance Σ and $EX = \mu$, a $t \times r$ constant matrix, and S has a Wishart distribution with $ES = n\Sigma$. Now let C be the matrix defined in Theorem 1. Let

$$C'SC = \begin{pmatrix} W_0 & D_1 & \cdots & D_m \\ D_1' & V_{11} & \cdots & V_{1m} \\ \vdots & \vdots & & \vdots \\ D_m' & V_{m1} & \cdots & V_{mm} \end{pmatrix}, \qquad C'X = \begin{pmatrix} Y_0 \\ Z_1 \\ \vdots \\ Z_m \end{pmatrix}, \qquad C'\mu = \begin{pmatrix} \delta_0 \\ \tau_1 \\ \vdots \\ \tau_m \end{pmatrix}$$

where W_0 is $q \times q$, V_{ij} is $(k_i - 1)p_i \times (k_j - 1)p_j$, Y_0 and δ_0 are $q \times r$ and Z_i and τ_i are $p_i(k_i - 1) \times r$. Let

$$V_{ii} = (V_{ii}(j, n)), \quad (V_{ii}(j, n): p_i \times p_i), \qquad j, n = 1, \cdots, k_i - 1,$$
$$Z_i = (Z_i(j)), \quad (Z_i(j): p_i \times r), \qquad \tau_i = (\tau_i(j)), \quad (\tau_i(j): p_i \times r),$$
$$j = 1, \cdots, m.$$

Let

$$W_i = \textstyle\sum_{j=1}^{k-1} V_{ii}(j, j), \qquad Y_i = (Z_i(1), \cdots, Z_i(k_i - 1)),$$
$$\delta_i = (\tau_i(1), \cdots, \tau_i(k_i - 1)).$$

Then, by a proof similar to that of Theorems 1 and 2 in Arnold (1973), we have

THEOREM 2. *If $X(t \times r) \sim N(\mu, \Sigma)$, $S \sim W(n, \Sigma)$ and Σ has pattern E, then $W_0, W_1, \cdots, W_m, Y_0, Y_1, \cdots, Y_m$ defined above are mutually independent, jointly sufficient, and*

$$Y_0(q \times r) \sim N(\delta_0, \Xi_0), \qquad Y_i(p_i \times (k_i - 1)r) \sim N(\delta_i, \Xi_i),$$
$$W_0 \sim W(n, \Xi_0), \qquad W_i \sim W(n(k_i - 1), \Xi_i).$$

THEOREM 3. (a) *μ has pattern D if and only if $\delta_i = 0$, $i = 1, \cdots, m$.*

(b) *$\mu = 0$ if and only if $\delta_i = 0$, $i = 0, 1, \cdots, m$.*

(c) *μ is unrestricted if and only if δ_i are unrestricted, $i = 0, 1, \cdots, m$.*

(d) *If μ has pattern D then $\mu'\Sigma^{-1}\mu \in F$ if and only if $\delta_0'\Xi_0^{-1}\delta_0 \in F$.*

See Arnold (1970) for explicit expressions for Y_0, W_i, $i = 0, 1, \cdots, m$. Y_i depends on which orthogonal matrix with the given first row is chosen. However, most invariant procedures will be functions of $Y_i Y_i'$ only and this does not depend on the choice.

3. Applications. In this section some examples are given using the previous theorems. The examples are in no way exhaustive, but are intended merely to illustrate different kinds of problems that can be solved using these methods. Once we have shown that a problem is a product of other problems, we will go to the next problem.

3.1. In this section the mean and covariance matrix are assumed patterned under both null and alternative hypotheses. We are testing

$$H: \mu'\Sigma^{-1}\mu \in G \qquad \mu \text{ has pattern } D, \quad \Sigma \text{ has pattern } E$$
$$A: \mu'\Sigma^{-1}\mu \in F \qquad \mu \text{ has pattern } D, \quad \Sigma \text{ has pattern } E.$$

This format is general enough to include the Hotelling's T^2, MANOVA and multivariate classification problems (see Arnold (1973)).

After making the transformation of Theorem 2, we get the product of testing

$$H: \delta_0'\Xi_0^{-1}\delta_0 \in G$$
$$A: \delta_0'\Xi_0^{-1}\delta_0 \in F,$$

and the trivial problem of testing

$$H: \delta_i = 0, \quad \Xi_i > 0 \qquad i = 1, 2, \cdots, m$$
$$A: \delta_i = 0, \quad \Xi_i > 0 \qquad i = 1, 2, \cdots, m.$$

This result shows that most problems where μ has pattern D and Σ has pattern E can be transformed to a product of a trivial problem and a problem similar to the original problem except that the mean vectors and covariance matrices are no longer patterned.

3.2. In this section we look at what happens when we only assume that the covariance matrix is patterned. Under these conditions we typically get a product of $m + 1$ problems, none of which is trivial. As an example we look at the MANOVA problem when we make the additional assumption that Σ is patterned under both hypotheses. A canonical form for the MANOVA problem is the following (see Lehmann (1959), pages 293–296):

$$P': X_1(t \times r) \sim N(\mu_1, \Sigma), \quad S \sim W(n, \Sigma),$$
$$X_2(t \times s) \sim N(\mu_2, \Sigma),$$
$$H: \mu_1 = \infty, \quad -\infty < \mu_1 < \infty, \quad \Sigma > 0$$
$$A: -\infty < \mu_1 < \infty, \quad -\infty < \mu_2 < \infty, \quad \Sigma > 0.$$

If we assume in addition that Σ has pattern E, under both hypotheses we get the product of the following $m + 1$ unrestricted MANOVA problems

$$P_0: Z_1(q \times r) \sim N(\delta_{01}, \Xi_0), \quad W_0 \sim W(n, \Xi_0),$$
$$Z_2(q \times s) \sim N(\delta_{02}, \Xi_0),$$

$$H_0: \delta_{01} = 0\,, \quad -\infty < \delta_{02} < \infty\,, \quad \Xi_0 > 0\,,$$
$$A_0: -\infty < \delta_{01} < \infty\,, \quad -\infty < \delta_{02} < \infty\,, \quad \Xi_0 > 0\,,$$

and

$$P_i: Y_i(p_i \times (k_i - 1)r) \sim N(\delta_{i1}, \Xi_i)\,, \quad W_i \sim W(n(k_i - 1)\,, \Xi_i)\,,$$
$$Y_{i2}(p_i \times (k_i - 1)s) \sim N(\delta_{i2}, \Xi_i)\,,$$
$$H_i: \delta_{i1} = 0\,, \quad -\infty < \delta_{i2} < \infty\,, \quad \Xi_i > 0\,,$$
$$A_i: -\infty < \delta_{i1} < \infty\,, \quad \infty < \delta_{i2} < \infty\,, \quad \Xi_i > 0\,,$$

$i, 1, \cdots, m$, so again this is a product of unpatterned problems that have been studied.

3.3. In the following examples we assume that $p_i = p_j$, $k_i = k_j$, for all i and j. That is $\Sigma = (\Sigma_{ij})$ where Σ_{ij} is $pk \times pk$, $i, j = 1, \cdots, m$, and $\Sigma_{ii} = (A_{ii} - B_{ii}) * I + B_{ii} * E$, $\Sigma_{ij} = B_{ij} * E$. The first problem we consider is testing $\Sigma_{ii} = \Sigma_{jj}$, $\Sigma_{ij} = \Sigma_{kn}$ and $\mu_i = \mu_j$ for all $i \neq j$, $k \neq n$. If

$$X = (X_i)\,, \quad (X_i: pk \times r) \qquad i = 1, \cdots, n\,,$$

this is the same as asking if the X_i are interchangeable when we know that Σ has pattern E. When we make the transformation suggested, we get the product of

$$P_0': Y_0(mp \times 1) \sim N(\delta_0, \Xi_0)\,, \quad W_0 \sim W(n, \Xi_0)\,.$$
$$H_0': \delta_0 \text{ has pattern } A_k\,, \quad \Xi_0 \text{ has pattern } B_k\,, \quad \Xi_0 > 0\,,$$
$$A_0': -\infty < \delta_0 < \infty\,, \quad \Xi_0 > 0\,,$$

and

$$P_1: Y_i(p \times (k - 1)) \sim N(\delta_i, \Xi_i)\,, \quad W_i \sim W(n(k - 1), \Xi_i)\,, \qquad i - 1, \cdots, m\,,$$
$$H_1': \delta_1 = \cdots = \delta_m\,, \quad \Xi_1 = \cdots = \Xi_m > 0\,,$$
$$A_1': -\infty < \delta_i < \infty\,, \quad \Xi_i > 0\,, \qquad i = 1, \cdots, m\,,$$

where A_k and B_k are the generalized intraclass correlation patterns for means and variances defined in Arnold (1973). If we assume in addition that μ has pattern D we get the product of P_0' and a new problem P_2' where we are testing

$$H_2': \delta_i = 0\,, \quad \Xi_1 = \cdots = \Xi_m\,,$$
$$A_2': \delta_i = 0\,, \quad \Xi_i > 0\,, \qquad i = 1, \cdots, m\,.$$

If we only test that $\Sigma_{ii} = \Sigma_{jj}$, $\Sigma_{ij} = \Sigma_{kn}$, we get the product of two problems P_3' and P_4' related to P_0' and P_1' where we are testing

$$P_3' \quad H_3': \quad -\infty < \delta_0 < \infty\,, \quad \Xi_0 \text{ has pattern } B_k\,, \quad \Xi_0 > 0\,,$$
$$A_3': \quad -\infty < \delta_0 < \infty\,, \quad \Xi_0 > 0\,,$$

and

$$P_4' \quad H_4': -\infty < \delta_i < \infty\,, \quad \Xi_1 = \cdots = \Xi_m > 0\,,$$
$$A_4': -\infty < \delta_i < \infty\,, \quad \Xi_i > 0\,, \qquad i = 1, \cdots, m\,.$$

P_0' and P_3' have been studied by Olkin (1970), P_2' and P_4' are two forms of the problem of testing the equality of covariance matrices, while P_2' is the problem of testing the equality of normal distributions; and so all these problems have been studied.

4. Summary. This paper has shown that many problems in which the covariance matrix has the generalized compound symmetry model can be factored into a product of problems in which the covariance matrices are no longer patterned.

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