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A NOTE ON FIRST EXIT TIMES WITH APPLICATIONS TO SEQUENTIAL ANALYSIS[1]

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In this paper, we prove certain theorems about the first exit time $N = \inf\{n \geqq 1 : S_n T_n + R_n \notin (-a, b)\}$, where S_n is the partial sum of i.i.d. random variables with zero mean and finite positive variance, and R_n, T_n are two sequences of random variables satisfying certain conditions. Such exit times arise in the analysis of the stopping rules of invariant sequential probability ratio tests, and our theorems are then applied to study the stopping rules of these tests.

1. Introduction. In recent years, the sample size distribution of invariant sequential probability ratio tests (SPRT) of composite hypotheses have been studied by a number of authors. Wijsman's papers [8], [9], [10] contain an extensive list of references on the subject. Asymptotic approximations for the moments of the stopping rule N have been explicitly evaluated in particular cases. For the rank-order SPRT in the two-sample problem of testing $H: F = G$ versus $K: F = G^A$, where $0 < A \neq 1$ is a known constant, Savage and Sethuraman [6] have shown that given $\varepsilon > 0$, there exists $0 < \rho < 1$ such that

$$P[|n^{-1}l_n - S(A, F, G)| \geqq \varepsilon] = O(\rho^n) \tag{1}$$

where l_n is the log likelihood ratio of the rank-order at stage n and

$$S(A, F, G) = \log 4A - 2 - \int \log (F(x) + AG(x))(dF(x) + dG(x)) \,. \tag{2}$$

Since we stop as soon as $l_n \notin (-a, b)$, it is easy to see from (1) that if $S(A, F, G) \neq 0$, then $Ee^{tN} < \infty$ for $t \leqq \theta(\theta > 0)$ and as $\min(a, b) \to \infty$,

$$\begin{aligned} EN^\beta &\sim (b/S(A, F, G))^\beta \qquad \text{if} \quad S(A, F, G) > 0 \\ EN^\beta &\sim (a/|S(A, F, G)|)^\beta \qquad \text{if} \quad S(A, F, G) < 0 \end{aligned} \tag{3}$$

for any $\beta > 0$.

Now let $X_1, X_2, \cdots$ be i.i.d. random variables with a common distribution P. To test the null hypothesis H_0 that P is $N(\zeta, \sigma^2)$ with $\zeta/\sigma = \gamma_0$ versus the alternative hypothesis H_1 that P is $N(\zeta, \sigma^2)$ with $\zeta/\sigma = \gamma_1$ ($\gamma_0 \neq \gamma_1$), the sequential t-test stops at stage $N = \inf\{n \geqq 1 : \log L_n \notin (-a, b)\}$ where L_n is the likelihood

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ratio of the maximal invariant at stage n. Define

$$f(u) = \tfrac{1}{2}[u + (u^2 + 4)^{\frac{1}{2}}], \qquad g(u) = \tfrac{1}{2}uf(u) + \log f(u) \tag{4}$$

$$\Psi(y) = g(\gamma_1 y) - g(\gamma_0 y) - \tfrac{1}{2}\gamma_1^2 + \tfrac{1}{2}\gamma_0^2 \tag{5}$$

$$\bar{X}_n = n^{-1} \textstyle\sum_1^n X_i, \qquad v_n^2 = n^{-1} \textstyle\sum_1^n X_i^2 . \tag{6}$$

Then there exists a constant d such that

$$|\log L_n - n\Psi(\bar{X}_n/v_n)| \leqq d, \qquad n = 1, 2, \cdots \tag{7}$$

(cf. [4], [9]). Letting $\lambda = EX_1/(EX_1^2)^{\frac{1}{2}}$, we obtain from (7) that if $0 < E|X_1|^{2(\beta+1)} < \infty$, then as $\min(a, b) \to \infty$,

$$\begin{aligned} EN^\beta &\sim (b/\Psi(\lambda))^\beta \qquad \text{if} \quad \Psi(\lambda) > 0 \\ EN^\beta &\sim (a/\Psi(\lambda))^\beta \qquad \text{if} \quad \Psi(\lambda) < 0 \end{aligned} \tag{8}$$

(cf. [4]).

The asymptotic approximations considered above require the assumption that $\Psi(\lambda) \neq 0$ for the sequential t-test and that $S(A, F, G) \neq 0$ for the rank-order SPRT of Savage and Sethuraman. In Section 3 below, we shall examine the situations when $\Psi(\lambda) = 0$ and $S(A, F, G) = 0$. We recall that for Wald's SPRT which stops as soon as $\prod_1^n (f_1(X_i)/f_0(X_i)) \notin (A, B)$ in testing a simple null f_0 versus a simple alternative f_1, Wald's lemma for squared sums can be applied to find an asymptotic approximation for the expected sample size when $E \log(f_1(X)/f_0(X)) = 0$. Unlike the case of Wald's SPRT, the log likelihood ratio of the maximal invariant in the invariant SPRT's considered above fails to be a random walk. Nevertheless, expressing the log likelihood ratio of the maximal invariant as a random walk plus a remainder term and analyzing the order of magnitude of the remainder term, we can obtain the asymptotic distribution and moments of the stopping rule by making use of certain results on first exit times which we develop in Section 2.

2. The asymptotic distribution and moments of first exit times.

THEOREM 1. *Suppose $X_1, X_2, \cdots$ are* i.i.d. *random variables such that $EX_1 = 0$, $EX_1^2 = \sigma^2 > 0$. Let $S_n = X_1 + \cdots + X_n$ and let R_n, T_n be two sequences of random variables. Define $N = \inf\{n \geqq 1 : S_n T_n + R_n \notin (-a, b)\}$. Let $\lambda \neq 0$, $0 < \nu < 1$.*

(i) *If $\lim_{n\to\infty} T_n = \lambda$* a.e. *and $\lim_{n\to\infty} n^{-\frac{1}{2}} R_n = 0$* a.e., *then as $a \to \infty$ and $b \to \infty$ such that $a/(a+b) \to \nu$, $\lambda^2\sigma^2(a+b)^{-2}N$ converges in distribution to $\tau = \inf\{t \geqq 0 : W(t) \notin (-\nu, 1-\nu)\}$, where $W(t)$, $t \geqq 0$, is the standard Wiener process.*

(ii) *Suppose $E|X_1|^{2+\eta} < \infty$ for some $\eta > 0$ and $EL^\gamma(\delta, \varepsilon) < \infty$ for some $\gamma > 0$, $\varepsilon > 0$ and $\delta < \frac{1}{2}$, where $L(\delta, \varepsilon) = \sup\{n \geqq 1 : |R_n| \geqq n^\delta$ or $|T_n| \leqq \varepsilon\}$ $(\sup \varnothing = 0)$. Then $EN^\gamma < \infty$. If furthermore $\lim_{n\to\infty} T_n = \lambda$* a.e., *then as $a \to \infty$ and $b \to \infty$ such that $a/(a+b) \to \nu$,*

$$EN^\gamma \sim (|\lambda|\sigma)^{-2\gamma}(a+b)^{2\gamma} E\tau^\gamma . \tag{9}$$

PROOF. Let $U_n = S_n T_n + R_n$. If $\lim_{n\to\infty} T_n = \lambda$ a.e. and $\lim_{n\to\infty} n^{-\frac{1}{2}} R_n = 0$ a.e., then the process $(r^{-\frac{1}{2}}U_{[rt]}/\lambda\sigma, 0 \leqq t \leqq 1)$ converges weakly to $(W(t), 0 \leqq t \leqq 1)$ as $r \to \infty$. First assume that $\lambda > 0$. Set $c = a + b$. As $c \to \infty$, we have $a = \nu c(1 + o(1))$, $b = (1 - \nu)c(1 + o(1))$ and so

$$P[N > (\lambda\sigma)^{-2}c^2t] = P[\max_{s\leq t} U_{[(\lambda\sigma)^{-2}c^2s]} < b, \min_{s\leq t} U_{[(\lambda\sigma)^{-2}c^2s]} > -a] \to P[\max_{s\leq t} W(s) < 1 - \nu, \min_{s\leq t} W(s) > -\nu] = P[\tau > t] .$$

Hence $(\lambda\sigma)^2c^{-2}N$ converges in distribution to τ. If $\lambda < 0$, then a similar argument shows that $(\lambda\sigma)^2c^{-2}N$ converges in distribution to $\tau^* = \inf\{t \geqq 0: W(t) \notin (-(1-\nu), \nu)\}$, and obviously, τ and τ^* have the same distribution.

To prove (ii), we note that the assumption $EL^\gamma(\delta, \varepsilon) < \infty$ implies $P[L(\delta, \varepsilon) < \infty] = 1$, and so $\lim_{n\to\infty} n^{-\frac{1}{2}}R_n = 0$ a.e. Hence if $\lim_{n\to\infty} T_n = \lambda$ a.e., then by (i), $(\lambda\sigma)^2c^{-2}N$ converges in distribution to τ as $c \to \infty$. We now show that under the assumptions of (ii), the family $\{(c^{-2}N)^\gamma, c \geqq 1\}$ is uniformly integrable. Let $M_c = \inf\{n \geqq L(\delta, \varepsilon) + 1 : |S_n| \geqq \varepsilon^{-1}(c + n^\delta)\}$. Then $N \leqq M_c$ for $a > 0$, $b > 0$. We shall show that

$$\lim_{t\to\infty} \sup_{c\geqq c_0} E(c^{-2}M_c)^\gamma I_{[M_c > c^2t]} = 0 . \tag{10}$$

First we note that

$$E(c^{-2}M_c)^\gamma I_{[M_c>c^2t]} = t^\gamma P[M_c > c^2t] + \gamma \int_t^\infty u^{\gamma-1}P[M_c > c^2u]\, du . \tag{11}$$

Let $k \geqq 2$ be an integer such that

$$\tfrac{1}{2}k\eta > \gamma \qquad \text{and} \qquad k(\tfrac{1}{2} - \delta) > \gamma . \tag{12}$$

For $u \geqq 1$, defining $n_i = n_i(u) = [ic^2u/k]$, $i = 1, \cdots, k$, $n_0 = 0$ and $S_1' = S_{n_1}$, $S_i' = S_{n_i} - S_{n_{i-1}}$ $(2 \leqq i \leqq k)$, we have

$$\begin{aligned} P[M_c > c^2u] &\leqq P[L(\delta, \varepsilon) + 1 > c^2u/2k] \\ &\quad + P[|S_n| < \varepsilon^{-1}(c + n^\delta) \ \text{ for } \ c^2u \geqq n \geqq c^2u/2k] \\ &\leqq P[L(\delta, \varepsilon) + 1 > c^2u/2k] + \textstyle\prod_{i=1}^k P[|S_i'| < 2\varepsilon^{-1}(c + n_i^\delta)] . \end{aligned} \tag{13}$$

Without loss of generality, we can assume that $\eta \leqq 1$. Then for $u \geqq 1$ and $c^2 \geqq 4k$, we have $n_i - n_{i-1} \geqq c^2u/2k \geqq 2$, and so by a theorem of Esseen [2],

$$\begin{aligned} &P[|S_i'| < 2\varepsilon^{-1}(c + n_i^\delta)] \\ &\quad = P[\sigma^{-1}(n_i - n_{i-1})^{-\frac{1}{2}}|S_i'| < 2\varepsilon^{-1}\sigma^{-1}(n_i - n_{i-1})^{-\frac{1}{2}}(c + n_i^\delta)] \\ &\quad \leqq P[|N(0, 1)| < 2\varepsilon^{-1}\sigma^{-1}(n_i - n_{i-1})^{-\frac{1}{2}}(c + n_i^\delta)] + \zeta(n_i - n_{i-1})^{-\eta/2} \end{aligned}$$

where ζ is a positive constant. Since $P[|N(0, 1)| \leqq x] \leqq x$ for $x > 0$, it then follows that for $u \geqq 1$ and $c^2 \geqq 4k$,

$$\begin{aligned} &\textstyle\prod_{i=1}^k P[|S_i'| < 2\varepsilon^{-1}(c + n_i^\delta)] \\ &\quad \leqq \{2\varepsilon^{-1}\sigma^{-1}(c^2u/2k)^{-\frac{1}{2}}(c + c^{2\delta}u^\delta) + \zeta(c^2u/2k)^{-\eta/2}\}^k . \end{aligned} \tag{14}$$

From (12), it is clear that

$$\begin{aligned} \lim_{t\to\infty} \sup_{c^2\geqq k} \{&t^\gamma(c^2t)^{-k/2}(c + c^{2\delta}t^\delta)^k + t^\gamma(c^2t)^{-\eta k/2} \\ &+ \textstyle\int_t^\infty u^{\gamma-1}[(c^2u)^{-k/2}(c + c^{2\delta}u^\delta)^k + (c^2u)^{-\eta k/2}]\, du\} = 0 . \end{aligned} \tag{15}$$

By assumption, $EL^{\gamma}(\delta, \varepsilon) < \infty$, and therefore

$$(16) \qquad \lim_{t\to\infty} \sup_{c \geqq c_0} \{t^{\gamma} P[L(\delta, \varepsilon) + 1 > c^2 t/2k] + \gamma \textstyle\int_t^{\infty} u^{\gamma-1} P[L(\delta, \varepsilon) + 1 > c^2 u/2k]\, du\} = 0\,.$$

From (11), (13), (14), (15) and (16), (10) follows immediately. □

THEOREM 2. *Let $X_1, X_2, \cdots$ be* i.i.d. *random variables such that $P[X_1 \neq 0] > 0$. Let S_n, R_n, T_n and N be defined as in Theorem* 1, *and define* $L_1(\Delta, \varepsilon) = \sup\{n \geqq 1 : |R_n| \geqq \Delta$ or $|T_n| \leqq \varepsilon\}$ ($\sup \emptyset = 0$), $\Delta, \varepsilon > 0$.

(i) *Suppose there exist $\theta > 0$, $\Delta > 0$ and $\varepsilon > 0$ such that $E \exp(\theta L_1(\Delta, \varepsilon)) < \infty$. Then N is exponentially bounded*, i.e., *$P[N > n] = O(\rho^n)$ for some $1 > \rho > 0$.*

(ii) *Let $\gamma > 0$. Suppose $EL_1^{\gamma}(\Delta, \varepsilon) < \infty$ for some $\Delta > 0$, $\varepsilon > 0$. Then $EN^{\gamma} < \infty$. Suppose furthermore that $EX_1 = 0$, $EX_1^2 = \sigma^2$ and $\lim T_n = \lambda$* a.e., *where λ is a nonzero constant. Then* (9) *holds as $a \to \infty$ and $b \to \infty$ such that $a/(a + b) \to \nu$, where* $0 < \nu < 1$.

PROOF. Without loss of generality, we can assume $a > 0$, $b > 0$ and let $c = a + b$. Since $N \leqq \inf\{n \geqq L_1(\Delta, \varepsilon) + 1 : |S_n| \geqq \varepsilon^{-1}(c + \Delta)\}$, we have

$$(17) \qquad \begin{aligned} P[N > n] &\leqq P[L_1(\Delta, \varepsilon) + 1 \geqq \tfrac{1}{2}n] \\ &\quad + P[|S_j| < \varepsilon^{-1}(c + \Delta) \text{ for all } \tfrac{1}{2}n \leqq j \leqq n] \\ &= A_n + B_n\,, \quad \text{say.} \end{aligned}$$

Since $P[X_1 \neq 0] > 0$, we obtain by an argument due to Stein [7] that $B_n = O(\rho^n)$ for some $0 < \rho < 1$. Hence N is exponentially bounded if $L_1(\Delta, \varepsilon)$ is exponentially bounded, and $EN^{\gamma} < \infty$ if $EL_1^{\gamma}(\Delta, \varepsilon) < \infty$.

Now suppose that $EX_1 = 0$ and $EX_1^2 = \sigma^2 > 0$. For $c > 0$, define $M_c = \inf\{n \geqq L_1(\Delta, \varepsilon) + 1 : |S_n| \geqq c\}$. By the central limit theorem, we can choose $c_0 > 1$ such that for all $c \geqq c_0$, $P[S_{[c^2]} \geqq 2c] \geqq p > 0$. We shall now show that for $n = 1, 2, \cdots$ and $c \geqq c_0$,

$$(18) \qquad P[M_c \geqq 2c^2 n] \leqq (1 - p)^n + P[L_1(\Delta, \varepsilon) + 1 \geqq n]\,.$$

For $n \geqq 1$ and $c \geqq c_0$, define $n_i = [c^2 n] + i[c^2]$, $i = 1, 2, \cdots$ and set $S_1' = S_{n_1}$, $S_i' = S_{n_i} - S_{n_{i-1}}$. We note that

$$(19) \qquad \begin{aligned} P[|S_j| < c \text{ for } c^2 n \leqq j \leqq 2c^2 n] &\leqq P[|S_i'| < 2c \text{ for } i = 1, \cdots, n] \\ &= \textstyle\prod_{i=1}^{n} P[|S_i'| < 2c] \leqq (1 - p)^n\,. \end{aligned}$$

Since $P[M_c \geqq 2c^2 n] \leqq P[L_1(\Delta, \varepsilon) + 1 \geqq n] + P[|S_j| < c$ for $c^2 n \leqq j \leqq 2c^2 n]$ for $c > 1$, (18) follows easily from (19). From (18), it is clear that if $EL_1^{\gamma}(\Delta, \varepsilon) < \infty$, then the family $\{(c^{-2}M_c)^{\gamma} : c \geqq c_0\}$ is uniformly integrable, and so if $\lim_{n\to\infty} T_n = \lambda$ a.e., then the asymptotic formula (9) holds. □

3. Applications to invariant sequential probability ratio tests. For the sequential t-test described in Section 1, we stop at stage

$$(20) \qquad N = \inf\{n \geqq 1 : \log L_n \notin (-a, b)\}$$

where L_n is the likelihood ratio of the maximal invariant based on the first n observations $X_1, \cdots, X_n$ and satisfies (7). Let $\mu = EX_1$, $\rho^2 = EX_1^2$, $\lambda = \mu/\rho$ and suppose that $\Psi(\lambda) = 0$. We note that Ψ is of class C^∞ and $\Psi'(y) \neq 0$ for all y (cf. [9]). Therefore

$$n\Psi(\bar{X}_n/v_n) = n(\bar{X}_n/v_n - \lambda)\Psi'(\hat{\lambda}_n) = (\sum_1^n X_i - n\lambda v_n)\Psi'(\hat{\lambda}_n)/v_n$$

where $\hat{\lambda}_n$ lies between $\bar{X}_n/v_n$ and λ. Furthermore,

$$v_n = (n^{-1} \sum_1^n X_i^2)^{\frac{1}{2}} = \rho + (n^{-1} \sum_1^n X_i^2 - \rho^2)/2\hat{\rho}_n$$

with $\hat{\rho}_n$ lying between v_n and ρ. Hence

$$\begin{aligned} n\Psi(\bar{X}_n/v_n) &= (\Psi'(\hat{\lambda}_n)/v_n)\{(\sum_1^n X_i - n\mu) - \lambda(\sum_1^n X_i^2 - n\rho^2)/2\hat{\rho}_n\} \\ &= \{\Psi'(\hat{\lambda}_n)/(\hat{\rho}_n v_n)\}\{\hat{\rho}_n(\sum_1^n X_i - n\mu) - \tfrac{1}{2}\lambda(\sum_1^n X_i^2 - n\rho^2)\} \\ &= T_n\{(\sum_1^n Y_i - nEY_1) + (\hat{\rho}_n - \rho)(\sum_1^n X_i - n\mu)\} \end{aligned}$$

where $T_n = \Psi'(\hat{\lambda}_n)/(\hat{\rho}_n v_n)$, $Y_i = \rho X_i - \frac{1}{2}\lambda X_i^2$ and so $EY_1 = \frac{1}{2}\rho\mu$.
Hence letting

$$R_n = (\log L_n - n\Psi(\bar{X}_n/v_n)) + T_n(\hat{\rho}_n - \rho)(\sum_1^n X_i - n\mu)\,, \tag{21}$$

we have

$$\log L_n = T_n \sum_1^n (Y_i - EY_1) + R_n\,. \tag{22}$$

By making use of Theorem 1, we can then obtain the asymptotic distribution and moments of N.

THEOREM 3. *Suppose $X_1, X_2, \cdots$ are* i.i.d. *random variables such that $EX_1 = \mu$, $EX_1^2 = \rho^2$ and*

$$P[\rho X_1 - \tfrac{1}{2}\lambda X_1^2 = \tfrac{1}{2}\rho\mu] < 1 \tag{23}$$

where $\lambda = \mu/\rho$ and suppose that $\Psi(\lambda) = 0$ with Ψ defined by (5). *Assume that $E|X_1|^{2p} < \infty$ for some $p > 2$. Let N be defined by* (20), *and let Φ denote the distribution function of the standard normal distribution. Set $\sigma^2 = E(\rho X_1 - \frac{1}{2}\lambda X_1^2 - \frac{1}{2}\rho\mu)^2$. Then for any $0 < \nu < 1$, we have as $a \to \infty$ and $b \to \infty$ such that $a/(a+b) \to \nu$,*

$$\begin{aligned} \forall t > 0\,, \quad & P[N > (\sigma\rho^{-2}\Psi'(\lambda))^{-2}(a+b)^2 t] \\ & \to \sum_{k=-\infty}^{\infty} \{\Phi(t^{-\frac{1}{2}}(2k+1-\nu)) - \Phi(t^{-\frac{1}{2}}(2k-\nu)) \\ & \qquad - \Phi(t^{-\frac{1}{2}}(2k+1+\nu)) + \Phi(t^{-\frac{1}{2}}(2k+\nu))\} \\ & = U(t)\,, \quad \textit{say}; \end{aligned} \tag{24}$$

$$EN^\beta \sim \beta(a+b)^{2\beta}\rho^{4\beta}|\sigma\Psi'(\lambda)|^{-2\beta} \int_0^\infty t^{\beta-1}U(t)\,dt \qquad \textit{for} \quad 0 < \beta < p - 1\,. \tag{25}$$

LEMMA (cf. [1]). *Let $Z_1, Z_2, \cdots$ be* i.i.d. *random variables such that $EZ_1 = 0$ and $E|Z_1|^p < \infty$. Let $M(\alpha, \varepsilon) = \sup\{n \geqq 1 : |\sum_1^n Z_i| \geqq \varepsilon n^\alpha\}$* ($\sup \varnothing = 0$). *Then $EM^{p\alpha-1}(\alpha, \varepsilon) < \infty$ for all $\varepsilon > 0$ and all $\alpha > \frac{1}{2}$ with $p\alpha > 1$.*

PROOF OF THEOREM 3. First note that $\sigma^2 = \operatorname{Var} Y_1 > 0$ by (23) and $E|Y_1|^p < \infty$ with $p > 2$. Let $M_1(\varepsilon) = \sup\{n \geqq 1 : |v_n^2 - \rho^2| \geqq \varepsilon$ or $|\bar{X}_n - \mu| \geqq \varepsilon\}$. Then since $E|X_1|^{2p} < \infty$, $EM_1^{p-1}(\varepsilon) < \infty$ for all $\varepsilon > 0$ by the preceding lemma. It then follows that $EM_2^{p-1}(\varepsilon) < \infty$ for all $\varepsilon > 0$, where $M_2(\varepsilon) = \sup\{n \geqq 1 : |T_n - \rho^{-2}\Psi'(\lambda)| \geqq \varepsilon\}$. For $0 < \beta < p - 1$, we can choose $\alpha > \frac{1}{2}$ such that $\beta < p\alpha - 1$ and $\alpha < 1$. Let $\gamma = 1 - \alpha$ and choose $0 < \zeta < \gamma$. Define

$$M_3(\gamma, \varepsilon) = \sup\{n \geqq 1 : |v_n^2 - \rho^2| \geqq \varepsilon n^{-\gamma}\} = \sup\{n \geqq 1 : |\sum_1^n X_i^2 - n\rho^2| \geqq \varepsilon n^{\alpha}\}.$$

Since $E|X_1|^{2p} < \infty$, it follows from the preceding lemma that $EM_3^{p\alpha-1}(\gamma, \varepsilon) < \infty$ and so $EM_3^{\beta}(\gamma; \varepsilon) < \varepsilon$ for all $\varepsilon > 0$. Therefore $EM_4^{\beta}(\gamma, \varepsilon) < \infty$ for all $\varepsilon > 0$, where we define

$$M_4(\gamma, \varepsilon) = \sup\{n \geqq 1 : |\hat{\rho}_n - \rho| \geqq \varepsilon n^{-\gamma}\};$$
$$M_5(\zeta, \varepsilon) = \sup\{n \geqq 1 : |\sum_1^n X_i - n\mu| \geqq \varepsilon n^{\frac{1}{2}+\zeta}\}.$$

By the preceding lemma, we obtain that $EM_5^{2p(\frac{1}{2}+\zeta)-1}(\zeta, \varepsilon) < \infty$ and so $EM_5^{\beta}(\zeta, \varepsilon) < \infty$ for all $\varepsilon > 0$. Let $\delta = \frac{1}{2} + \zeta - \gamma$. Then $\delta < \frac{1}{2}$. Define $M_6(\delta) = \sup\{n \geqq 1 : |R_n| \geqq n^{\delta}\}$. Using the finiteness of $EM_2^{\beta}(\varepsilon)$, $EM_4^{\beta}(\gamma, \varepsilon)$ and $EM_5^{\beta}(\zeta, \varepsilon)$, we obtain from (7) and (21) that $EM_6^{\beta}(\delta) < \infty$. From (22), Theorem 1 is applicable to N, and noting that $P[\tau > t] = U(t)$ (cf. [3], page 329), where τ is as defined in Theorem 1, we obtain the asymptotic formulas (24) and (25) from Theorem 1. □

Theorem 1 can similarly be used to study the stopping time of the rank-order SPRT of Savage and Sethuraman in the case when $S(A, F, G) = 0$, where $S(A, F, G)$ is defined by (2), since here we again have the representation of the log likelihood ratio l_n in terms of the partial sum of i.i.d. random variables plus a remainder term whose order of magnitude we can analyze. This representation, which is a special case of a more general representation theorem for generalized Chernoff-Savage statistics, together with related results on other sequential rank tests, will be treated in [4].

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