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ASYMPTOTIC EXPANSIONS FOR THE JOINT AND MARGINAL DISTRIBUTIONS OF THE LATENT ROOTS OF THE COVARIANCE MATRIX

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Let nS be an $m \times m$ matrix having the Wishart distribution $W_m(n, \Sigma)$. For large n and simple latent roots of Σ, it is known that the latent roots of S are asymptotically independently normal. In this paper an expansion, up to and including the term of order n^{-1}, is given for the joint density function of the roots of S in terms of normal density functions. Expansions for the marginal distributions of the roots are also given, valid when the corresponding roots of Σ are simple.

1. Introduction and summary. Let S be the covariance matrix formed from a sample of size $n + 1$ drawn from an m-variate normal distribution with population covariance matrix Σ (assumed to be positive definite); then nS has the Wishart distribution (see e.g. T. W. Anderson [2], page 157). Let $l_1 > l_2 > \cdots > l_m > 0$ and $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_m > 0$ denote the latent roots of S and Σ respectively. It is known (Girshick [6], T. W. Anderson [3]) that if λ_i is a simple root then, for large n, l_i is asymptotically independent of the other sample roots and the limiting distribution of $(n/2)^{\frac{1}{2}}(l_i/\lambda_i - 1)$ is standard normal $N(0, 1)$. We will assume throughout this paper that all the roots of Σ are simple. The extreme roots l_1 and l_m are first considered in Section 2. Sugiyama [15] has shown that the distribution function of l_1 can be expressed in a form involving a confluent hypergeometric function ${}_1F_1$ of matrix argument. It is shown that the distribution function of l_m can be expressed in terms of another confluent hypergeometric function defined earlier by Muirhead [13]. Then a system of partial differential equations (pde's) is used to expand the two distribution functions up to and including terms of order n^{-1}.

In Section 3 an expansion is given, up to and including the term of order n^{-1}, for the joint density function of $l_1, \cdots, l_m$ in terms of normal density functions. This expansion then yields an expansion for the marginal density function of l_i which has also been obtained by Sugiura [14] using another method.

2. Expansions of the extreme root distributions. We consider first the largest root l_1. Sugiyama [15], [16] has shown that the distribution function of l_1 can

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be expressed in the form

$$P(l_1 < y) = \left[\frac{\Gamma_m(p)(\frac{1}{2}ny)^{\frac{1}{2}mn}}{\Gamma_m(\frac{1}{2}n + p)}\right](\det \Sigma)^{-\frac{1}{2}n}\,{}_1F_1(\tfrac{1}{2}n; \tfrac{1}{2}n + p; -\tfrac{1}{2}ny\Sigma^{-1}) , \tag{2.1}$$

where $p = (m + 1)/2$, $\Gamma_m(a) = \pi^{m(m-1)/4} \prod_{i=1}^m \Gamma(a - (i - 1)/2)$ and ${}_1F_1$ is a confluent hypergeometric function of matrix argument (see Herz [7], Constantine [5]). Since (2.1) depends on Σ only via its latent roots we can regard Σ as being diagonal, i.e. $\Sigma = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_m)$. An expansion for large n has been obtained for the ${}_1F_1$ function in (2.1) by Muirhead [12]; however this expansion is of limited interest since it is valid only over the range $0 \leqq y < \lambda_m$ and one would usually be interested in the upper tail of the distribution. Using (2.1) Sugiyama [16] has obtained an approximation to $P(l_1 < y)$ in terms of a product of χ^2 probabilities.

Now assume that $\lambda_1, \cdots, \lambda_m$ are all distinct; from (2.1) the distribution function of $x_1 = (n/2)^{\frac{1}{2}}(l_1/\lambda_1 - 1)$ can be written as

$$P(x_1 < x) = [\Gamma_m(p)/\Gamma_m(\tfrac{1}{2}n + p)](\det R)^{n/2}\,{}_1F_1(\tfrac{1}{2}n; \tfrac{1}{2}n + p; -R) , \tag{2.2}$$

where $R = \operatorname{diag}(r_1, r_2, \cdots, r_m)$ with $r_i = [n/2 + (n/2)^{\frac{1}{2}}x]z_i$, $z_i = \lambda_1/\lambda_i$ $(i = 1, 2, \cdots, m)$. (Note that $z_1 = 1$ is a dummy variable and the R.H.S. of (2.2) is a function of $x, z_2, \cdots, z_m$.) A system of pde's satisfied by the ${}_1F_1$ function has been given by Muirhead [12]. Starting with this system it can be readily verified that $P \equiv P(x_1 < y)$ satisfies each of the m pde's

$$\begin{aligned}\frac{\partial^2 P}{\partial x^2} &+ x\frac{\partial P}{\partial x} + \left(\frac{2}{n}\right)^{\frac{1}{2}}\Big[2x\frac{\partial^2 P}{\partial x^2} + (1 + x^2 - \tfrac{1}{2}A_1)\frac{\partial P}{\partial x} - x\textstyle\sum_{k=2}^m z_k \frac{\partial P}{\partial z_k} \\ &- 2\textstyle\sum_{k=2}^m z_k \frac{\partial^2 P}{\partial x \partial z_k}\Big] + \frac{2}{n}\Big[x^2\frac{\partial^2 P}{\partial x^2} + x(1 - \tfrac{1}{2}A_1)\frac{\partial P}{\partial x} \\ &+ \textstyle\sum_{k=2}^m z_k\left(1 + \tfrac{1}{2}A_1 - \frac{1}{2(1 - z_k)}\right)\frac{\partial P}{\partial z_k} - 2x\sum_{k=2}^m z_k \frac{\partial^2 P}{\partial x \partial z_k} \\ &+ \textstyle\sum_{k=2}^m \sum_{j=2}^m z_j z_k \frac{\partial^2 P}{\partial z_k \partial z_j}\Big] = 0\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}(z_i - 1)\frac{\partial P}{\partial z_i} &+ \left(\frac{2}{n}\right)^{\frac{1}{2}}\left[\frac{1}{2(1 - z_i)}\frac{\partial P}{\partial x} + xz_i\frac{\partial P}{\partial z_i}\right] \\ &+ \frac{2}{n}\Big[z_i\frac{\partial^2 P}{\partial z_i^2} + \frac{x}{2(1 - z_i)}\frac{\partial P}{\partial x} + (1 - \tfrac{1}{2}A_i)\frac{\partial P}{\partial z_i} \\ &- \frac{1}{2(1 - z_i)}\textstyle\sum_{k=2}^m z_k \frac{\partial P}{\partial z_k} - \frac{1}{2}\sum_{j=2, j\neq i}^m \frac{z_j}{z_i - z_j}\frac{\partial P}{\partial z_j}\Big] = 0 \\ &\qquad (i = 2, 3, \cdots, m) ,\end{aligned} \tag{2.4}$$

where

$$A_i = \textstyle\sum_{j=1, j\neq i}^m \frac{z_i}{z_j - z_i} \qquad (i = 1, 2, \cdots, m) . \tag{2.5}$$

We now look for a solution of these m pde's (2.3) and (2.4) of the form

$$P = \Phi(x) + \sum_{k=1}^{\infty} (2/n)^{k/2} Q_k , \tag{2.6}$$

where $\Phi(\cdot)$ denotes the standard normal distribution function and the Q_k are functions of $x, z_2, \cdots, z_m$. (That P possesses such an expansion follows from results in the next section.) We substitute the series (2.6) into (2.3) and (2.4) and equate coefficients of powers of $(2/n)^{\frac{1}{2}}$ on the L.H.S.'s to zero. Equating the constant term in (2.3) to zero gives

$$d^2\Phi(x)/dx^2 + x\, d\Phi(x)/dx = 0 ,$$

verifying that $\Phi(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} \exp(-t^2/2)\, dt$ is indeed the correct limiting distribution function. Now equate the coefficient of $(2/n)^{\frac{1}{2}}$ in (2.4) to zero. We obtain

$$(z_i - 1)\partial Q_1/\partial z_i + (1 - z_i)^{-1}\varphi(x)/2 = 0 , \qquad (i = 2, 3, \cdots, m)$$

where $\varphi(\cdot)$ denotes the standard normal density function. This may be solved to give Q_1 in the form

$$Q_1 = -A_1\varphi(x)/2 + f(x) , \tag{2.7}$$

where A_1 is given by (2.5) and $f(x)$ is a function of x alone, which has yet to be determined. Equating the coefficient of $(2/n)^{\frac{1}{2}}$ in (2.3) to zero gives

$$\partial^2 Q_1/\partial x^2 + x\, \partial Q_1/\partial x + [1 - x^2 - \tfrac{1}{2}A_1]\varphi(x) = 0 . \tag{2.8}$$

Substituting (2.7) in (2.8) gives

$$d^2f/dx^2 + x\, df/dx + (1 - x^2)\varphi(x) = 0 ,$$

the complete solution of which is

$$f(x) = (1 - x^2)\varphi(x)/3 + k_1\Phi(x) + k_2 ,$$

where k_1 and k_2 are arbitrary constants. The boundary conditions $P(x_1 < \infty) = 1$ and $P(x_1 < -\infty) = 0$ may be used to show that $k_1 = k_2 = 0$. Hence we have

$$Q_1 = -\tfrac{1}{6}\varphi(x)[2H_2(x) + 3A_1H_0(x)] , \tag{2.9}$$

where $H_j(x)$ denotes the Hermite polynomial of degree j (tabulated to $j = 10$ in Kendall and Stuart [9], page 155). Similarly, equating the coefficient of $2/n$ in the L.H.S.'s of (2.3) and (2.4) to zero and solving the resulting equations gives

$$Q_2 = -\tfrac{1}{72}\varphi(x)[4H_5(x) + 18H_3(x) + 12A_1H_3(x) - 18B_1H_1(x) + 9A_1^2H_1(x)] , \tag{2.10}$$

where

$$A_1 = \sum_{j=2}^{m} (z_j - 1)^{-1} , \qquad B_1 = \sum_{j=2}^{m} (z_j - 1)^{-2} .$$

Coefficients of higher powers of $(2/n)^{\frac{1}{2}}$ in (2.6) may be obtained in a similar manner if required. The expansion is summarized in the following:

THEOREM 2.1. *The distribution function of* $x_1 = (n/2)^{\frac{1}{2}}(l_1/\lambda_1 - 1)$, *when the latent* *roots of* Σ *are simple, can be expanded for large n as*

$$P(x_1 < x) = \Phi(x) + (2/n)^{\frac{1}{2}}Q_1 + (2/n)Q_2 + O(n^{-\frac{3}{2}}), \tag{2.11}$$

where $\Phi(\cdot)$ *denotes the standard normal distribution function and* Q_1, Q_2 *are given by* (2.9), (2.10) *respectively.*

Consider now the distribution of the smallest root l_m. We first derive an exact expression for its distribution function. Since nS is $W_m(n, \Sigma)$ we have

$$P(l_m > y) = [(\tfrac{1}{2}n)^{\frac{1}{2}mn}(\det\Sigma)^{-\frac{1}{2}n}/\Gamma_m(\tfrac{1}{2}n)] \times \int_{S>yI}\exp(-\tfrac{1}{2}n\operatorname{tr}(\Sigma^{-1}S))\det S^{\frac{1}{2}n-p}\,dS, \tag{2.12}$$

where $p = (m+1)/2$. Making the transformation $T = y^{-1}S - I$ it is easily seen that (2.12) becomes

$$P(l_m > y) = [\Gamma_m(p)/\Gamma_m(\tfrac{1}{2}n)](\tfrac{1}{2}ny)^{\frac{1}{2}mn}(\det\Sigma)^{-\frac{1}{2}n}\exp(-\tfrac{1}{2}ny\operatorname{tr}\Sigma^{-1}) \times \Psi(p, \tfrac{1}{2}n + p; \tfrac{1}{2}ny\Sigma^{-1}), \tag{2.13}$$

where

$$\Psi(a, c; R) \equiv_{\text{def.}} [1/\Gamma_m(a)]\int_{S>0}\exp(-\operatorname{tr}(RS))(\det S)^{a-p}\det(I+S)^{c-a-p}\,dS.$$

The function Ψ is another confluent hypergeometric function of matrix argument (see Muirhead [13]).

Another expression for $P(l_m > y)$ has been obtained by Khatri [10], in the case when $n/2 - p$ is a nonnegative integer, as a finite series of zonal polynomials.

Now put $x_m = (n/2)^{\frac{1}{2}}(l_m/\lambda_m - 1)$. From (2.13) we have

$$P(x_m > x) = [\Gamma_m(p)/\Gamma_m(\tfrac{1}{2}n)](\det R)^{\frac{1}{2}n}\exp(-\operatorname{tr}R)\Psi(p, \tfrac{1}{2}n + p; R),$$

where $R = \operatorname{diag}(r_1, r_2, \cdots, r_m)$ with $r_i = (n/2 + (n/2)^{\frac{1}{2}}x)z_i$, $z_i = \lambda_m/\lambda_{m-i+1}$ ($i = 1, 2, \cdots, m$). (Again, $z_1 = 1$ is a dummy variable.) Using the system of pde's satisfied by the Ψ function given by Muirhead [13] it can readily be shown that the distribution function of x_m, $P \equiv P(x_m < x)$, satisfies each of the m pde's (2.3) and (2.4). The only difference here is that now $z_i = \lambda_m/\lambda_{m-i+1}$ instead of λ_1/λ_i as it was in the largest root distribution. Hence

THEOREM 2.2. *The distribution function of* $x_m(n/2)^{\frac{1}{2}}(l_m/\lambda_m - 1)$, *when the latent roots of* Σ *are simple, can be expanded for large n as*

$$P(x_m < x) = \Phi(x) + (2/n)^{\frac{1}{2}}Q_1 + (2/n)Q_2 + O(n^{-\frac{3}{2}}),$$

where $z_i = \lambda_m/\lambda_{m-i+1}$ *in* Q_1 *and* Q_2 *given by* (2.9) *and* (2.10) *respectively.*

From the general form of the expansion for the marginal distribution of l_i obtained in the next section, it may be conjectured that the distribution functions of each of the variables $(n/2)^{\frac{1}{2}}(l_i/\lambda_i - 1)$ ($i = 1, 2, \cdots, m$) satisfy the system of pde's (2.3) and (2.4), with appropriate changes in the definitions of the z_i. The authors have not been able to show this.

3. Expansion of the joint distribution. In this section we derive an expansion for the joint density function of $l_1, \cdots, l_m$ when each of the roots $\lambda_1, \cdots, \lambda_m$ is assumed to be simple. The joint density function of $l_1, \cdots, l_m$ can be expressed in the form (see James [8])

$$\pi^{\frac{1}{2}m^2}(\tfrac{1}{2}n)^{\frac{1}{2}mn}[\Gamma_m(\tfrac{1}{2}n)\Gamma_m(\tfrac{1}{2}m)]^{-1}\prod_{i=1}^m l_i^{n/2-p}\lambda_i^{-n/2}\prod_{i<j}^m (l_i - l_j) \times {}_0F_0(-\tfrac{1}{2}nL, \Sigma^{-1}), \tag{3.1}$$

where $p = (m+1)/2$, $L = \operatorname{diag}(l_1, \cdots, l_m)$, $\Sigma = \operatorname{diag}(\lambda_1, \cdots, \lambda_m)$ and ${}_0F_0$ is a hypergeometric function with two argument matrices. The ${}_0F_0$ function in (3.1) has been expanded for large n by G. Anderson [1] by expressing it as an integral over the orthogonal group. After making the transformation from $l_1, \cdots, l_m$ to $x_1, \cdots, x_m$ where $x_i = (n/2)^{\frac{1}{2}}(l_i/\lambda_i - 1)$ $(i = 1, 2, \cdots, m)$, Anderson's method can be adapted to expand the resulting ${}_0F_0$ function for large n. However it is simpler, and equivalent, to transform Anderson's expansion directly. In [1] it is shown that the joint density function can be expressed as

$$k_1\prod_{i=1}^m[\lambda_i^{(m-n-1)/2}l_i^{n/2-p}\exp(-nl_i/2\lambda_i)]\prod_{i<j}^m[(l_i - l_j)/(\lambda_i - \lambda_j)]^{\frac{1}{2}}\cdot G, \tag{3.2}$$

where

$$k_1 = (n/2)^{mn/2-m(m-1)/4}/\prod_{i=1}^m\Gamma((n-i+1)/2)$$

and

$$G = 1 + (2n)^{-1}\sum_{i<j}^m\lambda_i\lambda_j(\lambda_i - \lambda_j)^{-1}(l_i - l_j)^{-1} + O(n^{-2}). \tag{3.3}$$

(Anderson did not show in general that the remainder term in (3.3) is of order n^{-2}; this has been shown by Chikuse (unpublished).) Now put $x_i = (n/2)^{\frac{1}{2}}(l_i/\lambda_i - 1)$ $(i = 1, 2, \cdots, m)$. From (3.2) the joint density function of $x_1, \cdots, x_m$ can be expressed as

$$k_2F_1F_2[1 + (2n)^{-1}\sum_{i<j}^m\lambda_i\lambda_j/(\lambda_i - \lambda_j)^2 + O(n^{-\frac{3}{2}})], \tag{3.4}$$

where

$$k_2 = (n/2)^{mn/2-m(m+1)/4}\exp(-mn/2)/\prod_{i=1}^m\Gamma((n-i+1)/2),$$

$$F_1 = \prod_{i=1}^m[(1 + (2/n)^{\frac{1}{2}}x_i)^{n/2-p}\exp(-(n/2)^{\frac{1}{2}}x_i)]$$

and

$$F_2 = \prod_{i<j}^m[1 + (2/n)^{\frac{1}{2}}(x_i\lambda_i - x_j\lambda_j)/(\lambda_i - \lambda_j)]^{\frac{1}{2}}.$$

It remains to expand k_2, F_1 and F_2 in (3.4) for large n. For example, by expanding the gamma functions for large n it follows that

$$k_2 = (2\pi)^{-m/2}[1 - (24n)^{-1}m(2m^2 + 3m - 1) + O(n^{-2})].$$

The functions F_1 and F_2 can be easily expanded in terms of powers of $n^{-\frac{1}{2}}$; however these expansions, up to and including the terms of order n^{-1}, are quite lengthy and are omitted here. Substituting these expansions in (3.4) gives an expansion of the joint density function of $x_1, \cdots, x_m$. This final result is summarized in the following

THEOREM 3.1. *The joint density function of* $x_i = (n/2)^{\frac{1}{2}}(l_i/\lambda_i - 1)$ $(i = 1, 2, \cdots,$

m), where $\lambda_1, \cdots, \lambda_m$ *are simple roots of* Σ, *may be expanded for large n as*

$$\prod_{i=1}^m \varphi(x_i) \cdot \Big\{1 + (2/n)^{\frac{1}{2}} \sum_{i=1}^m P_{1i}(x_i) + (2/n)\Big(\sum_{i=1}^m P_{2i}(x_i) + \sum_{i<j}^m P_{1i}(x_i)P_{1j}(x_j) + \tfrac{1}{2}\sum_{i<j}^m \frac{x_i x_j \lambda_i \lambda_j}{(\lambda_i - \lambda_j)^2}\Big) + O(n^{-\frac{3}{2}})\Big\}, \tag{3.5}$$

where $\varphi(\cdot)$ *denotes the standard normal density function,*

$$P_{1i}(x) = \tfrac{1}{6}[2H_3(x) + 3A_i H_1(x)], \tag{3.6}$$

$$P_{2i}(x) = \tfrac{1}{72}[4H_6(x) + 18H_4(x) + 12A_i H_4(x) - 18B_i H_2(x) + 9A_i^2 H_2(x)], \tag{3.7}$$

$H_j(x)$ *is the Hermite polynomial of degree j, and*

$$A_i = \sum_{j=1, j\neq i}^m \lambda_j/(\lambda_i - \lambda_j), \qquad B_i = \sum_{j=1, j\neq i}^m \lambda_j^2/(\lambda_i - \lambda_j)^2. \tag{3.8}$$

Note that A_i is the same as in (2.5).

By integrating out the other variables in (3.5) an expansion of the marginal density function of x_i can be obtained.

COROLLARY. *The marginal density function of* $x_i = (n/2)^{\frac{1}{2}}(l_i/\lambda_i - 1)$, *where* λ_i *is a simple root of* Σ, *may be expanded for large n as*

$$\varphi(x_i)\{1 + (2/n)^{\frac{1}{2}}P_{1i}(x_i) + (2/n)P_{2i}(x_i) + O(n^{-\frac{3}{2}})\}. \tag{3.9}$$

where $P_{1i}(x_i)$ *and* $P_{2i}(x_i)$ *are given by* (3.6) *and* (3.7) *respectively.*

The expansion (3.9), in the cases $i = 1$ and m, agrees with the expansions for the extreme root distributions given in the previous section. Sugiura [14] has also obtained (3.9) using another method.

Asymptotic moments of l_i can be obtained from (3.9); we obtain

$$\begin{aligned} &E(l_i) = \lambda_i + A_i\lambda_i/n + O(n^{-2}), \\ &\mathrm{Var}\,(l_i) = 2\lambda_i^2/n - 2\lambda_i^2 B_i/n^2 + O(n^{-3}), \\ &\kappa_3(l_i) = 8\lambda_i^3/n^2 + O(n^{-3}), \qquad \kappa_4(l_i) = 48\lambda_i^4/n^3 + O(n^{-4}), \end{aligned} \tag{3.10}$$

where $\kappa_3(l_i)$ and $\kappa_4(l_i)$ denote the third and fourth cumulants of l_i and A_i, B_i are given by (3.8). From (3.5) we obtain

$$\mathrm{Cov}\,(l_i, l_j) = 2[\lambda_i\lambda_j/(\lambda_i - \lambda_j)]^2/n^2 + O(n^{-3}). \tag{3.11}$$

These expressions agree with results obtained by Lawley [11] without using the asymptotic normality. In fact, it can be readily verified that the expansion (3.9) for the marginal density function of x_i can also be obtained by substituting the expressions (3.10) for the first four moments of l_i in the general Edgeworth expansion given in Kendall and Stuart [9], page 164. Similarly, the expansion (3.5) of the joint density function of $x_1, \cdots, x_m$ could also have been obtained using (3.10) and (3.11) in a multivariate Edgeworth expansion (see Chambers [4]).

In [1], Anderson showed that, if λ_i is a simple root of Σ, then for large n,

nl_i/λ_i is approximately distributed as χ^2 with n degrees of freedom. It was pointed out by the referee that if, instead of $x_i = (n/2)^{\frac{1}{2}}(l_i/\lambda_i - 1)$ we consider $\tilde{x}_i = (n/2)^{\frac{1}{2}}(l_i - \lambda_i - A_i\lambda_i/n)/\lambda_i(1 - B_i/n)^{\frac{1}{2}}$, so that $E(\tilde{x}_i) = O(n^{-\frac{3}{2}})$ and $\text{Var}(\tilde{x}_i) = 1 + O(n^{-2})$, then the series obtained agrees, through terms of order n^{-1}, with the Edgeworth series for $(\chi_n^2 - n)/(2n)^{\frac{1}{2}}$. This suggests that a χ^2 approximation might be a little sharper than the normal approximation.

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REFERENCES

[1] Anderson, G. A. (1965). An asymptotic expansion for the distribution of the latent roots of the estimated covariance matrix. *Ann. Math. Statist.* **36** 1153–1173.

[2] Anderson, T. W. (1958). *An Introduction to Multivariate Statistical Analysis.* Wiley, New York.

[3] Anderson, T. W. (1963). Asymptotic theory for principal component analysis. *Ann. Math. Statist.* **34** 122–148.

[4] Chambers, J. M. (1967). On methods of asymptotic approximation for multivariate distributions. *Biometrika* **54** 367–383.

[5] Constantine, A. G. (1963). Some noncentral distribution problems in multivariate analysis. *Ann. Math. Statist.* **34** 1270–1285.

[6] Girshick, M. A. (1939). On the sampling theory of roots of determinantal equations. *Ann. Math. Statist.* **10** 203–224.

[7] Herz, C. S. (1955). Bessel functions of matrix argument. *Ann. of Math.* **61** 474–523.

[8] James, A. T. (1964). Distributions of matrix variates and latent roots derived from normal samples. *Ann. Math. Statist.* **35** 475–501.

[9] Kendall, M. G. and Stuart, A. (1969). *The Advanced Theory of Statistics*, **1**. Hafner, New York.

[10] Khatri, C. G. (1972). On the exact finite series distribution of the smallest or the largest root of matrices in three situations. *J. Multivariate Anal.* **2** 201–207.

[11] Lawley, D. N. (1956). Tests of significance for the latent roots of covariance and correlation matrices. *Biometrika* **43** 128–136.

[12] Muirhead, R. J. (1970). Systems of partial differential equations for hypergeometric functions of matrix argument. *Ann. Math. Statist.* **41** 991–1001.

[13] Muirhead, R. J. (1970). Asymptotic distributions of some multivariate tests. *Ann. Math. Statist.* **41** 1002–1010.

[14] Sugiura, N. (1973). Derivatives of the characteristic root of a symmetric or a Hermitian matrix with two applications in multivariate analysis. *Commun. Statist.* **1** 393–417.

[15] Sugiyama, T. (1967). On the distribution of the largest latent root of the covariance matrix. *Ann. Math. Statist.* **38** 1148–1151.

[16] Sugiyama, T. (1972). Approximation for the distribution of the largest latent root of a Wishart matrix. *Austral. J. Statist.* **14** 17–24.

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