*The Annals of Statistics*
1975, Vol. 3, No. 4, 916-925

# ON THE ASYMPTOTIC DISTRIBUTIONS OF SOME STATISTICS USED FOR TESTING $\Sigma_1 = \Sigma_2$[1]

BY KOCHERLAKOTA SUBRAHMANIAM

*University of Manitoba*

The asymptotic non-null distributions of four statistics, used for testing the hypothesis $H_0: \Sigma_1 = \Sigma_2$, are developed. These expansions are obtained by the partial differential equations method of Muirhead (*Ann. Math. Statist.* **41** 1002-1010). The procedure permits a simple extension of the results of Pillai and Nagarsenker (*J. Multivariate Anal.* **2** 96-114).

**1. Introduction.** Consider two random matrices $S_1$, $S_2$ which are independently distributed as Wishart $(n_i, p, \Sigma_i)$, $i = 1, 2$. Here the distribution of some statistics derived from the roots of the matrix $S_1 S_2^{-1}$, $0 < f_1 \leqq f_2 \leqq \cdots \leqq f_p < \infty$, are examined. Let the roots of the matrix $\Sigma_1 \Sigma_2^{-1}$ be $0 < \lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_p < \infty$, and let $\Lambda = \text{diag}(\lambda_1, \cdots, \lambda_p)$. Defining $\theta_j = f_j/(1 + f_j)$, $j = 1, 2, \cdots, p$, Pillai and Nagarsenker [7] have obtained the $h$th moment of the statistic

$$Y = \prod_{j=1}^{p} \theta_j^{a}(1 - \theta_j)^b \tag{1.1}$$

as

$$K|\Lambda|^{-n_1/2} {}_2F_1[(n/2), (n_1/2) + ah; (n_1 + n_2)/2 + (a + b)h; I - \Lambda^{-1}], \tag{1.2}$$

where

$$K = \{\Gamma_p(n/2)\Gamma_p(n_1/2 + ah)\Gamma_p(n_2/2 + bh)\}/\{\Gamma_p(n_2 + (a + b)h)\Gamma_p(n_1/2)\Gamma_p(n_2/2)\}.$$

In what follows,

$$n = n_1 + n_2, \qquad M = I - \Lambda^{-1}, \qquad n_1 = nk_1, \qquad \text{and} \qquad n_2 = nk_2.$$

By specializing to the case where (i) $a = n_1/2$, $b = n_2/2$, (ii) $a = 1$, $b = 0$, (iii) $a = 0$, $b = 1$, one can generate various statistics used for testing the hypothesis $H_0: \Sigma_1 = \Sigma_2$. These are considered in Sections 2, 3, and 4, respectively.

The asymptotic distributions are derived here by using the partial differential equations technique developed by Muirhead in [4] and exemplified to various situations in [5] and [6]. An earlier paper [3] by him details the equations satisfied by the hypergeometric function of matrix argument. These and related results have been extensively used in the present paper.

**2. Asymptotic expansions for $-2 \ln W$, $-2\rho \ln W$.** Anderson ([1], page 254) has discussed the null distribution of $-2\rho \ln W$, where

$$W = \left\{\frac{n^n}{n_1^{n_1} n_2^{n_2}}\right\}^{p/2} \prod_{j=1}^{p} \theta_j^{n_1/2}(1 - \theta_j)^{n_2/2} \tag{2.1}$$

Received April 1974; revised December 1974.

[1] Research supported by NRC grant.

*AMS* 1970 *subject classifications.* Primary 62H10; Secondary 62E15.

*Key words and phrases.* Non-null distributions, asymptotic expansions, hypergeometric functions of matrix argument, partial differential equations.

with $\rho$ being defined as

$$1 - \rho = \{n_1^{-1} + n_2^{-1} - 1\}(2p^2 + 3p - 1)/12(p + 1) .$$

Pillai and Nagarsenker [7] look at the non-null distribution of the statistics $-2 \ln W$, $-2\rho \ln W$ under two different conditions. We shall discuss the asymptotic distributions of these statistics in this section.

2.1. *Asymptotic distribution of* $-2 \ln W$. Referring to (1.2) in the special case when $a = n_1/2$, $b = n_2/2$ with $h = -2it$, we have the characteristic function (ch.f.) of $-2 \ln W$ as

$$\phi(t, \Lambda) = \phi_0(t)\phi_A(t, \Lambda) \tag{2.2}$$

where $\phi_0(t)$ is the ch.f. in the null case and $\phi_A(t, \Lambda)$ is the factor due to $\Lambda \neq I$. The expansion for $\phi_0(t)$ is given by

$$\begin{aligned}\phi_0(t) = g^{-f/2}[1 &+ n^{-1}r(g^{-1} - 1) + n^{-2}\{s(1 - g^{-2}) + r^2(g^{-1} - 1)^2/2\} \\ &+ n^{-3}\{u(g^{-3} - 1) + rs(g^{-1} - 1)(g^{-2} - 1) + r^3(g^{-1} - 1)^3/6\}] ,\end{aligned} \tag{2.3}$$

where $g = (1 - 2it)$, $f = p(p + 1)/2$, and

$$\begin{aligned} r &= p(2p^2 + 3p - 1)(k_1^{-1} + k_2^{-1} - 1)/24 , \\ s &= -f(p + 2)(p - 1)(k_1^{-2} + k_2^{-2} - 1)/24 , \\ u &= p\{(p + 1)(2p + 1)(3p^2 + 3p - 11) + 14\}(k_1^{-3} + k_2^{-3} - 1)/720) . \end{aligned}$$

This expansion can be obtained readily by applying equation (16) of Anderson ([1], page 205) to $\phi_0(t)$.

The remainder of this section is devoted to the expansion of $\phi_A(t, \Lambda)$ which is given by

$$\phi_A(t, \Lambda) = |\Lambda|^{-n_1/2}{}_2F_1[n/2, n_1 g/2; ng/2; M] . \tag{2.4}$$

Writing $\Lambda = \{I - 2P/n\}^{-1}$ and $M = 2P/n$, with $P$ being a fixed matrix, it is seen that $\lim_{n\to\infty} \phi_A(t, \Lambda) = 1$ and hence

$$\begin{aligned}\phi_A(t, \Lambda) = \exp\{H(t, \Lambda)\} &\approx 1 + Q_1/n + n^{-2}\{Q_2 + Q_1^2/2\} \\ &+ n^{-3}\{Q_3 + Q_1Q_2 + Q_1^3/6\} + O(n^{-4}) .\end{aligned}$$

If $\mu_1, \mu_2, \cdots, \mu_p$ are the roots of $M$, then the function $|I - M|^{-nk_1/2} \exp\{H(t, M)\}$ satisfies the system of partial differential equations (1.3) of Muirhead [3]. It is necessary to transform this system into the ones in terms of $\rho_1, \cdots, \rho_p$, the roots of $P$. Note that $\rho_j = (n/2)\mu_j$. Without loss of generality, the first equation satisfied by $H$ is considered:

$$\begin{aligned} \rho_1(1 - 2\rho_1/n)\left[\frac{\partial^2 H}{\partial \rho_1^2} + \left(\frac{\partial H}{\partial \rho_1}\right)^2\right] &+ \frac{1}{2}\frac{\partial H}{\partial \rho_1}[4\rho_1 k_1 + \{ng - (p - 1)\} \\ &- 2\rho_1\{1 + k_1 g - (p - 3)/n\} + \textstyle\sum_{j\neq 1} \rho_1(1 - 2\rho_1/n)/(\rho_1 - \rho_j)] \\ &- \tfrac{1}{2}\textstyle\sum_{j\neq 1} \dfrac{\rho_j(1 - 2\rho_j/n)}{(\rho_1 - \rho_j)}\left(\dfrac{\partial H}{\partial \rho_j}\right) \\ &= 2it(k_1 - k_1^2)\rho_1/(1 - 2\rho_1/n) . \end{aligned} \tag{2.5}$$

Substituting $H(t, M) = \sum_{k=1}^{\infty} Q_k(t, P)/n^k$ and equating like powers of $n^{-1}$ in (2.5), the differential equations for $Q_1$, $Q_2$, $Q_3$ are

$$g\frac{\partial Q_1}{\partial \rho_1} = 4it(k_1 - k_1^2)\rho_1 ,$$

$$g\frac{\partial Q_2}{\partial \rho_1} + 2\Big[\rho_1\frac{\partial^2 Q_1}{\partial \rho_1^2} + \frac{\partial Q_1}{\partial \rho_1}\{2\rho_1 k_1 - (p-1)/2 - \rho_1 - \rho_1 k_1 g - \textstyle\sum_{j\neq 1}\rho_1/2(\rho_1 - \rho_j)\}$$
$$- \textstyle\sum_{j\neq 1}\{\rho_1/2(\rho_1 - \rho_j)\}\displaystyle\frac{\partial Q_1}{\partial \rho_j}\Big] = 8it(k_1 - k_1^2)\rho_1^2$$

and

$$g\frac{\partial Q_3}{\partial \rho_1} + 2\Big[\rho_1\Big\{\frac{\partial^2 Q_2}{\partial \rho_1^2} + \Big(\frac{\partial Q_1}{\partial \rho_1}\Big)^2\Big\} + \frac{\partial Q_2}{\partial \rho_1}\{2\rho_1 k_1 - (p-1)/2 - \rho_1 - k_1\rho_1 g$$
$$+ \textstyle\sum_{j\neq 1}\rho_1/2(\rho_1 - \rho_j)\} - \textstyle\sum_{j\neq 1}\{\rho_j/2(\rho_1 - \rho_j)\}\displaystyle\frac{\partial Q_2}{\partial \rho_j}\Big]$$
$$+ 2\Big[-2\rho_1^2\frac{\partial^2 Q_1}{\partial \rho_1^2} + \frac{\partial Q_1}{\partial \rho_1}\{(p-3)\rho_1 - \textstyle\sum_{j\neq 1}\rho_1^2/(\rho_1 - \rho_j)\}$$
$$+ \textstyle\sum_{j\neq 1}\{\rho_j/(\rho_1 - \rho_j)\}\displaystyle\frac{\partial Q_1}{\partial \rho_j}\Big] = 16it(k_1 - k_1^2)\rho_1^3 .$$

Imposing the boundary conditions $Q_1(0, P) = Q_2(0, P) = Q_3(0, P) = 0$ and the symmetry in $\rho_1, \cdots, \rho_p$ of the functions $Q_1$, $Q_2$, $Q_2$, we have

$$Q_1 = (k_1 - k_1^2)(g^{-1} - 1)\sigma_2 ,$$
$$Q_2 = 4(k_1 - k_1^2)[-\sigma_3(1 + k_1)/3 + g^{-1}\{k_1\sigma_3 + (\sigma_1^2 + \sigma_2)/4\}$$
$$- g^{-2}\{\sigma_3(2k_1 - 1)/3 + (\sigma_1^2 + \sigma_2)/4\}] ,$$
$$Q_3 = -2(k_1 - k_1^2)[(1 + k_1 + k_1^2)\sigma_4 - 2g^{-1}\{3k_1^2\sigma_4 + 4k_1\sigma_3 + k_1\sigma_1\sigma_2\}$$
$$- g^{-2}\{(6k_1 - 10k_1^2)\sigma_4 + (2 - 6k_1)\sigma_3 - 3\sigma_2/2 - (p-2)\sigma_1^2/2$$
$$- (6k_1 - 2)\sigma_1\sigma_2\} - g^{-3}\{(5k_1 - 5k_1^2 - 1)\sigma_4 - 2(2k_1 - 1)\sigma_3$$
$$- 3\sigma_2/2 - (p-2)\sigma_1^2/2 - 2(2k_1 - 1)\sigma_1\sigma_2\}] .$$

Note here that $\int (\sigma_2 + 2\rho_i\sigma_1)\,d\rho_i = \sigma_1\sigma_2$ (symmetric in $\rho_1, \cdots, \rho_p$). The quantity $\sigma_j = \rho_1^j + \rho_2^j + \cdots + \rho_p^j$.

Substituting for $\phi_0(t)$ and $\phi_A(t, \Lambda)$ in $\phi(t)$, to $O(n^{-4})$,

$$\phi(t) = g^{-f/2}[1 + n^{-1}(\alpha_0 + g^{-1}\alpha_1)$$
$$+ n^{-2}(\alpha_2 + \alpha_3 g^{-1} + \alpha_4 g^{-2}) + n^{-3}\textstyle\sum_{j=0}^{3}\alpha_{5+j}g^{-j}] + O(n^{-4}) , \tag{2.6}$$

with $\alpha$'s being defined as

$$k = k_1(1 - k_1) , \qquad \alpha_0 = -(r + k\sigma_2) , \qquad \alpha_1 = -\alpha_0 ,$$
$$\alpha_2 = -4k(1 + k_1)\sigma_3/3 + (r + k\sigma_2)^2/2 - s ,$$
$$\alpha_3 = k(4k_1\sigma_3 + \sigma_1^2 + \sigma_2) - (r + k\sigma_2)^2 ,$$
$$\alpha_4 = -k[4(2k_1 - 1)\sigma_3/3 + \sigma_1^2 + \sigma_2] + (r + k\sigma_2)^2/2 + s ,$$

$$\alpha_5 = -u - 2k(1 + k_1 + k_1^2)\sigma_4 + (r + k\sigma_2)[4(1 + k_1)k\sigma_3/3 + s - (r + k\sigma_2)^2/6] ,$$

$$\alpha_6 = 4kk_1[3k_1\sigma_4 + \sigma_3 + \sigma_1\sigma_2] + (r + k\sigma_2)[(r + k\sigma_2)^2/2 - s - 4k(1 + 4k_1)\sigma_3/3 - (\sigma_1^2 + \sigma_2)k] ,$$

$$\alpha_7 = 4k[(3k_1 - 5k_1^2)\sigma_4 + \sigma_3(1 - 3k_1) - 3\sigma_2/4 - (p - 2)\sigma_1^2/4 - (3k_1 - 1)\sigma_1\sigma_2] - (r + k\sigma_2)[s + (r + k\sigma_2)^2/2 - 4k\{\sigma_3(5k_1 - 1)/3 + (\sigma_1^2 + \sigma_2)/2\}] ,$$

and

$$\alpha_8 = u - 4k[(5k_1 - 5k_1^2 - 1)\sigma_4/2 - (2k_1 - 1)\sigma_3 - 3\sigma_2/4 - (p - 2)\sigma_1^2/4 - (2k_1 - 1)\sigma_1\sigma_2] + (r + k\sigma_2)[s + (r + k\sigma_2)^2/6 - k\{4(2k_1 - 1)\sigma_3/3 + (\sigma_1^2 + \sigma_2)\}] .$$

Inverting (2.6), the distribution function of $-2 \ln W$ is

$$P\{-2 \ln W \leqq z\} = D_f(z) + n^{-1}\{\alpha_0 D_f(z) + \alpha_1 D_{f+2}(z)\} + n^{-2}\{\alpha_2 D_f(z) + \alpha_3 D_{f+2}(z) + \alpha_4 D_{f+4}(z)\} + n^{-3} \sum_{j=0}^{3} \alpha_{5+j} D_{f+2j}(z) + O(n^{-4}) \tag{2.7}$$

where $D_\nu(z) = P\{\chi_\nu^2 \leqq z\}$.

Pillai and Nagarsenker [7] have obtained this distribution function to the order $O(n^{-3})$. Their results in equation (5.35) agree with (2.7) up to the term $n^{-2}$.

2.2. *Distribution of* $-2\rho \ln W$. As before, we can obtain the ch.f. of $-2\rho \ln W$ as

$$\phi(t) = \phi_0(t)\phi_A(t, \Lambda) \tag{2.8}$$

where $\phi_0(t)$ is the ch.f. in the null case and $\phi_A(t, \Lambda)$ is the term due to non-null nature of the distribution ($\Lambda \neq I$). The $\phi_0(t)$ factor can be expanded as in Anderson ([1], page 255), so that the term involving $n^{-1}$ is not present in the expansion. This is achieved by choosing $\rho$ appropriately.

The expansion of the statistic $-2\rho \ln W$ is given in terms of $m$ and $P$ where

$$P = mM/2 , \qquad m = \rho n - 2\alpha$$

and

$$\alpha = (k_1^{-1} + k_2^{-1} - 1)(2p^2 + 3p - 1)/12(p + 1) .$$

It can be shown that to order $O(m^{-4})$,

$$\phi_0(t) = g^{-f/2}[1 + m^{-2}w_2(g^{-2} - 1) + m^{-3}w_3(g^{-3} - 1) + O(m^{-4})] . \tag{2.9}$$

In (2.9), the coefficients $w_2$, $w_3$ are obtained from equation (11) of Anderson ([1], page 205). Writing $\tau_j = (k_1^{-j} + k_2^{-j} - 1)$,

$$w_2 = f[(p - 1)(p + 2)\tau_2 - \tau_1^2(2p^2 + 3p - 1)^2/6(p + 1)^2]/24 ,$$

$$w_3 = p[(2p^2 + 3p - 1)^3\tau_1^3/108(p + 1)^2 - (p - 1)(p + 2)(2p^2 + 3p - 1)\tau_1\tau_2/6 + (p - 1)(6p^3 + 21p^2 + 11p - 19)\tau_3/30]/24 - p/45 .$$

The non-null component is

$$\phi_A(t, \Lambda) = |\Lambda|^{-(m+2\alpha)k_1/2}\,{}_2F_1[m/2 + \alpha, mgk_1 + \alpha k_1; mg/2 + \alpha; M] .$$

As before, if $M = 2P/m$, $P$ being fixed, $\lim_{m\to\infty} \phi_A(t, 2P/m) = 1$. Hence $\phi_A(t, \Lambda)$ can be similarly expanded. The partial differential equations for $H(t, M)$, in terms of $\rho_1, \rho_2, \cdots, \rho_p$, where $\rho_i = m\mu_i/2$, can be written quite easily. An examination of these equations shows that the equations for $Q_1, Q_2, Q_3$ in this case can be obtained from those in Section 2.1,

$$Q_1 = Q_1^0\,, \qquad Q_2 = Q_2^0 - 2\alpha k(g^{-2} - g^{-1})\sigma_2 + 2\alpha k(g^{-1} - 1)\sigma_2\,,$$
$$Q_3 = Q_3^0 - 2\alpha g^{-1}[Q_2 - 2(1 - k_1)k(g^{-1} - 1)\sigma_3/3] + 4\alpha k(g^{-1} - 1)\sigma_3/3\,.$$

Here $Q_1^0, Q_2^0, Q_3^0$ are the solutions obtained in the previous section.

Substituting for the various quantities and solving the equations, the distribution function is obtained as

$$\begin{aligned} P\{-2\rho \ln W \leqq z\} = D_f(z) &+ m^{-1}[\beta_0 D_f(z) + \beta_1 D_{f+2}(z)] \\ &+ m^{-2}[\beta_2 D_f(z) + \beta_3 D_{f+2}(z) + \beta_4 D_{f+4}(z)] \\ &+ m^{-3}[\textstyle\sum_{j=0}^{3} \beta_{5+j} D_{f+2j}(z)] + O(m^{-4})\,. \end{aligned} \tag{2.10}$$

The coefficients in (2.10) are given by, ($k = k_1(1 - k_1)$),

$$\beta_0 = -k\sigma_2\,, \qquad \beta_1 = k\sigma_2\,, \qquad \beta_2 = k^2\sigma_2^2/2 - 4k(1 + k_1)\sigma_3/3 - 2\alpha k\sigma_2 - w_2\,,$$
$$\beta_3 = -k^2\sigma_2^2 + 4k\{k_1\sigma_3 + \sigma_1^2/4 + (1 + 4\alpha)\sigma_2/4\}\,,$$
$$\beta_4 = k^2\sigma_2^2/2 - 4k\{(2k_1 - 1)\sigma_3/3 + \sigma_1^2/4 + (1 + 2\alpha)\sigma_2/4\} + w_2\,,$$
$$\begin{aligned}\beta_5 = -k[&2(1 + k_1 + k_1^2)\sigma_4 + 8\alpha\sigma_3/3 - 4k(1 + k_1)\sigma_2\sigma_3/3 \\ &- 2k\alpha\sigma_2^2 + k^2\sigma_3^2/6 - w_2\sigma_2] - w_3\,,\end{aligned}$$
$$\begin{aligned}\beta_6 = 4k[&3k_1^2\sigma_4 + k_1(1 + 4\alpha/3)\sigma_3 + \alpha^2\sigma_2 - (4k_1 + 1)k\sigma_2\sigma_3/3 - k\sigma_1^2\sigma_2/4 \\ &- k(1 + 6\alpha)\sigma_2^2/4 + k^2\sigma_2^3/8 - w_2\sigma_2/4 + k\sigma_1\sigma_2 + (2\alpha/3)\sigma_3]\,,\end{aligned}$$
$$\begin{aligned}\beta_7 = 4k[&(3k_1 - 5k_1^2)\sigma_4 + (1 - 3k_1 + 2\alpha/3 - 8k_1\alpha/3)\sigma_3 - (p - 2 + 2\alpha)\sigma_1^2/4 \\ &- (3 + 2\alpha + 8\alpha^2)\sigma_2/4 - (3k_1 - 1)\sigma_1\sigma_2 \\ &+ k\sigma_2\{(5k_1 - 1)\sigma_3/3 + (3\alpha + 1)\sigma_2/2 + \sigma_1^2/2 - k\sigma_2^2/8\} - w_2\sigma_2/4]\,,\end{aligned}$$
$$\begin{aligned}\beta_8 = 4k[&(5k_1 - 5k_1^2 - 1)\sigma_4/2 - (2k_1 - 1)(1 - 2\alpha/3)\sigma_3 - (p - 2 - 2\alpha)\sigma_1^2/4 \\ &- (3 - 2\alpha - 4\alpha^2)\sigma_2/4 - (2k_1 - 1)\sigma_1\sigma_2 \\ &- k\sigma_2\{(2k_1 - 1)\sigma_3/3 + \sigma_1^2/4 + (1 + 2\alpha)\sigma_2/4 + k\sigma_2^2/24\} + w_2\sigma_2/4] + w_3\,.\end{aligned}$$

The equation (2.10) agrees with equation (5.51) of [7] up to term $m^{-2}$.

**3. Asymptotic distribution of $Y = \prod_{j=1}^{p} \theta_j$.** The ch.f. of $L_1 = n^{\frac{1}{2}} \ln \{Y/k_1^p\}$ has been shown by Pillai and Nagarsenker [7] to be

$$\phi(t) = \phi_0(t)\phi_A(t, \Lambda)\,, \tag{3.1}$$

where

$$\phi_0(t) = k_1^{-it(np)^{\frac{1}{2}}}\{\Gamma_p[n/2]\Gamma_p[n_1/2 + itn^{\frac{1}{2}}]\}/\{\Gamma_p[n_1/2]\Gamma_p[n/2 + itn^{\frac{1}{2}}]\}\,.$$

The asymptotic expansion, to order $O(n^{-2})$, for $\phi_0(t)$ is

$$\begin{aligned}(3.2)\qquad &\exp(-pT_1t^2)[1 - n^{-\frac{1}{2}}\{(it)T_1f + 2pT_2(it)^3/3\} \\ &+ n^{-1}\{(it)^2(T_2f + f^2T_1^2/2) + (it)^4(2p/3)(T_3 + T_1T_2f) + (it)^6(2p^2T_2^2/9)\} \\ &- n^{-\frac{3}{2}}\{(it)(2p^3 - 9p^2 + 11p)T_2/12 + (it)^3f(4T_3/3 + fT_1T_2 + fT_1^2/6) \\ &+ (it)^5p(4T_4/5 + 2fT_2^2/3 + 2fT_1T_3/3 + f^2T_1^2T_2/3) \\ &+ (it)^7(4T_2p^2/9)(T_3 + T_1T_2/2) + (it)^9(4p^3T_2^3/81)\}] + O(n^{-2}),\end{aligned}$$

with $T_i = k_1^{-i} - 1$, $f = p(p+1)/2$. The expansion (3.2) is obtained by using the usual Barnes' approximation to logarithmic gamma function.

The second term of (3.1) is the non-null component given by

$$\phi_A(t, \Lambda) = |\Lambda|^{-n_1/2}{}_2F_1[n/2, n_1/2 + itn^{\frac{1}{2}}; n/2 + itn^{\frac{1}{2}}; M],$$

where $M = I - \Lambda^{-1} = 2P/n$, $P$ being a fixed matrix.

Writing $\phi_A(t, \Lambda) = \exp\{H(T, \Lambda)\}$ and substituting in the equations satisfied by the hypergeometric function of matrix argument, we have the partial differential equations for $H$, in terms of the roots $\rho_1, \cdots, \rho_p$ of $P$. Expanding $H(t, 2P/n)$ in terms of series $\sum_{k=1}^{\infty} Q_k(t, P)/n^{k/2}$, the differential equations for $Q_1, Q_2, Q_3$ are obtained. Following the procedure as before, to $O(n^{-2})$,

$$\begin{aligned}(3.3)\qquad \phi(t) = \exp(-pT_1t^2)[1 &- n^{-\frac{1}{2}}\{itA_1 + (it)^3A_2\} \\ &+ n^{-1}\{(it)^2A_3 + (it)^4A_4 + (it)^6A_5\} - n^{-\frac{3}{2}} \textstyle\sum_{j=1}^{5} (it)^{2j-1}A_{j+5}] \\ &+ O(n^{-2}),\end{aligned}$$

where

$$\begin{aligned}A_1 &= T_1f + 2(k_1 - 1)\sigma_1, \qquad A_2 = 2pT_2/3, \\ A_3 &= fT_2 + f^2T_1^2/2 + 2(k_1 - 1)\sigma_1\{2 + (k_1 - 1)\sigma_1\} + 2f(k_1 - 1)\sigma_1T_1, \\ A_4 &= (2p/3)[T_3 + fT_1T_2 + 2(k_1 - 1)\sigma_1T_2], \\ A_5 &= 2p^2T_2^2/9, \qquad A_6 = p(2p^2 + 3p - 1)T_2/12 - 2(k_1 - 1)^2\sigma_2, \\ A_7 &= f[4T_3/3 + fT_1T_2 + fT_1^3/6] + 8(k_1 - 1)\sigma_1[1 + (k_1 - 1)\sigma_1 + (k_1 - 1)^2\sigma_1^2/6] \\ &\quad + 2f(k_1 - 1)\sigma_1[T_2 + fT_1^2/2 + 2T_1 + (k_1 - 1)\sigma_1T_1], \\ A_8 &= 2p[2T_4/5 + fT_2^2/3 + fT_1T_3/3 + f^2T_1^2T_2/6] \\ &\quad + (4p/3)(k_1 - 1)\sigma_1[T_3 + fT_1T_2 + 2T_2 + (k_1 - 1)\sigma_1T_2], \\ A_9 &= (4p^2/9)[T_2T_3 + fT_1T_2^2/2] + 4p^2(k_1 - 1)\sigma_1T_2^2/9, \qquad A_{10} = 4p^3T_2^3/81.\end{aligned}$$

Writing $x = L_1/(2pT_1)^{\frac{1}{2}}$, its density function is given by:

$$\begin{aligned}(3.4)\qquad f(x) = \psi(x) &+ n^{-\frac{1}{2}}[qA_1\psi'(x) + q^3A_2\psi^{(3)}(x)] \\ &+ n^{-1}[q^2A_3\psi^{(2)}(x) + q^4A_4\psi^{(4)}(x) + q^6A_5\psi^{(6)}(x)] \\ &+ n^{-\frac{3}{2}}[\textstyle\sum_{j=1}^{5} q^{2j-1}A_{j+5}\psi^{(2j-1)}(x)] + O(n^{-2}),\end{aligned}$$

where

$$q = (2pT_1)^{-\frac{1}{2}}, \qquad \psi^{(r)}(x) = (d/dx)^r\psi(x)$$

and $\psi(x)$ is the standard normal pdf. The distribution function of (3.3) agrees with (6.15) of Pillai and Nagarsenker [7] up to the term $n^{-1}$.

**4. Asymptotic distribution of $Y = \prod_{j=1}^{p} (1 - \theta_j)$.** The ch.f. for the statistic $L_2 = n^{\frac{1}{2}} \ln \{Y/k_2^p\}$ has been given by Pillai and Nagarsenker [7] as

$$\phi(t) = \phi_0(t)\phi_A(t, \Lambda) , \tag{4.1}$$

where

$$\phi_0(t) = k_2^{-it(np)^{\frac{1}{2}}}\{\Gamma_p[n/2]\Gamma_p[n_2/2 + itn^{\frac{1}{2}}]\}/\{\Gamma_p[n/2 + itn^{\frac{1}{2}}]\Gamma_p[n_2/2]\} ,$$

and

$$\phi_A(t, \Lambda) = |\Lambda|^{-nk_1/2}\,{}_2F_1[n/2, nk_1/2; n/2 + itn^{\frac{1}{2}}; M] ,$$

with $M = I - \Lambda^{-1} = 2P/n$. As before $\phi_0(t)$ can be expanded to $O(n^{-2})$ as

$$\begin{aligned}\phi_0(t) = \exp(-pR_1t^2)[1 &- n^{-\frac{1}{2}}\{(it)fR_1 + (it)^3 2pR_2/3\} \\ &+ n^{-1}\{(it)^2 f(R_2 + fR_1^2/2) + (it)^4(2p/3)(R_3 + fR_1R_2) + (it)^6(2p^2R_2^2/9)\} \\ &- n^{-\frac{3}{2}}\{(it)p(2p^2 - 9p + 11)R_2/12 + (it)^3 f(4R_3/3 + fR_1R_2 + fR_1^2/6) \\ &+ (it)^5 p(4R_4/5 + 2fR_2^2/3 + 2fR_1R_3/3 + f^2R_1^2R_2/3) \\ &+ (it)^7(2p^2R_2/9)(2R_3 + fR_1R_2) + (it)^9(4p^3R_2^3/81)\}] + O(n^{-2}) ,\end{aligned} \tag{4.2}$$

where $R_i = (k_2^{-i} - 1)$.

Setting up the partial differential equations for the $H$-function and expanding the latter in powers of $n^{-\frac{1}{2}}$ as $\Sigma_k Q_k(t, P)/n^{k/2}$, we have

$$\begin{aligned}\phi(t) = \exp(-pR_1t^2)[1 &- n^{-\frac{1}{2}}\{(it)B_1 + (it)^3B_2\} \\ &+ n^{-1}\{(it)^2B_3 + (it)^4B_4 + (it)^6B_5\} - n^{-\frac{3}{2}}\{\textstyle\sum_{j=1}^{5} (it)^{2j-1}B_{5+j}\}] \\ &+ O(n^{-2}) ,\end{aligned} \tag{4.3}$$

where

$$\begin{aligned} B_1 &= 2k_1\sigma_1 + fR_1 , \qquad B_2 = 2pR_2/3 , \\ B_3 &= 2k_1\sigma_1(2 + k_1\sigma_1) + f[R_2 + fR_1^2/2 + 2k_1\sigma_1R_1] , \\ B_4 &= (2p/3)[R_3 + fR_1R_2 + 2k_1\sigma_1R_2] , \qquad B_5 = 2p^2R_2^2/9 , \\ B_6 &= p[2p^2 + 3p - 1]R_2/12 - 2k_1(k_1 - 2)\sigma_2 , \\ B_7 &= 8k_1\sigma_1(1 + k_1\sigma_1 + k_1^2\sigma_1^2/6) + f[4R_3/3 + fR_1R_2 + f^2R_1^3/6] \\ &\quad + 2fk_1\sigma_1(R_2 + fR_1^2/2) + 4fR_1(k_1\sigma_1/2 + 1)k_1\sigma_1 , \\ B_8 &= p[4R_4/5 + 2fR_2^2/3 + 2fR_1R_3/3 + f^2R_1^2R_2/3] \\ &\quad + 2k_1\sigma_1[2p/3][R_3 + fR_1R_2] + 2k_1\sigma_1(2 + k_1\sigma_1)(2pR_2/3) , \\ B_9 &= (4p^2/9)[R_2R_3 + fR_1R_2^2 + k_1\sigma_1R_2^2] , \qquad B_{10} = 4p^3R_2^3/81 . \end{aligned}$$

The pdf of $x = L_2/(2pR_1)^{\frac{1}{2}}$ is given by

$$\begin{aligned} f(x) = \psi(x) &+ n^{-\frac{1}{2}}[q_1B_1\psi'(x) + q_1^3B_2\psi^{(3)}(x)] \\ &+ n^{-1}[q_1^2B_3\psi^{(2)}(x) + q_1^4B_4\psi^{(4)}(x) + q_1^6B_5\psi^{(6)}(x)] \\ &+ n^{-\frac{3}{2}}[\textstyle\sum_{j=1}^{5} q_1^{(2j-1)}B_{j+5}\psi^{(2j-1)}(x)] + O(n^{-2}) ,\end{aligned} \tag{4.4}$$

where

$$\psi^{(r)} = (d/dx)^r\psi(x) , \qquad q_1 = (2pR_1)^{-\frac{1}{2}} ,$$

and $\psi(x)$ is the standard normal pdf. This pdf agrees with (7.11) of Pillai and Nagarsenker to the term $n^{-1}$.

TABLE 1

*Power of the tests for $p = 2$, $\alpha = .05$*

| $\lambda_1$ | $\lambda_2$ | $n = 24$ | | | | | $n = 48$ | | | | | $n = 60$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $k_1$ | | | | | $k_1$ | | | | | $k_1$ | | | | |
| | | $\frac{3}{4}$ | $\frac{2}{3}$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{4}$ | $\frac{3}{4}$ | $\frac{2}{3}$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{4}$ | $\frac{3}{4}$ | $\frac{2}{3}$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{4}$ |
| | | | | | | | | $-2 \ln W$ | | | | | | | | |
| 1.0 | 1.0 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 |
| 1.0 | 2.0 | .083 | .084 | .087 | .090 | .089 | .143 | .156 | .174 | .178 | .170 | .177 | .198 | .223 | .227 | .214 |
| 1.0 | 4.0 | .153 | .148 | .150 | .177 | .187 | .367 | .402 | .332 | .496 | .486 | .495 | .555 | .638 | .680 | .653 |
| 1.5 | 1.5 | .078 | .081 | .081 | .076 | .072 | .123 | .135 | .146 | .139 | .128 | .148 | .166 | .183 | .174 | .158 |
| 1.5 | 3.0 | .148 | .147 | .143 | .147 | .146 | .332 | .368 | .410 | .417 | .392 | .441 | .500 | .567 | .571 | .530 |
| | | | | | | | | $-2\rho \ln W$ | | | | | | | | |
| 1.0 | 1.0 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 |
| 1.0 | 2.0 | .097 | .105 | .129 | .138 | .132 | .123 | .169 | .266 | .293 | .262 | .149 | .223 | .375 | .408 | .356 |
| 1.0 | 4.0 | .118 | .118 | .350 | .435 | .409 | .254 | .590 | .906[1] | .867[2] | .874[1] | .447 | .601[2] | .999 | .839[5] | .909[3] |
| 1.5 | 1.5 | .088 | .090 | .099 | .102 | .098 | .110 | .137 | .190 | .197 | .176 | .132 | .178 | .264 | .270 | .234 |
| 1.5 | 3.0 | .107 | .148 | .270 | .330 | .309 | .257 | .545 | .941[3] | .815[4] | .841[3] | .397 | .818 | .961[3] | .988[3] | .803[5] |
| | | | | | | | | $L_1$ | | | | | | | | |
| 1.0 | 1.0 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 |
| 1.0 | 2.0 | .131 | .146 | .164 | .166 | .159 | .188 | .213 | .240 | .237 | .223 | .214 | .244 | .276 | .270 | .252 |
| 1.0 | 4.0 | .204 | .234 | .271 | .273 | .260 | .322 | .375 | .432 | .422 | .389 | .379 | .444 | .512 | .494 | .451 |
| 1.5 | 1.5 | .172 | .194 | .219 | .216 | .203 | .265 | .305 | .345 | .332 | .304 | .310 | .358 | .406 | .387 | .352 |
| 1.5 | 3.0 | .292 | .339 | .390 | .382 | .354 | .496 | .585 | .673 | .636 | .570 | .600 | .710 | .818 | .764 | .677 |
| | | | | | | | | $L_2$ | | | | | | | | |
| 1.0 | 1.0 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 |
| 1.0 | 2.0 | .039 | .038 | .037 | .035 | .034 | .042 | .044 | .047 | .043 | .039 | .046 | .051 | .054 | .049 | .044 |
| 1.0 | 4.0 | .058 | .060 | .062 | .055 | .049 | .085 | .097 | .105 | .092 | .078 | .103 | .119 | .130 | .113 | .095 |
| 1.5 | 1.5 | .049 | .051 | .051 | .046 | .042 | .067 | .075 | .081 | .072 | .062 | .079 | .091 | .099 | .087 | .074 |
| 1.5 | 3.0 | .093 | .102 | .108 | .093 | .079 | .160 | .186 | .203 | .176 | .146 | .198 | .232 | .256 | .221 | .183 |

[1] $\lambda_2 = 3.25$, [2] $\lambda_2 = 3.00$, [3] $\lambda_2 = 2.75$, [4] $\lambda_2 = 2.50$, [5] $\lambda_2 = 2.25$

TABLE 2

*Power of the tests for $p = 3$, $\alpha = .05$*

| $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $n = 24$ | | | | | $n = 48$ | | | | | $n = 60$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | $k_1$ | | | | | $k_1$ | | | | | $k_1$ | | | | |
| | | | $\frac{3}{4}$ | $\frac{2}{3}$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{4}$ | $\frac{3}{4}$ | $\frac{2}{3}$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{4}$ | $\frac{3}{4}$ | $\frac{2}{3}$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{4}$ |
| | | | | | | | | $-2 \ln W$ | | | | | | | | | |
| 1.0 | 1.0 | 1.0 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 |
| 1.0 | 1.0 | 2.0 | .073 | .067 | .058 | .061 | .065 | .111 | .108 | .105 | .115 | .119 | .134 | .133 | .135 | .146 | .149 |
| 1.0 | 1.0 | 4.0 | .123 | .085 | .042 | .073 | .105 | .262 | .230 | .207 | .273 | .311 | .349 | .326 | .318 | .397 | .432 |
| 1.0 | 1.5 | 1.5 | .073 | .071 | .064 | .058 | .057 | .100 | .102 | .102 | .097 | .093 | .116 | .122 | .124 | .120 | .113 |
| 1.0 | 1.5 | 3.0 | .128 | .101 | .061 | .065 | .079 | .245 | .231 | .213 | .236 | .247 | .318 | .316 | .314 | .342 | .347 |
| | | | | | | | | $-2\rho \ln W$ | | | | | | | | | |
| 1.0 | 1.0 | 1.0 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 |
| 1.0 | 1.0 | 2.0 | .091 | .082 | .097 | .114 | .119 | .058 | .096 | .024 | .249 | .226 | .063 | .135 | .313 | .369 | .321 |
| 1.0 | 1.0 | 2.25 | .087 | .079 | .111 | .141 | .146 | .044 | .113 | .300 | .374 | .332 | .058 | .187 | .495 | .583 | .497 |
| 1.0 | 1.0 | 2.5 | .079 | .074 | .126 | .171 | .176 | .028 | .136 | .417 | .521 | .456 | .057 | .255 | .718 | .837 | .703 |
| 1.0 | 1.5 | 1.5 | .089 | .076 | .077 | .086 | .092 | .064 | .084 | .140 | .159 | .144 | .072 | .114 | .211 | .232 | .200 |
| | | | | | | | | $L_1$ | | | | | | | | | |
| 1.0 | 1.0 | 1.0 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 |
| 1.0 | 1.0 | 2.0 | .109 | .121 | .135 | .135 | .125 | .151 | .169 | .190 | .190 | .179 | .170 | .191 | .215 | .213 | .200 |
| 1.0 | 1.0 | 4.0 | .161 | .183 | .213 | .217 | .203 | .243 | .280 | .323 | .322 | .302 | .282 | .326 | .375 | .369 | .344 |
| 1.0 | 1.5 | 1.5 | .138 | .154 | .173 | .169 | .153 | .203 | .231 | .259 | .252 | .232 | .234 | .267 | .300 | .289 | .265 |
| 1.0 | 1.5 | 3.0 | .219 | .253 | .292 | .289 | .264 | .356 | .415 | .475 | .456 | .416 | .424 | .496 | .566 | .536 | .484 |
| | | | | | | | | $L_2$ | | | | | | | | | |
| 1.0 | 1.0 | 1.0 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 | .050 |
| 1.0 | 1.0 | 2.0 | .063 | .052 | .045 | .039 | .037 | .051 | .049 | .046 | .034 | .036 | .052 | .052 | .049 | .042 | .038 |
| 1.0 | 1.0 | 4.0 | .085 | .074 | .064 | .053 | .046 | .085 | .088 | .085 | .071 | .059 | .094 | .100 | .100 | .082 | .068 |
| 1.0 | 1.5 | 1.5 | .076 | .065 | .056 | .047 | .042 | .071 | .072 | .069 | .058 | .050 | .077 | .080 | .079 | .066 | .055 |
| 1.0 | 1.5 | 3.0 | .120 | .108 | .098 | .079 | .066 | .141 | .152 | .153 | .125 | .103 | .164 | .180 | .185 | .152 | .124 |

**5. Numerical comparisons.** In this section some comparisons of the power of the tests based on the four statistics are presented. As the pattern of the performance would not be affected by the level of significance $\alpha$, it was decided to confine the numerical evaluation to $\alpha = .05$. Various combinations of the noncentrality parameters $\lambda_i$, sample size ratios $k_i$ and sample sizes were examined for $p = 2, 3$. Computationally the problem gets complicated for larger values of $p$, without really exhibiting any new trends. The results are obtained for the percentage points evaluated from the asymptotic central distributions and hence, to this extent, are approximate. Also, the values of $\lambda_2$ in some instances are smaller due to the fact that in these cases, the power tended to become greater than one. The pattern of the results is not violated due to this selection of the values of $\lambda_2$. The results of our evaluation are summarized in Tables 1 and 2 for $p = 2, 3$ respectively.

An examination of the tables shows that while all the four tests are unbiased, their relative performance is markedly different for various regions. For small values of $n$ (the total sample size), $L_1$ seems to perform much better than any of the other tests. But as $n$ increases, the other tests, with the exception of $L_2$, seem to improve considerably. Of all the four test procedures examined here, $L_2$ is by far the weakest. For increase in $p$, as one would expect, the power diminishes considerably for all the four procedures. For brevity, only small departures from the hypothesis of $\Lambda = I$ are tabled; the performance of all the tests improves for larger departures.

**Acknowledgments.** The author thanks the referee for making corrections on the original draft of the paper.

## REFERENCES

[1] ANDERSON, T. W. (1958). *An Introduction to Multivariate Statistical Analysis.* Wiley, New York.

[2] ERDÉLYI, A. (1956). *Asymptotic Expansions.* Dover, New York.

[3] MUIRHEAD, R. J. (1970). Systems of partial differential equations for hypergeometric functions of matrix argument. *Ann. Math. Statist.* **41** 991–1001.

[4] MUIRHEAD, R. J. (1970). Asymptotic distributions of some multivariate tests. *Ann. aMth. Statist.* **41** 1002–1010.

[5] MUIRHEAD, R. J. (1972). On the tests of independence between two sets of variates. *Ann. Math. Statist.* **43** 1491–1497.

[6] MUIRHEAD, R. J. (1972). The asymptotic noncentral distribution of Hotelling's generalized $T_0^2$. *Ann. Math. Statist.* **43** 1671–1677.

[7] PILLAI, K. C. S. and NAGARSENKER, B. N. (1972). On the distributions of a class of statistics in multivariate analysis. *J. Multivariate Anal.* **2** 96–114.

DEPARTMENT OF STATISTICS
THE UNIVERSITY OF MANITOBA
WINNIPEG, MANITOBA
CANADA R3T 2N2