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EXPONENTIALLY BOUNDED STOPPING TIME OF THE SEQUENTIAL t-TEST[1]

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Let N be the stopping time of the sequential t-test, based on the i.i.d. sequence $Z_1, Z_2, \cdots$, for testing that the ratio of mean to standard deviation in a normal population equals γ_1 agaist the alternative that it equals γ_2. Let P be the actual distribution of the Z_i (not necessarily normal). It is proved that if $\gamma_1^2 \neq \gamma_2^2$ and P is an arbitrary unbounded distribution, then there exist constants $c > 0$ and $\rho < 1$ such that $P(N > n) < c\rho^n$, $n = 1, 2, \cdots$.

Let $Z, Z_1, Z_2, \cdots$ be independent and identically distributed real valued random variables, with common distribution P. Define

$$T_n = \frac{\frac{1}{n} \sum_1^n Z_i}{\left(\frac{1}{n} \sum_1^n Z_i^2\right)^{\frac{1}{2}}} . \tag{1}$$

If it is assumed that P is normal with mean ζ and standard deviation σ, then the distribution of the sequence $(T_1, T_2, \cdots)$ depends only on $\gamma = \zeta/\sigma$. For testing the hypothesis $H_1: \gamma = \gamma_1$ against the alternative $H_2: \gamma = \gamma_2$ the sequential t-test is the sequential probability ratio test based on $(T_1, T_2, \cdots)$. That is, if L_n is the log probability ratio of $(T_1, \cdots, T_n)$, sampling continues as long as

$$l_1 < L_n < l_2 \tag{2}$$

for some chosen stopping bounds l_1, l_2, and H_1 (H_2) is accepted the first time that $L_n \leqq l_1$ $(\geqq l_2)$. Let N be the smallest $n \geqq 1$ such that (2) is violated; N will be called the stopping time.

The true distribution P does not necessarily belong to the above normal model. It is desired to prove that for every conceivable distribution P and any choice of stopping bounds l_1, l_2 there exist constants $c > 0$ and $\rho < 1$ such that

$$P(N > n) < c\rho^n , \qquad n = 1, 2, \cdots . \tag{3}$$

We shall, however, exclude P such that $P(Z = 0) = 1$, since for $Z_1, Z_2, \cdots$ degenerate at 0, T_n given by (1) is undefined. If for every choice of stopping bounds l_1, l_2 there exist c, ρ to satisfy (3) we shall say that N is exponentially bounded under P. If not, P will be called obstructive. In [1], [5] it was shown that if P is such that Z^2 has finite moment generating function in a neighborhood of 0 and P does not belong to a certain family of two-point distributions,

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then N is exponentially bounded under P. Furthermore, those two-point distributions mentioned in the previous sentence were shown in [6] to be obstructive. It had been speculated that N is exponentially bounded under every unbounded P, i.e. every P such that $P(|Z| > B) > 0$ for every real B. However, Lai [2] demonstrated the existence of unbounded obstructive distributions P in the case $\gamma_1 = -\gamma_2$. He also proved exponential boundedness of N in the case $\gamma_1^2 \neq \gamma_2^2$ under additional conditions on P. We shall show now that if $\gamma_1^2 \neq \gamma_2^2$, then no conditions on P are needed to obtain (3).

THEOREM. *Let* $Z, Z_1, Z_2, \cdots$ *be* i.i.d. *with unbounded distribution* P *and let* N *be the smallest integer* $n \geqq 1$ *for which* (2) *is violated. If* $\gamma_1^2 \neq \gamma_2^2$ *then* N *is exponentially bounded under* P.

PROOF. As in [2], Theorem 3, the problem can be redefined as follows: Let N be the smallest integer $n \geqq 1$ such that

$$n|T_n - \lambda| < \tfrac{1}{2}d \tag{4}$$

is violated, where λ is a given nonzero number depending on γ_1 and γ_2; to prove (3) for every $d > 0$. Without loss of generality we shall assume $\lambda > 0$. It is convenient to denote

$$t = \tfrac{1}{2}\lambda\,. \tag{5}$$

Consider the function

$$f(y) = y\left(1 - \left(\frac{y}{1+y}\right)^{\frac{1}{2}}\right), \quad y \geqq 0\,. \tag{6}$$

It is elementary to show that $f(y)$ increases monotonically to the limit $\frac{1}{2}$ as $y \to \infty$. Let $y_0 > 1$ be such that

$$f(y_0) > \tfrac{3}{8}\,. \tag{7}$$

Choose integers r and n_0 such that

$$r > \frac{8d}{t}, \tag{8}$$

with t defined in (5), and

$$n_0 > y_0 r\,. \tag{9}$$

Let B and B_1 be positive numbers such that

$$P(|Z| < \tfrac{1}{8}Bt) = p_1 > 0\,, \tag{10}$$

$$P\left(B\frac{4}{t} < |Z| < \frac{1}{r}B_1\right) = p_2 > 0\,. \tag{11}$$

(The choice for B is always possible because P is unbounded.) Finally, choose B_2 such that

$$B_2 > y_0 B_1^2\,. \tag{12}$$

Let A_n be the event $[N > n]$ and C_n the event

$$C_n = [\sum_1^n Z_i^2 > B_2] , \tag{13}$$

C_n^c its complement. Since Z is not degenerate at 0, it follows from [4] that the sequence $\{PC_n^c, n = 1, 2, \cdots\}$ is exponentially bounded:

$$PC_n^c < c_1 \rho_1^n , \qquad n = 1, 2, \cdots \tag{14}$$

for some $c_1 > 0$, $\rho_1 < 1$. The task is to prove

$$PA_n < c\rho^n , \qquad n = 1, 2, \cdots , \quad \text{for some } c > 0, \rho < 1 . \tag{15}$$

Following the ideas of Stein [4] and Sethuraman [3] it is proved in the lemma at the end of the paper that it suffices to show that there exists $p > 0$ and n_0 such that for $n \geqq n_0$

$$P(A_{n+r} C_{n+r} \mid A_n C_n) \leqq 1 - p . \tag{16}$$

From the definition (13) of C_n it follows that $C_n \subset C_{n+1}$. Therefore, (16) is the same as $P(A_{n+r} \mid A_n C_n) \leqq 1 - p$, or as

$$P(A_{n+r}^c \mid A_n C_n) \geqq p . \tag{17}$$

Denoting

$$L_n = n(T_n - \lambda) , \tag{18}$$

we have (consulting (4)) that given A_n, the event A_{n+r}^c is implied by $|L_{n+r} - L_n| > d$. Thus, the left hand side of (17) is at least as large as

$$P(|L_{n+r} - L_n| > d \mid A_n C_n) . \tag{19}$$

From (18) follows

$$L_{n+r} - L_n = (n + r)(T_{n+r} - T_n) + r(T_n - \lambda) . \tag{20}$$

It will be assumed in the following that $n \geqq n_0$ and that A_n happens. Since (4) is then satisfied we have $|T_n - \lambda| < d/2n \leqq d/2n_0$. From (8), (9), and $y_0 > 1$ it is deduced that $d/2n_0 < \frac{1}{16}t < \frac{1}{8}t$. The inequality $|T_n - 2t| < \frac{1}{8}t$ implies first of all $T_n > t$, and secondly $r|T_n - \lambda| < \frac{1}{8}rt$. Then if the first term on the right hand side of (20) is in absolute value $> \frac{1}{4}rt$ we have $|L_{n+r} - L_n| > \frac{1}{8}rt > d$, by (8). The inequality $(n + r)|T_{n+r} - T_n| > \frac{1}{4}rt$ is, in turn, implied by $n|T_{n+r} - T_n| > \frac{1}{4}rt$, which is implied by $n(T_n - T_{n+r}) > \frac{1}{4}rt$. Therefore, (19) is at least as large as

$$P(n(T_n - T_{n+r}) > \tfrac{1}{4}rt \mid A_n C_n) \tag{21}$$

and it suffices to show that (21) is $\geqq p$ for some $p > 0$.

It is convenient to denote

$$V = \sum_{n+1}^{n+r} Z_i , \qquad W = \sum_{n+1}^{n+r} Z_i^2 , \qquad X = \sum_1^n Z_i^2 . \tag{22}$$

Then, employing the function f given in (6), the following expression is computed from (1):

$$n(T_n - T_{n+r}) = rT_n f\left(\frac{n}{r}\right) + \left(\frac{n}{n+r}\right)^{\frac{1}{2}} T_n \frac{W}{n^{-1}X} f\left(\frac{X}{W}\right) - \left(\frac{n}{n+r}\frac{X}{X+W}\right)^{\frac{1}{2}} \frac{V}{(n^{-1}X)^{\frac{1}{2}}} . \tag{23}$$

The first term on the right hand side in (23) is $> rtf(n_0/r) > rtf(y_0) > \frac{3}{8}rt$, using (9) and (7). The second term on the right hand side in (23) is $\geqq 0$. Therefore, if $P(Z < 0) = p_0 > 0$, then $P(V < 0) \geqq p_0^r = p > 0$ so that the right hand side of (23) is $> \frac{3}{8}rt$ with probability at least p. Using the independence of $(Z_1, \cdots, Z_n)$ and $(Z_{n+1}, \cdots, Z_{n+r})$ it is seen that (21) is at least p so that the theorem is proved if $P(Z < 0) > 0$.

Assume now that $P(Z \geqq 0) = 1$, then also $P(V \geqq 0) = 1$. In that case (23) can be replaced by the inequality

$$n(T_n - T_{n+r}) > \tfrac{3}{8}rt + \left(\frac{n}{n+r}\right)^{\frac{1}{2}} \frac{tW}{n^{-1}X} f\left(\frac{X}{W}\right) - \frac{V}{(n^{-1}X)^{\frac{1}{2}}} . \tag{24}$$

We shall treat separately the two cases $X/n > B^2$ and $\leqq B^2$.

Case 1: $X/n > B^2$. From (10) and $Z \geqq 0$ follows $P(Z_{n+i} < \frac{1}{8}Bt, i = 1, \cdots, r) = p_1^r = p > 0$, so that $P(V < \frac{1}{8}Brt) \geqq p$. If the event $[V < \frac{1}{8}Brt]$ happens then the third term on the right hand side in (24) is $> -\frac{1}{8}rt$, so that the right hand side is $\geqq \frac{1}{4}rt$, and it follows that (21) is $\geqq p$.

Case 2: $X/n \leqq B^2$. From (11) follows that the event

$$\frac{4r}{t}B < V < B_1 \quad . \tag{25}$$

happens with probability $p \geqq p_2^r > 0$. From the definition (22) of V and W we have $V^2 \leqq rW$. This, together with the lower bound for V in (25), implies that $V(n/X)^{\frac{1}{2}} < tWn^{\frac{1}{2}}/(4BX^{\frac{1}{2}}) \leqq ntW/(4X)$, the last inequality due to the conditioning $X/n \leqq B^2$. On the right hand side in (24) the third term is therefore $> -ntW/(4X)$. In the second term on the right hand side, $(n/(n+r))^{\frac{1}{2}} \geqq (n_0/(n_0+r))^{\frac{1}{2}} > \frac{2}{3}$, using (9) and $y_0 > 1$. Furthermore, $W \leqq V^2$, and $V < B_1$ by the upper bound in (25), so $W < B_1^2$. Since the conditioning $A_n C_n$ implies C_n, which by the definitions (13) and (22) implies $X > B_2$, it follows that $f(X/W) > f(B_2/B_1^2) > f(y_0) > \frac{3}{8}$, using (12) and (7). The second term on the right hand side in (24) is therefore $\geqq ntW/(4X)$ so that the sum of second and third terms is $\geqq 0$. Thus, the right hand side of (24) is $\geqq \frac{3}{8}rt$ so that (21) is at least p. This concludes the proof of the theorem.

The following lemma was needed in the proof the theorem.

LEMMA. *Let* $\{A_n, n = 1, 2, \cdots\}$ *be a nonincreasing sequence of events and* $\{C_n, n = 1, 2, \cdots\}$ *a sequence of events such that*

$$PA_n C_n^c < c_1 \rho_1^n , \qquad n = 1, 2, \cdots , \quad \textit{for some} \quad c_1 > 0 , \quad \rho_1 < 1 . \tag{26}$$

If there exist positive integers n_0 *and* r, *and a positive number* p, *such that*

$$P(A_{n+r}C_{n+r} \mid A_n C_n) < 1 - p \qquad \text{if} \quad n \geqq n_0\,, \tag{27}$$

then

$$PA_n < c\rho^n\,, \qquad n = 1, 2, \cdots, \quad \text{for some} \quad c > 0\,, \quad \rho < 1\,. \tag{28}$$

PROOF. For any integer $n > 0$ we may write

$$A_{2nr} = D_n \cup E_n\,, \tag{29}$$

in which

$$D_n = A_{2nr} \bigcap_{nr}^{2nr} C_i\,, \tag{30}$$

$$E_n = A_{2nr} \bigcup_{nr}^{2nr} C_i^c\,. \tag{31}$$

Using the fact that $A_{2nr} \subset A_i$ for $i \leqq 2nr$, (31) provides the inclusion $E_n = \bigcup_{nr}^{2nr} A_{2nr} C_i^c \subset \bigcup_{nr}^{2nr} A_i C_i^c$, so that $PE_n \leqq \sum_{nr}^{2nr} PA_i C_i^c$. Combining this with (26) yields $PE_n \leqq \sum_{nr}^{2nr} c_1 \rho_1^n \leqq \sum_{nr}^{\infty} c_1 \rho_1^n = c_1(1 - \rho_1)^{-1}\rho_1^{nr}$, so that for some $c_2 > 0$, $\rho_2 < 1$

$$PE_n < c_2 \rho_2^{2nr}\,, \qquad n = 1, 2, \cdots\,. \tag{32}$$

In (30) we also use $A_{2nr} \subset A_i$ for $i \leqq 2nr$ so that $D_n = \bigcap_{nr}^{2nr} A_i C_i$. It follows that

$$PD_n = PA_{nr} C_{nr} \prod_{j=1}^{n} P(A_{(n+j)r} C_{(n+j)r} \mid A_{(n+j-1)r} C_{(n+j-1)r})\,. \tag{33}$$

Now taking n such that $nr \geqq n_0$, (33) combined with (27) implies $PD_n < (1 - p)^n$. Thus, there exists $C_3 > 0$ and $\rho_3 < 1$ such that

$$PD_n < c_3 \rho_3^{2nr}\,, \qquad n = 1, 2, \cdots\,. \tag{34}$$

In view of (29), (32), and (34) there exists $c_4 > 0$ and $\rho_4 < 1$ such that

$$PA_{2nr} < c_4 \rho_4^{2nr}\,, \qquad n = 1, 2, \cdots\,. \tag{35}$$

The desired result (28) follows easily from (35) as in [4], with $\rho = \rho_4$ and $c = c_4 \rho_4^{-2r}$.

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