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ON FUNCTIONS OF ORDER STATISTICS FOR MIXING PROCESSES[1]

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Recent results of Shorack (*Ann. Math. Statist.* (1972) 412–427) on the asymptotic normality of functions of order statistics are extended to stationary ϕ-mixing processes and to a class of strong mixing processes. The results of this paper are based on the weak convergence of empirical processes relative to the metric d_q as developed in Fears and Mehra (*Ann. Statist.* (1974) 586–596). Some remarks on trimmed and Winsorized means in the strong mixing case are also included.

1. Introduction and notation. The asymptotic normality of functions of order statistics for sequences of i.i.d. rv's has been studied by a number of authors, e.g., Bickel (1967), Chernoff, Gastwirth and Johns (1967), Stigler (1969) and Shorack (1972). In the last of these papers, Shorack uses very effectively the weak convergence properties of empirical processes developed in [9] to prove two quite general—and so far the best—theorems of this kind for the case of independent rv's. The object of the present paper is to prove similar results for stationary sequences $\{X_n : n = 1, 2, \cdots\}$ of rv's under ϕ-mixing and strong mixing types of dependence. In this connection, we refer the reader also to a paper by Gastwirth and Rubin (1969) where the authors have studied, using different methods, certain types of functions of order statistics from stationary Gaussian processes (see Section 4 of [6]). No attempt will be made to discuss their results here.

Let M_1^k and M_{k+n}^∞ be σ-algebras generated, respectively, by $\{X_i : 1 \leqq i \leqq k\}$ and $\{X_i : i \geqq k + n\}$. Then $\{X_n\}$ is ϕ-mixing if

$$\sup\{|P(B \mid A) - P(B)| : A \in M_1^k, B \in M_{k+n}^\infty\} \leqq \phi_n , \tag{1.1}$$

and strong-mixing if

$$\sup\{|P(B \cap A) - P(B)P(A)| : A \in M_1^k, B \in M_{k+n}^\infty\} \leqq \alpha_n , \tag{1.2}$$

for all positive integers k and n, where ϕ_n and α_n are non-increasing functions of positive integers with $0 \leqq \phi_n, \alpha_n \leqq 1$ and $\lim_{n\to\infty} \phi_n = \lim_{n\to\infty} \alpha_n = 0$. (In (1.1) $|P(B \mid A) - P(B)|$ is defined to be zero for $P(A) = 0$.) For a discussion of mixing conditions see Ibragimov [7] and Rosenblatt [11].

The function spaces $C = C[0, 1]$ with supremum metric ρ and $D = D[0, 1]$

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with Skorohod metric d are as given in [2]. If Y_n and Y are random functions, we write $Y_n \Rightarrow_\delta Y$ to denote the weak convergence of Y_n to Y relative to the metric δ, as $n \to \infty$. Where no confusion is possible K, K_ϕ, K_α, etc., denote generic constants.

Throughout below $\xi_1, \xi_2, \cdots$ will denote uniform [0, 1] rv's. Let $\{\xi_n : n \geqq 1\}$ be a stationary ϕ-mixing (strong mixing) sequence of such variables and define $\Gamma_n(t) = n^{-1} \sum_{i=1}^n I_{[\xi_i \leqq t]}$, $0 \leqq t \leqq 1$. Let $U_n(t) = n^{\frac{1}{2}}[\Gamma_n(t) - t]$ and $V_n(t) = n^{\frac{1}{2}}[\Gamma_n^{-1}(t) - t]$, $0 \leqq t \leqq 1$, be the corresponding empirical and quantile processes. Further set for $0 \leqq s, t \leqq 1$

$$\sigma(s, t) = [(s \wedge t) - st] + \sum_{j=2}^\infty [F_{1j}(s, t) - st] + \sum_{j=2}^\infty [F_{1j}(t, s) - st] , \tag{1.3}$$

where $F_{1j}(s, t) = P[\xi_1 \leqq s, \xi_j \leqq t]$, and whenever $|\sigma(s, t)| < \infty$, define a tied down Gaussian random function $\{U_0(t) : 0 \leqq t \leqq 1\}$ in D by

$$E[U_0(t)] = 0 , \qquad E[U_0(s)U_0(t)] = \sigma(s, t) . \tag{1.4}$$

For a fixed θ ($0 < \theta \leqq \frac{1}{2}$), we also define

$$q(t, \theta) = K[t(1 - t)]^{\frac{1}{2}-\theta} , \qquad 0 \leqq t \leqq 1 , \tag{1.5}$$

where K is a constant.

2. Functions of OS under ϕ-mixing. In this section we shall establish that the two main theorems of Shorack [13], concerning the asymptotic normality of functions of order statistics, remain valid under ϕ-mixing with only slight variations in conditions. For the sake of brevity we shall explicitly prove only two simple versions of these theorems for the case $g_n = g$. (See Remark 2.1.) Let $\xi_{n1}, \xi_{n2}, \cdots, \xi_{nn}$ denote the OS based on the first $n\xi_i$'s of a uniform stationary ϕ-mixing $\{\xi_i\}$ and $0 < p_1 < p_2 < \cdots < p_\kappa$ be κ teal numbers, where κ ($\geqq 0$) remains fixed. Consider the statistic

$$T_n = n^{-1} \sum_{i=1}^n c_{ni}\, g(\xi_{ni}) + \sum_{k=1}^\kappa d_{nk}\, g(\xi_{n,[np_k]+1}) , \tag{2.1}$$

where $g \in \mathscr{G} =$ the class of functions on (0, 1) which are of bounded variation on $(\eta, 1 - \eta)$ for every $\eta > 0$, and c_{ni}'s and d_{nk}'s are suitable constants. If $g = F^{-1}$, with F a continuous distribution function (df), T_n is a linear function of OS from some ϕ-mixing stationary process $\{X_n\}$, with $X_n = F^{-1}(\xi_n)$. It is also important to note that given a random process $\{X_n\}$, with each marginal df $F_n = \mathscr{L}(X_n)$ continuous, there exists a uniform process $\{\xi_n\}$ such that $\mathscr{L}(\{X_n)\} = \mathscr{L}\{F_n^{-1}(\xi_n)\})$; if $\{X_n\}$ is stationary ϕ-mixing, so is $\{\xi_n\}$. Thus the present approach based on the transformation $X_n = F_n^{-1}(\xi_n)$ does not restrict any further the generality of these theorems in the ϕ-mixing case. (The preceding notation and remarks apply also to the strong mixing case in Section 3.)

Throughout we shall keep the notation of [13], unless it is necessary to do otherwise. We also use the following condition on the mixing coefficient ϕ.

$$\sum_{n=1}^\infty n^2 \phi^{\frac{1}{2}}(n) < \infty . \tag{2.2}$$

We refer to [5] for the proof of the following proposition.

PROPOSITION 2.1. *Let* $q = q(t, \theta)$ *for a fixed* $0 < \theta \leqq \frac{1}{2}$ *and let a stationary* ϕ-*mixing sequence* $\{\xi_n\}$ *satisfy the condition* (2.2). *Then* $(U_n/q) \Rightarrow_d (U_0/q)$ *where* U_0 *is a Gaussian random function given by* (1.4). *Further* $P[(U_0/q) \in C] = P[U_0 \in C] = 1$.

COROLLARY 2.1. *Under the conditions of Proposition* 2.1,

$$V_n \Rightarrow_d V_0 \qquad \text{where} \quad V_0 = -U_0 .$$

PROOF. Since $U_n \Rightarrow_d U_0$ under the conditions of the proposition, the proof follows from Theorem 1 of Ver Vaat [17]. □

Let $\rho_q(f, g) = \rho(f/q, g/q)$ and similarly for d_q. Since (D, d) is a complete separable metric space, the conclusion of Proposition 2.1 can be strengthened as in [5] (Corollary 2.1) to

$$d_q(\tilde{U}_n, \tilde{U}_0) \rightarrow_{\text{a.s.}} 0 \qquad \text{and} \qquad \rho_q(\tilde{U}_n, \tilde{U}_0) \rightarrow_{\text{a.s.}} 0 , \tag{2.3}$$

as $n \rightarrow \infty$, where $\tilde{U}_n$ and $\tilde{U}_0$ are now processes equivalent in the sense of Skorohod (see [5]) to U_n and U_0 respectively. We shall, in the sequel, deal only with these special processes with $\sim$ suppressed for convenience.

We now turn to the convergence of the quantile process $\{V_n(t) : 0 \leqq t \leqq 1\}$ relative to the general metrics ρ_q and d_q. Let f^* denote the restriction of any function f on $[0, 1]$ to $[1/n, 1 - (1/n)]$ (and 0 elsewhere).

LEMMA 2.1. *Under the hypotheses of Proposition* 2.1

$$\rho_q(V_n^*, V_0) \rightarrow_p 0 , \tag{2.4}$$

as $n \rightarrow \infty$, *where* V_0 *is as defined in Corollary* 2.1 *and* q *by* (1.5).

PROOF. First note that (2.4) holds if

$$\rho_q(U_n^*(\Gamma_n^{-1}), U_0) \rightarrow_p 0 , \qquad \text{as} \quad n \rightarrow \infty , \tag{2.5}$$

where we have set $U_n^*(\Gamma_n^{-1}) = (U_n(\Gamma_n^{-1}))^*$. For, we have $V_0 = -U_0$ and

$$V_n(t) = -U_n(\Gamma_n^{-1}(t)) + n^{\frac{1}{2}}(\Gamma_n \circ \Gamma_n^{-1}(t) - t) ,$$

so that

$$|V_n^*(t) - V_0(t)| \leqq |U_n^*(\Gamma_n^{-1}) - U_0(t)| + n^{\frac{1}{2}}|(\Gamma_n \circ \Gamma_n^{-1}(t) - t)^*| .$$

We thus have

$$\rho_q(V_n^*, V_0) \leqq \rho_q(U_n^*(\Gamma_n^{-1}), U_0) + n^{\frac{1}{2}}\left[q\left(\frac{1}{n}\right)\right]^{-1} \rightarrow 0 ,$$

as $n \rightarrow \infty$, in view of (2.5) and the fact that $nq^{-2}(1/n) \rightarrow 0$, as $n \rightarrow \infty$. Thus (2.4) holds if (2.5) is satisfied. The proof of (2.5) is similar to that of Theorem 2.2 of [9] since for $\varepsilon > 0$, $0 < \theta < \frac{1}{2}$, there exists a $\beta > 0$ such that for sufficiently large n

$$P\left[\Gamma_n^{-1}(t) \leqq \beta t^{1-\theta} \ \text{ for } \ t \geqq \frac{1}{n}\right] \geqq 1 - \varepsilon \tag{2.6}$$

holds (see (3.4) of [5]), as noted in the proof of Theorem 3.1 of [5]. □

The following two lemmas correspond to Lemma A.3 of [13].

LEMMA 2.2. *Let $\{\xi_n\}$ be stationary ϕ-mixing satisfying* (2.2). *Then, for given* $\varepsilon > 0$, *there exist a* $\lambda = \lambda_\varepsilon$, $0 < \lambda < 1$, *and a set* $S_{n,\varepsilon}$ *with* $P[S_{n,\varepsilon}] > 1 - \varepsilon$ *such that on* $S_{n,\varepsilon}$

$$\lambda t \leqq \Gamma_n^{-1}(t) \leqq 1 - \lambda(1-t) \qquad \textit{for all} \quad 0 \leqq t \leqq 1 . \tag{2.7}$$

PROOF. By symmetry of the graphs of Γ_n and Γ_n^{-1}, it is easy to see that (2.7) is equivalent to

$$-\lambda'(1-t) \leqq \Gamma_n(t) - t \leqq \lambda' t \qquad \text{for all} \quad 0 \leqq t \leqq 1 , \tag{2.8}$$

where $\lambda' = (1/\lambda) - 1 > 0$. For $0 < t \leqq \frac{1}{2}$ ($\frac{1}{2} \leqq t < 1$) (2.8) is implied by

$$|\Gamma_n(t) - t| \leqq t\lambda' \qquad (|\Gamma_n(t) - t| \leqq (1-t)\lambda') . \tag{2.9}$$

We, therefore, need to prove only that for some $\lambda' > 0$ and n sufficiently large

$$P[|\Gamma_n(t) - t| \leqq t\lambda' \text{ for } 0 < t \leqq \tfrac{1}{2}] \geqq 1 - (\varepsilon/2) ; \tag{2.10}$$

that the inequality in parenthesis in (2.9) holds for $\frac{1}{2} \leqq t < 1$ with probability $\geqq 1 - (\varepsilon/2)$ follows by symmetry. Consequently, (2.8) will hold with probability $\geqq 1 - \varepsilon$ and the proof would be complete.

Let $W_n(t) = \Gamma_n(t) - t$ and $\eta_i = s[(I_{[\xi_i \leqq t]}/t) - (I_{[\xi_i \leqq s]}/s)]$ for $0 < s < t \leqq \frac{1}{2}$, $i = 1, 2, \cdots, n$. Since $|\eta_i| \leqq 1$ and $E(\eta_1^2) = \frac{1}{2} \cdot E|\eta_1| = s^2(s^{-1} - t^{-1})$, so that proceeding as for (2.11) of [5], we obtain

$$E|\textstyle\sum_{i=1}^n \eta_i|^4 \leqq K_\phi[n^2E^2(\eta_1^2) + nE(\eta_1^2)] ,$$

which implies (since $\sum_{i=1}^n \eta_i = ns[(W_n(t)/t) - (W_n(s)/s)]$) that

$$E\left|\frac{W_n(t)}{t} - \frac{W_n(s)}{s}\right|^4 \leqq K_\phi\left[\frac{1}{n^2}\left(\frac{1}{s} - \frac{1}{t}\right)^2 + \frac{1}{n^3s^2}\left(\frac{1}{s} - \frac{1}{t}\right)\right] ; \tag{2.11}$$

similar arguments yield

$$E\left|\frac{W_n(t)}{t}\right|^4 \leqq K_\phi\left[\frac{1}{n^2t^2} + \frac{1}{n^3t^3}\right] . \tag{2.12}$$

Further, for integers $1 \leqq j < k \leqq M/2$, where M is assumed to satisfy

$$0 < a_0 \leqq n/M \leqq b_0 < \infty \qquad (a_0, b_0 \text{ to be chosen later}) , \tag{2.13}$$

set $s = j/M$ and $t = k/M$. Since $(k/j(k-j)) \leqq (k/(k-1)) \leqq 2$, we have from (2.11)

$$E\left|\frac{W_n(k/M)}{(k/M)} - \frac{W_n(j/M)}{(j/M)}\right|^4 \leqq K_\phi\left(\frac{1}{j} - \frac{1}{k}\right)^2 \tag{2.14}$$

and, for $1 \leqq k \leqq M/2$, from (2.12)

$$E\left|\frac{W_n(k/M)}{(k/M)}\right|^4 \leqq K_\phi\left(\frac{1}{k}\right)^2 . \tag{2.15}$$

Now considering the variables

$$\frac{W_n(1/M)}{(1/M)}, \qquad \frac{W_n(i/M)}{(i/M)} - \frac{W_n((i-1)/M)}{((i-1)/M)}, \qquad 2 \leqq i \leqq M/2,$$

and applying Theorem 12.2 of Billingsley [2] to (2.14) and (2.15) (with $u_1 = 1$, $u_l = (1/(l-1) - (1/l))$ for $l \geqq 2$), we obtain for $c > 1$

$$P\left[\max_{1 \leqq i \leqq M/2} \frac{|W_n(i/M)|}{(i/M)} \geqq c\right] \leqq \frac{K_\phi}{c^4}\left[1 + \left(1 - \frac{2}{M}\right)\right]^2. \tag{2.16}$$

Now proceeding as for (2.15) of [5] we get

$$\sup_{(1/M) \leqq t \leqq \frac{1}{2}} \left|\frac{W_n(t)}{t}\right| \leqq 3 \max_{1 \leqq i \leqq (M/2)-1} \left|\frac{W_n(i/M)}{(i/M)}\right| + 1. \tag{2.17}$$

Also, proceeding as for (2.19) of [5] and using (2.13) we get

$$P\left[\sup_{0 < t < (1/M)} \left|\frac{W_n(t)}{t}\right| \geqq c\right] \leqq \left(\frac{n}{M}\right) \leqq b_0. \tag{2.18}$$

Now let $b_0 = \varepsilon/4$, $a_0 = \varepsilon/8$ and choose $c = c(\varepsilon, \phi)$ so large that $K_\phi/(c-1)^4 \leqq \varepsilon/4$. Since for suitably large n and M (2.13) would hold for above choices of a_0 and b_0, (2.10) follows from (2.16)—(2.18) with $\lambda' = 2c$. The proof is complete. □

LEMMA 2.3. *Let $\{\xi_n\}$ be as in Lemma 2.2. Then, for given $\varepsilon > 0$ and τ_1, τ_2 with $0 < \tau_1, \tau_2 < \frac{1}{2}$, there exist a $\beta = \beta(\tau_1, \tau_2, \varepsilon)$ $(0 < \beta < \frac{1}{2})$ and, for sufficiently large n, a set $S_{n,\varepsilon}$ such that on $S_{n,\varepsilon}$*

$$\beta t^{1+\tau_1} \leqq \Gamma_n(t) \leqq 1 - \beta(1-t)^{1+\tau_2} \qquad \textit{for} \quad 0 < \Gamma_n(t) < 1. \tag{2.19}$$

PROOF. Clearly it is sufficient to prove, for given $\varepsilon > 0$ and τ, $0 < \tau < \frac{1}{2}$, the existence of a β such that for sufficiently large n

$$P[\Gamma_n(t) \geqq \beta t^{1+\tau} \text{ for } \Gamma_n(t) > 0] \geqq 1 - \varepsilon. \tag{2.20}$$

Now by the "symmetry" of the graphs of Γ_n and Γ_n^{-1} again, for any strictly increasing continuous function p on [0, 1] we have

$$\begin{aligned} \left\{\Gamma_n^{-1}(t) \leqq p(t) \text{ for } t \geqq \frac{1}{n}\right\} &= \{t \leqq \Gamma_n(p(t)) \text{ for } p(t) \geqq \xi_{n1}\} \\ &= \{p^{-1}(t) \leqq \Gamma_n(t) \text{ for } t \geqq \xi_{n1}\}. \end{aligned} \tag{2.21}$$

In view of (2.21), with $p(t) = t^{1-\tau'}/c$, $\tau' = (\tau/(1+\tau))$ and $c = \beta^{1-\tau'}$, (2.20) is equivalent to

$$P\left[\Gamma_n^{-1}(t) \leqq t^{1-\tau'}/c \text{ for } t \geqq \frac{1}{n}\right] \geqq 1 - \varepsilon, \tag{2.22}$$

which is the same as (2.6) with $\beta = 1/c$, $\theta = \tau'$. □

The first theorem. Let J_n, J and μ_n be defined as in [13] page 413, and set for

$0 < t < 1$

(2.23) $$B(t) = Kt^{-b_1}(1-t)^{-b_2}, \quad \text{where} \quad b_1, b_2 > 0, \quad \text{and}$$
$$L(t) = Kt^{-\frac{1}{2}+r_1}(1-t)^{-\frac{1}{2}+r_2}, \quad \text{where} \quad r_1 > b_1 \quad \text{and} \quad r_2 > b_2.$$

Further let

(2.24) $$\sigma^2 = \int_0^1 \int_0^1 \sigma(t, s)J(s)J(t)\,dg(s)\,dg(t) + 2\sum_{k=1}^m d_k g'(p_k) \int_0^1 \sigma(t, p_k)J(t)\,dg(t) + \sum_{k=1}^m \sum_{j=1}^m d_k d_j g'(p_j)g'(p_k)\sigma(p_j\, p_k),$$

where $\sigma(s, t)$ is defined by (1.3).

THEOREM 2.1. *Let $\{\xi_n\}$ be a stationary ϕ-mixing uniform process satisfying* (2.2). *If assumptions* 1, 2 *and* 4 *of* [13] (page 413) *hold, with $g_n = g$ and $D(t)$ replaced by $L(t)$ of* (2.23), *then*

$$n^{\frac{1}{2}}(T_n - \mu_n) \Rightarrow N(0, \sigma^2),$$

where T_n is defined by (2.1), *and σ^2 given by* (2.24) *is finite.*

PROOF. Since the proof is analogous to that of Theorem 1 of [13], we shall touch only those points where the present proof departs from that of [13]: Considering the decomposition of $n^{\frac{1}{2}}[T_n - \mu_n]$ on page 414 of [13], note that, in view of Lemma 2.3, we have in the present case

(2.25) $$\chi_{n\varepsilon}|A_n^*| \leqq Kt^{-b_1(1+\tau_1)}(1-t)^{-b_2(1+\tau_2)} = KB_{\tau_1,\tau_2},$$

where $\tau_1 = r_1 - b_1$, $\tau_2 = r_2 - b_2$ and $\chi_{n\varepsilon}$ is the indicator function of the set $S_{n,\varepsilon}$ of Lemma 2.3. In view of (2.3) and $P[(U_0/q) \in C] = 1$, (2.25) permits the application of the dominated convergence theorem, as in [13], to conclude that $\chi_{n\varepsilon} S_n \to S$ a.s. and therefore $S_n \to_p S$. That $\gamma_{n1}, \gamma_{n2} \to_p 0$ is also similar: From (2.22) we obtain, for an arbitrary $\delta' > 0$ and $0 < \tau' < 2\delta'/(1 + 2\delta')$, $P[n\xi_{n1}^{1+2\delta'} > \eta] \to 0$, as $n \to 0$, for every $\eta > 0$. Consequently, since on the event $A_n = \{\xi_{n1} < (\frac{1}{2} \wedge p_1)\}$ with $P(A_n) \to 1$ as $n \to \infty$ we have

$$\gamma_{n1} \leqq K[n\xi_{n1}^{1+(\tau_1/(b-\frac{1}{2}))}]^{b_1-(\frac{1}{2})} + (n\xi_{n1}^{1+2\tau_1})^{\frac{1}{2}} \quad \text{for} \quad b_1 \geqq 1, \quad \text{and}$$
$$\gamma_{n1} \leqq n(n\xi_{n1}^{1+2\tau_1})^{\frac{1}{2}} \quad \text{for} \quad b_1 < 1,$$

it follows that, as $n \to \infty$, γ_{n1} and analogously $\gamma_{n2} \to_p 0$. The rest of the proof does not require any change. □

The second theorem. We now consider the statistic (2.1) with $\kappa = 0$ and g a fixed left continuous function on (0, 1) viz.,

(2.26) $$T_n = n^{-1}\sum_{k=1}^n c_{ni}\, g(\xi_{ni}),$$

where

$$c_{ni} = n\int_{(i-1)/n}^{(i/n)} C_n\,d\nu \quad (\text{with } \int_b^a \cdot\, d\nu = \int_{(a,b]} \cdot\, d\nu) \qquad 1 \leqq i \leqq n,$$

for some sequence $\{C_n\}$ of measurable functions on (0, 1) and a signed measure ν defined on (0, 1). Let C be a fixed measurable function on (0, 1) and set

(2.27) $$\mu = \int_0^1 gC\,d\nu \quad \text{and}$$
$$\sigma^2 = \int_0^1 \int_0^1 \sigma(t, s)g'(t)g'(s)C(t)C(s)\,d\nu(t)\,d\nu(s),$$

where $\sigma(t, s)$ is defined by (1.3). Let μ_n and σ_n denote the corresponding quantities in which C is replaced by C_n. Assume that

(2.28)
(i) g is absolutely continuous on $(\varepsilon, 1-\varepsilon)$ for all $\varepsilon > 0$, g' exists a.e. $|\nu|$ and $|g'| \leqq R$ a.e. w.r.t. ($|\nu|$ + Lebesgue measure), where R is a reproducing U-shaped function.
(ii) For large n, $|C_n| \leqq \psi$ a.e. $|\nu|$ with $\int_0^1 qR\psi\, d|\nu| < \infty$, where q is defined by (1.5).

THEOREM 2.2. *Let* $\{\xi_n\}$ *be as in Theorem* 2.1 *satisfying* (2.2). *If the assumption* (2.28) *above and assumptions* (E2a), (E3) *and* (E4) *of* [13] pages 420–421 *hold, then* $\sigma^2 < \infty$ *and*

$$n^{\frac{1}{2}}(T_n - \mu) \Rightarrow N(0, \sigma^2)\,, \qquad as\ n \to \infty\,, \tag{2.29}$$

where μ *and* σ^2 *are defined by* (2.27). *If only assumptions* (2.28) *above and* (E2a) *of* [13] *hold, then* $[n^{\frac{1}{2}}(T_n - \mu_n)/\sigma_n] \Rightarrow N(0, 1)$ *provided* $\liminf_{n\to\infty} \sigma_n^2 > 0$.

PROOF. Keeping the notation of [13], consider the decomposition of $n^{\frac{1}{2}}(T_n - \mu)$ given by (13) of [13]. Now from Lemma 2.1, we have $\rho_q(V_n^*, V_0) = o_p(1)$ $(V_0 = -U_0)$ and $\rho_q(V_0, 0) = O_p(1)$. Therefore, under the assumptions of the theorem $T_n^* \to_p T$, where $T = \int_0^1 V_0 g' C\, d\nu$, by the same arguments as in [13] provided we show that

$$\int_0^1 g|A_n^* - g'|\psi\, d|\nu| \to_p 0\,, \tag{2.30}$$

as $n \to \infty$. For (2.30) note that $|A_n| \leqq R \vee R(\Gamma_n^{-1})$, so that by Lemma 2.2

$$\chi_{S_{n,\varepsilon}}|A_n^*| \leqq K_\varepsilon R\,,$$

where $S_{n,\varepsilon}$ is the set of Lemma 2.2. Since $A_n \to g'$ a.e. $|\nu|$ for each $\omega \in \Omega$ and $P[S_{n\varepsilon}] \to 1$, (2.30) follows from the preceding inequality, the dominated convergence theorem and assumption (2.28). Since $\theta_n \to 0$ by assumption (E4), the proof of (2.29) would be complete if we show that $\gamma_{n1}, \gamma_{n2} \to_p 0$, as $n \to \infty$. To see this note that by assumption (E2a), $\gamma_{n1}^2 = g^2(\xi_{n1})c_{n1}^2/n$ does not exceed

$$\frac{K^2}{(n\xi_{n1})^2}\frac{c_{n1}^2}{n^{1-2\alpha}}\frac{1}{(1-\xi_{n1})^{2\alpha}}\,,$$

which converges to zero in probability as $n \to \infty$ by assumption (E2a) and the fact that $(n\xi_1)^{-1} = O_p(1)$. This last fact follows from $P[\xi_{n1} \leqq (\varepsilon/n)] \leqq P[\bigcup_{i=1}^n \{\xi_i \leqq (\varepsilon/n)\}] = \varepsilon$ for every $\varepsilon > 0$. Thus γ_{n1}, and analogously γ_{n2}, $\to_p 0$. This proves the first part; the proof of the second assertion is similar. □

REMARK 2.1. All variations, corollaries and applications of the main theorems of Shorack [13] remain valid in the present context with minor modifications. A version of Theorem 2.1, as in [13], in which g depends on n also holds under additional Assumptions 3 and 4 of [13] page 413.

3. Functions of OS under strong mixing. We shall show in this section that the results of Section 2 continue to hold for stationary strong mixing processes

$\{\xi_n\}$ provided an additional restriction on the covariances among ξ_i's (see (3.1) below) is satisfied and provided the mixing coefficient $\alpha(n) \downarrow 0$, as $n \to \infty$, at an appropriate rate. It is well known, however, (see [18]) that $U_n \Rightarrow_d U_0$, as $n \to \infty$, holds provided $\alpha(n) \downarrow 0$ sufficiently fast. We believe that the condition on α of Lemma 3.1 below is the best so far.

LEMMA 3.1. *Let $\{\xi_n\}$ be a stationary strong mixing uniform process with $\sum n^2[\alpha(n)]^\delta < \infty$ for some $0 < \delta < 1$. Then the series* (1.1) *converges absolutely and $U_n \Rightarrow_d U_0$, as $n \to \infty$, with $P[U_0 \in C] = 1$.*

PROOF. The proof can be easily accomplished using Lemma 2.1 of [4] and the method of proof of Theorem 22.1 of [2]. □

Let corr (ξ, η) denote the correlation between ξ and η. The condition we impose on $\{\xi_n\}$ is that for all ξ measurable $\mathscr{M}_1^k$ and all η measurable $\mathscr{M}_{k+n}^\infty$,

$$|\operatorname{corr}(\xi, \eta)| \leqq c\alpha^\delta(n), \qquad \text{for some} \quad 0 < \delta \leqq 1, \tag{3.1}$$

where c is an absolute constant independent of n holds. A well-known result of Sarmanov (see [8]) states that (3.1) is satisfied by all strong mixing stationary Gaussian processes with $\delta = 1$. The important case of Gaussian processes is thus covered by the results of this section (cf. Section 4 of [6]).

THEOREM 3.1. *Let $\{\xi_n\}$ be as in Lemma 3.1 with $\sum n^2\alpha^\delta(n) < \infty$ for a $0 < \delta < 1$, q be defined by* (1.5) *and assume that* (3.1) *holds. Then* (i) $\rho_q(U_n, U_0) \to_{\text{a.s.}} 0$ *and* (ii) $\rho_q(V_n^*, V_0) \to_p 0$ *as $n \to \infty$.*

PROOF. Suppose $\{Z_i\}$ is a stationary strong mixing process with $|Z_i| \leqq 1$ and set $S_n = \sum_{i=1}^n Z_i$. We will first show that the assumed conditions imply

$$E(S_n^4) \leqq K_\alpha[n^2E^2(Z_1^2) + nE(Z_1^2)], \tag{3.2}$$

which inequality is the assertion of Lemma 22.1 of [2]. Consequently, the conclusion of Proposition 2.1 and, therefore, all results of Section 2 remain valid in the present case with K_α in place of K_ϕ. It remains to prove (3.2): First note that

$$\begin{aligned} E(S_n^4) = {} & \textstyle\sum_{i=1}^n EZ_i^4 + 4\sum\sum_{i\neq j} E(Z_i^3Z_j) + 6\sum\sum_{i<j} EZ_i^2Z_j^2 \\ & + 12\textstyle\sum\sum\sum_{i<j\neq k} E(Z_iZ_jZ_k^2) \\ & + 24\textstyle\sum\sum\sum\sum_{i<j<k<l} E(Z_iZ_jZ_kZ_l)\,. \end{aligned} \tag{3.3}$$

Now letting $E(Z_i^2) = \tau$ and using the hypothesis $|Z_i| \leqq 1$ and (3.1), we obtain

$$\textstyle\sum_{i=1}^n E(Z_i^4) \leqq n\tau\,, \tag{3.4}$$

$$4|\textstyle\sum\sum_{i\neq j} E(Z_i^3Z_j)| + 6|\sum\sum_{i<j} E(Z_i^2Z_j^2)| \leqq 14n\tau c\sum_{n=1}^\infty \alpha^\delta(n) + 6n^2\tau^2\,, \tag{3.5}$$

$$\begin{aligned} & \textstyle\sum\sum_{i<j}\sum_k |E(Z_iZ_jZ_k^2)| \\ & \leqq \textstyle\sum_k EZ_k^2 \cdot \sum\sum_{i<j}|EZ_iZ_j| + \sum\sum_{i<j}\sum_k \min\{|\operatorname{Cov}(Z_iZ_j; Z_k^2)|, \\ & \qquad |\operatorname{Cov}(Z_i; Z_jZ_k^2)|, |\operatorname{Cov}(Z_iZ_k^2; Z_j)|\} \\ & \leqq n^2c\tau^2\textstyle\sum_{k=1}^\infty \alpha^\delta(k) + 6n\tau c\sum_{k=1}^\infty k\alpha^\delta(k)\,. \end{aligned} \tag{3.6}$$

Similarly we obtain

$$(3.7)\qquad |\sum\sum\sum\sum_{i<j<k<l} E(Z_0 Z_j Z_k Z_l)| \leqq n^2c^2\tau^2(\sum_{k=1}^{\infty} \alpha^\delta(k))^2 + 3c\tau n(\sum_{k=1}^{\infty} k^2\alpha^\delta(k)) .$$

From (3.3) to (3.7), the inequality (3.2) follows. □

Since Lemmas 2.2 and 2.3 also remain valid under the conditions of the preceding theorem, we can conclude the following:

THEOREM 3.2. *Let $\{\xi_n\}$ be a stationary strong mixing uniform process satisfying $\sum_{n=1}^{\infty} n^2\alpha^\delta(n) < \infty$ and* (3.1). *Then the conclusions of Theorems* 2.1 *and* 2.2 *remain valid under the same conditions.*

4. Trimmed and Winsorized means. The asymptotic joint normality of sample quantiles and weak convergence of the quantile process on $[\alpha, \beta]$ with $0 < \alpha < \beta < 1$, have been proved by Bickel [1] and Shorack [14] in the case of i.i.d. rv's and by Sen [12] in the case of ϕ-mixing sequences. As pointed out by Pyke [10], these results are direct consequences of the weak convergence of empirical processes and the Glivenko-Cantelli lemma so that they are valid even in the case of strong mixing under certain conditions (for example those of Lemma 3.1). However, in the ϕ-mixing case or the strong mixing Gaussian case we have stronger results: the weak convergence of quantile processes, proved by Shorack [14] (Theorem 1 of [14] and its variations) in the more general supremum and integral metrics, remain valid in the present case with the function q of Section 1. We omit the details since the proofs are similar to those of [14] in view of the results of our Sections 2 and 3. Further, in the theory of robust estimation the trimmed and Winsorized means appear very prominently. In the i.i.d. case their asymptotic normality is well known (see e.g. [1]). Theorem 4.1 below shows that this remains true in the case of strong mixing sequences since the limiting rv's, as is easily verified, have normal distribution.

Let $\{\xi_n\}$ be a uniform process and ξ_{ni}, $1 \leqq i \leqq n$ be the OS as defined in Section 2. Also let $g = F^{-1}$. For $0 < \alpha < \beta < 1$, consider the trimmed and Winsorized means

$$T_n = \frac{1}{\beta_n - \alpha_n} \sum_{i=\alpha_n+1}^{\beta_n} g(\xi_{ni}) ,$$

$$W_n = n^{-1}[\alpha_n g(\xi_{n\alpha_n}) + (n - \beta_n)g(\xi_{n(\beta_n+1)}) + \sum_{i=\alpha_n+1}^{\beta_n} g(\xi_{ni})] ,$$

where $\alpha_n = [n\alpha]$, $\beta_n = [n\beta]$. It is to be noted that a.s. convergence results proved in the theorem below hold only for the specially constructed processes; for the original processes, only the corresponding weak convergence is implied.

THEOREM 4.1. *Suppose $\{\xi_i\}$ satisfy the conditions of Lemma* 3.1. *If g is differentiable on $(\alpha - \varepsilon, \beta + \varepsilon)$ for some $\varepsilon > 0$, then*

$$n^{\frac{1}{2}}(T_n - \mu_T) \to_{\text{a.s.}} (\beta - \alpha) \int_\alpha^\beta g'(t)V_0(t)\, dt ,$$

$$n^{\frac{1}{2}}(W_n - \mu_W) \to_{\text{a.s.}} \int_\alpha^\beta g'(t)V_0(t)\, dt + \alpha g'(\alpha)V_0(\alpha) + (1 - \beta)g'(\beta)V_0(\beta) ,$$

as $n \to \infty$, *where* V_0 *is defined as in Corollary* 2.1, *and*

$$\mu_T = (\beta - \alpha)^{-1} \int_\alpha^\beta g(t)\,dt$$
$$\mu_W = \alpha g(\alpha) + (1 - \beta) g(\beta) + \int_\alpha^\beta g(t)\,dt\,.$$

PROOF. In view of Lemma 3.1 and the Glivenko–Cantelli lemma for stationary strong mixing rv's (see [16]), the proof is straightforward. The details are therefore omitted. □

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