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A CONJECTURE OF BERRY REGARDING A BERNOULLI TWO-ARMED BANDIT

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Two independent Bernoulli processes (arms) have unknown success probabilities ρ and λ. The initial (a priori) information about ρ and λ is expressed by probability distributions

$$dR(\rho) = C_R \rho^{r_0}(1-\rho)^{r_0'}\, d\mu(\rho) \quad \text{for the right arm,}$$

and

$$dL(\lambda) = C_L \lambda^{l_0}(1-\lambda)^{l_0'}\, d\mu(\lambda) \quad \text{for the left arm,}$$

where μ is any arbitrary measure on the unit interval. A specified number n of observations is made sequentially, the arm selected at each stage depending on the previous observations and the initial information. A conjecture of Berry states that if the initial information present about the right arm (given by $r_0 + r_0'$) is not greater than that present for the left arm ($l_0 + l_0'$) and the initial expected value of ρ is not less than that of λ, then for any n the advantage (in terms of expected number of successes) of taking the first observation on the right arm is never less than that for the left arm. A proof of this conjecture is given in this paper.

1. Introduction. In a recent paper dealing with the Bernoulli two-armed bandit Berry [1] has stated three conjectures A, B and C. In the following we prove the conjecture B. As we have to refer frequently to formulas in Berry's paper [1], for brevity they are denoted by starred numbers to distinguish them from equations in this paper. The symbols and terms used in the following have the same meanings and definitions as in [1]. A brief explanation of the notation and the structure of the problem is given below. For more details Sections 1 and 2 of [1] may be referred to.

Writing for convenience $r_0, r_0', \cdots$ in place of r, r', etc. in (2.1)*, (2.2)*, we take the initial (prior) probability distributions R and L of the Bernoulli parameters ρ and λ for the right arm $\mathscr{R}$ and the left arm $\mathscr{L}$ as

$$dR(\rho) = C_R \rho^{r_0}(1-\rho)^{r_0'}\, d\mu_{\mathscr{R}}(\rho)\,, \tag{1 i}$$

$$dL(\lambda) = C_L \lambda^{l_0}(1-\lambda)^{l_0'}\, d\mu_{\mathscr{L}}(\lambda)\,, \tag{1 ii}$$

where C_R, C_L are normalizing constants, $\mu_{\mathscr{R}}$, $\mu_{\mathscr{L}}$ arbitrary positive measures on [0, 1], and r_0, r_0', l_0, l_0' arbitrary real numbers, such that the integrals of the right-hand side of (1 i) and (1 ii) converge. $N_{\mathscr{R}} = r_0 + r_0'$ is the "amount of initial information" about $\mathscr{R}$ and $N_{\mathscr{L}} = l_0 + l_0'$ that about $\mathscr{L}$. $E(\rho \mid R)$ and $E(\lambda \mid L)$ are respectively the expectations of ρ and λ with respect to the distributions R and L. n is the total number of observations to be made and $\Delta_n(R, L)$

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the expected advantage obtained by taking the first observation on the right arm $\mathscr{R}$. Berry's conjecture B ([1], page 892) is the following.

CONJECTURE B. If $\mu_{\mathscr{R}} = \mu_{\mathscr{L}}$, $r_0 + r_0' \leqq l_0 + l_0'$, and $E(\rho \mid R) \geqq E(\lambda \mid L)$, then $\Delta_n(R, L) \geqq 0$ for all n.

2. Posterior expectations of ρ, λ. In accordance with the condition in the conjecture, we assume throughout the following that

$$\mu_{\mathscr{R}} = \mu_{\mathscr{L}} = \mu \quad \text{say.} \tag{2}$$

Let (r, r') denote a variable point in the real plane for which

$$\textstyle\int_0^1 \rho^r(1-\rho)^{r'}\, d\mu(\rho) < \infty \tag{3}$$

where μ is the measure in (2).

The region of the real plane on which (3) holds is called the *possibility region* for μ, ([1], page 874). (r_0, r_0') (l_0, l_0') in (1 i) and (1 ii) are obviously points in the possibility region of μ. Similarly, if (r, r') lies in the possibility region, so does the point $(r+m, r'+n)$ for any $m, n \geqq 0$ (provided that $\mu(0, 1) > 0$). We refer to the distribution in (3) as the distribution (r, r'). We denote by $E(\rho \mid r, r')$ the expectation of ρ with respect to the distribution

$$dF(\rho) = K\rho^r(1-\rho)^{r'}\, d\mu(\rho) \tag{4}$$

where $K = K(r, r')$ is the normalizing constant. $E(\rho \mid r, r')$ denotes the same function as $E(\rho \mid r, r', \mu)$ in [1]. We drop the symbol μ as the measure remains fixed throughout the following.

If μ is a one-point measure then $E(\rho \mid r, r')$ has a constant value independent of (r, r') and the conjecture is satisfied trivially. In the following, to the end of the proof of Lemma 2.2, we make the following assumption.

ASSUMPTION 2.1. The measure μ in (2) is not a one-point measure.

Next let

$$A(r, r') = \frac{-\dfrac{\partial E(\rho \mid r, r')}{\partial r'}}{\dfrac{\partial E(\rho \mid r, r')}{\partial r}}\,. \tag{5}$$

By Lemma 4.6 in [1] both the numerator and denominator in (5) are finite and positive, i.e.

$$\frac{\partial E(\rho \mid r, r')}{dr} > 0\,, \tag{6i}$$

$$\frac{\partial E(\rho \mid r, r')}{\partial r'} < 0\,. \tag{6ii}$$

LEMMA 2.1. *As the point (r, r') moves along a curve on which $E(\rho \mid r, r')$ has a constant value in the direction in which r and r' increase, $A(r, r')$ is non-decreasing.*

PROOF. As shown in the proof of Lemma 5.1 in [1], along the curve

$$E(\rho \mid r, r') = \text{constant} \tag{7}$$

$$\frac{dr}{dr'} = A(r, r') > 0 \tag{8}$$

by (6 i), (6 ii).

Let D denote the directional derivative of any function of (r, r') along the curve (7), in the direction in which r and r' increase. Putting

$$k = \left[1 + \left(\frac{dr}{dr'}\right)^2\right]^{-\frac{1}{2}} = [1 + A^2(r, r')]^{-\frac{1}{2}}, \tag{9}$$

we have by (8),

$$\begin{aligned} DA(r, r') &= k\left\{\frac{\partial A(r, r')}{\partial r} \cdot \frac{dr}{dr'} + \frac{\partial A(r, r')}{\partial r'}\right\} \\ &= k\left\{A(r, r')\frac{\partial A(r, r')}{\partial r} + \frac{\partial A(r, r')}{\partial r'}\right\}. \end{aligned} \tag{10}$$

Next by differentiating (5), we obtain after some manipulation,

$$\begin{aligned} DA(r, r') = -k_1 \Big\{ & A^2(r, r')\frac{\partial^2 E(\rho \mid r, r')}{\partial r^2} \\ & + 2A(r, r')\frac{\partial^2 E(\rho \mid r, r')}{\partial r\, \partial r'} + \frac{\partial^2 E(\rho \mid r, r')}{\partial r^2}\Big\} \end{aligned} \tag{11}$$

where

$$k_1 = k\left[\frac{\partial E(\rho \mid r, r')}{\partial r}\right]^{-1}. \tag{11 i}$$

Let E denote the expectations with respect to the distribution (r, r'). Then

$$E(\rho \mid r, r') = [\textstyle\int_0^1 \rho^{r+1}(1-\rho)^{r'}\, d\mu(\rho)][\int_0^1 \rho^{r}(1-\rho)^{r'}\, d\mu(\rho)]^{-1}. \tag{12}$$

By differentiating the right-hand side of (12) we obtain the second order derivatives with respect to r, r' of $E(\rho \mid r, r')$. Substituting these in (11) we obtain

$$DA(r, r') = -k_2 E\{[\rho - E(\rho)]H^2(\rho)\} \tag{13}$$

where by (11 i), (9) and (6 i)

$$k_2 = k_1\left[\frac{\partial E(\rho \mid r, r')}{\partial r}\right]^{-2} = [1 + A^2(r, r')]^{-\frac{1}{2}}\left[\frac{\partial E(\rho \mid r, r')}{\partial r}\right]^{-3} \geqq 0 \tag{14}$$

and

$$\begin{aligned} H(\rho) = {} & \frac{\partial E(\rho \mid r, r')}{\partial r'}[\log \rho - E(\log \rho)] \\ & - \frac{\partial E(\rho \mid r, r')}{\partial r}[\log(1-\rho) - E\log(1-\rho)]. \end{aligned} \tag{15}$$

Since

$$\frac{\partial E(\rho \mid r, r')}{\partial r'} = \operatorname{Cov}(\rho, \log(1-\rho))$$

and

$$\frac{\partial E(\rho \mid r, r')}{\partial r} = \operatorname{Cov}(\rho, \log \rho),$$

$H(\rho)$ in (13) is the same as the function defined in (4.11)*. $H(\rho)$ satisfies

$$E\{H(\rho)[\rho - E(\rho)]\} = 0. \tag{16}$$

Substituting for $H(\rho)$ by (15) and using (16) we obtain,

$$\begin{aligned} &E\{[\rho - E(\rho)]H^2(\rho)\} \\ &= E\left\{[\rho - E(\rho)]H(\rho)\left[\frac{\partial E(\rho \mid r, r')}{\partial r'} \log \rho - \frac{\partial E(\rho \mid r, r')}{\partial r} \log(1-\rho)\right]\right\} \\ &= \frac{\partial E(\rho \mid r, r')}{\partial r'} \cdot E\{[\rho - E(\rho)]H(\rho) \log \rho\} \\ &\quad - \frac{\partial E(\rho \mid r, r')}{\partial r} \cdot E\{[\rho - E(\rho)]H(\rho) \log(1-\rho)\}. \end{aligned} \tag{17}$$

In (17) $E\{[(\rho - E(\rho)]H(\rho) \log \rho\} \geqq 0$ by (4.11)* and $E\{[\rho - E(\rho)]H(\rho) \log(1 - \rho)\} \leqq 0$ by the observations below (4.15)*. Hence using (6i), (6ii), we have from (17),

$$E\{[\rho - E(\rho)]H^2(\rho)\} \leqq 0. \tag{18}$$

Since k_2 in (13) $\geqq 0$ by (14), it follows from (18), that

$$DA(r, r') \geqq 0. \tag{19}$$

This completes the proof of Lemma 2.1.

Note 2.1. The result (22) of the following lemma means that under the conditions of the lemma the directional derivative of $E(\rho \mid r, r')$ in the direction of the vector $(C, 1)$ is non-increasing as we move along the curve defined by (7) in the direction in which r increases.

LEMMA 2.2. *If (r, r'), (l, l') are points such that*

$$r' \leqq l', \tag{20i}$$

$$E(\rho \mid r, r') = E(\rho \mid l, l') \tag{20ii}$$

(and therefore $r \leqq l$), and C is a number satisfying

$$C \geqq A(l, l') \tag{21}$$

then,

$$C\frac{\partial E(\rho \mid r, r')}{\partial r} + \frac{\partial E(\rho \mid r, r')}{\partial r'} \geqq C\frac{\partial E(\rho \mid l, l')}{\partial l} + \frac{\partial E(\rho \mid l, l')}{\partial l'}. \tag{22}$$

PROOF. Using (5) we obtain that (22) is equivalent to

$$[C - A(r, r')] \frac{\partial E(\rho \mid r, r')}{\partial r} \geqq [C - A(l, l')] \frac{\partial E(\rho \mid l, l')}{\partial l} . \tag{23}$$

By (20i) and (20ii), Lemma 2.1 applies. Using that lemma and (21) we get

$$C - A(r, r') \geqq C - A(l, l') \geqq 0 . \tag{24}$$

Hence the lemma is proved by showing that

$$\frac{\partial E(\rho \mid r, r')}{\partial r} \geqq \frac{\partial E(\rho \mid l, l')}{\partial l} \geqq 0 . \tag{25}$$

By (4.9)*

$$\frac{\partial}{\partial r} A(r, r') \geqq 0 . \tag{26}$$

Hence by (5)

$$-\frac{\partial E(\rho \mid r, r')}{\partial r} \frac{\partial^2 E(\rho \mid r, r')}{\partial r\, \partial r'} + \frac{\partial E(\rho \mid r, r')}{\partial r'} \frac{\partial^2 E(\rho \mid r, r')}{\partial r^2} \geqq 0 . \tag{27}$$

Dividing (27) throughout by $\partial E(\rho \mid r, r')/\partial r$, which is positive by (6i), we obtain

$$\frac{\partial^2 E(\rho \mid r, r')}{\partial r^2} A(r, r') + \frac{\partial^2 E(\rho \mid r, r')}{\partial r\, \partial r'} \leqq 0 . \tag{28}$$

Hence,

$$D \frac{\partial E(\rho \mid r, r')}{\partial r} \leqq 0 , \tag{29}$$

where, again D denotes the derivative along the curve defined by (7). Hence (25) follows from (29) and (6i), and (25) combined with (24) yields (23).

This completes the proof of Lemma 2.2. We now prove the following.

THEOREM 2.1. *If* $r_0 \leqq l_0$, $r_0' \leqq l_0'$, $E(\rho \mid r_0, r_0') \geqq E(\rho \mid l_0, l_0')$, *and* m, n *are any nonnegative numbers such that*

$$E(\rho \mid l_0 + m, l_0' + n) \geqq E(\rho \mid r_0, r_0') , \tag{30}$$

then

$$E(\rho \mid r_0 + m, r_0' + n) \geqq E(\rho \mid l_0 + m, l_0' + n) . \tag{31}$$

COROLLARY 2.1. *The inequality* (31) *holds if* (30) *is substituted by*

$$E(\rho \mid r_0 + m, r_0' + n) \geqq E(\rho \mid r_0, r_0') . \tag{32}$$

Note 2.2. For any point (r, r') in the region of possibility of μ, obviously $E(\rho \mid r, r') = E(\lambda \mid r, r')$. Hence we freely interchange ρ and λ in this expression.

PROOF. If μ is a one-point measure the theorem is true trivially. In the following we therefore assume that μ is not a one-point measure. By the assumption in the theorem

$$(33)\qquad E(\rho \mid l_0, l_0') \leqq E(\rho \mid r_0, r_0')\,.$$

Consider a variable point (l, l') which moves from the point (l_0, l_0') along the line defined by

$$(34)\qquad l + l' = l_0 + l_0'$$

in the direction in which l increases, and consequently l' decreases. By (6i) and (6ii) $E(\rho \mid l, l')$ is strictly increasing and continuous as (l, l') moves along the line. It therefore follows from (33), that there exists a unique point (l_1, l_1') on the line defined by (34), such that

$$(35\text{i})\qquad E(\rho \mid l_1, l_1') = E(\rho \mid r_0, r_0')\,,$$

$$(35\text{ii})\qquad l_1 \geqq l_0\,; \qquad l_1' \leqq l_0'\,.$$

Since by assumption in the theorem $l_0 \geqq r_0$, it follows from (35ii) that

$$(36\text{i})\qquad l_1 \geqq r_0\,.$$

Hence by (8) and (35i)

$$(36\text{ii})\qquad l_1' \geqq r_0'\,.$$

Since by (35ii)

$$l_1 + m \geqq l_0 + m\,, \qquad l_1' + n \leqq l_0' + n$$

we obtain (recalling Note 2.2) by using (6i), (6ii), (30) and (35i)

$$(37)\qquad \begin{aligned} E(\rho \mid l_1 + m, l_1' + n) &\geqq E(\rho \mid l_0 + m, l_0' + n) \\ &\geqq E(\rho \mid r_0, r_0') \\ &= E(\rho \mid l_1, l_1')\,. \end{aligned}$$

Let P_1 be a point (l, l'), which moves continuously from the point (l_1, l_1') to the point $(l_1 + m, l_1' + n)$. We shall show that there exists a differentiable path T_1 at each point of which the slope of the tangent to T_1 is not less than the value of $A(l, l')$ at that point. Consider the path defined by

$$(38)\qquad \frac{dl}{dl'} = A(l, l') + \left[\frac{\partial E(\rho \mid l, l')}{\partial l}\right]^{-1} \cdot \frac{1}{n}\{E(\rho \mid l_1 + m, l_1' + n) - E(\rho \mid l_1, l_1')\}\,.$$

By differentiating $E(\rho \mid l, l')$ along the path we obtain using (5),

$$(39)\qquad \begin{aligned} \frac{d}{dl'} E(\rho \mid l, l') &= \frac{1}{n}\{E(\rho \mid l_1 + m, l_1' + n) - E(\rho \mid l_1, l_1')\} \\ &= \text{a constant.} \end{aligned}$$

Integrate both sides of (39) with respct to l' from $l' = l_1'$ to $l' = l_1' + n$. Let l^* be the value assumed by l when $l' = l_1' + n$. Cancelling out the common term $-E(\rho \mid l_1, l_1')$ on both sides we obtain that

$$(40)\qquad E(\rho \mid l^*, l_1' + n) = E(\rho \mid l_1 + m, l_1' + n)\,.$$

From (40), it follows by (6 i) that $l^* = l_1 + m$. Thus the path T_1 defined by (38) goes from the point (l_1, l_1') to the point $(l_1 + m, l_1' + n)$. As $\partial E(\rho \mid l, l')/\partial l$ is positive by (6 i) and the expression in braces in (38) is positive by (37), the path T_1 satisfies

$$\frac{dl}{dl'} \geqq A(l, l') \tag{41}$$

at each point of the path.

Next let P_2 be a point with coordinates (r, r'), such that as P_1 moves along T_1, P_2 moves along a path T_2, defined by

$$r' = r_0' + (l' - l_1'), \tag{42 i}$$

$$E(\rho \mid r, r') = E(\rho \mid l, l'). \tag{42 ii}$$

From (35 i), (42 i) and (42 ii) it follows that when P_1 starts at (l_1, l_1'), P_2 starts at the point (r_0, r_0'). As P_1 moves along the path T_1, r, r' and l are functions of l'. Differentiating both sides of (42 ii), with respect to l', we get, after using (42 i) that

$$\frac{\partial E(\rho \mid r, r')}{\partial r} \cdot \frac{dr}{dl'} + \frac{\partial E(\rho \mid r, r')}{\partial r'} = \frac{\partial E(\rho \mid l, l')}{dl} \frac{dl}{dl'} + \frac{\partial E(\rho \mid l, l')}{\partial l'}. \tag{43}$$

In (43) put

$$r = r_0 + (l - l_1) + v(l'), \tag{44}$$

so that

$$\frac{dr}{dl'} = \frac{dl}{dl'} + \frac{dv}{dl'}. \tag{45}$$

Substituting by (45) in (43), we obtain that

$$\frac{\partial E(\rho \mid r, r')}{\partial r} \cdot \frac{dv}{dl'} = \left\{\frac{\partial E(\rho \mid l, l')}{\partial l} \frac{dl}{dl'} + \frac{\partial E(\rho \mid l, l')}{\partial l'}\right\} - \left\{\frac{\partial E(\rho \mid r, r')}{\partial r'} \frac{dl}{dl'} + \frac{\partial E(\rho \mid r, r')}{dr'}\right\}. \tag{46}$$

We next show that (46) implies that

$$\frac{dv}{dl'} \leqq 0. \tag{47}$$

In the left-hand side of (46), by (6 i)

$$\frac{\partial E(\rho \mid r, r')}{\partial r} > 0. \tag{48}$$

In the right-hand side of (46) we apply Lemma 2.2. From (42 i) we have by (36 ii)

$$r' = l' - (l_1' - r_0') \leqq l' \tag{49 i}$$

and hence by (42 ii) and (8)

$$r \leqq l. \tag{49 ii}$$

By (49i), (42ii) and (41) the conditions (20i), (20ii) and (21) of Lemma 2.2 are respectively satisfied taking $C = dl/dl'$. Hence the right-hand side of (46) is $\leqq 0$ by (22).

At the start of the motion when P_1 is at (l_1, l_1'), P_2 is at (r_0, r_0'). Hence

$$r = r_0 \quad \text{when} \quad l = l_1 . \tag{50}$$

Substituting by (50) in (44) we obtain that in the initial position

$$v(l_1') = 0 . \tag{51}$$

(51) and (47) combined yield that throughout the motion

$$v(l') \leqq 0 . \tag{52}$$

In particular when P_1 reaches the point $(l_1 + m, l_1' + n)$, the coordinates of P_2 using (42i), (44) and (52) are given by

$$r' = r_0' + n , \tag{53i}$$

and

$$r^* = r_0 + m + v(l_1' + n) \leqq r_0 + m . \tag{53ii}$$

Hence using (53ii) and (6i) in the first step, (42ii) in the second and (35ii), (6i) and (6ii) in the third we obtain that

$$\begin{aligned} E(\rho \mid r_0 + m, r_0' + n) \geqq E(\rho \mid r^*, r_0' + n) &= E(\rho \mid l_1 + m, l_1' + n) \\ &\geqq E(\rho \mid l_0 + m, l_0' + n) \end{aligned} \tag{54}$$

as asserted in (31). This completes the proof of Theorem 2.1.

The proof of Corollary 2.1 follows immediately, since when (32) holds, if (30) holds, (31) holds by the theorem, and if (30) does not hold then

$$E(\rho \mid r_0, r_0') > E(\rho \mid l_0 + m, l_0' + n) ,$$

which combined with (32) gives (31). This completes the proof of Corollary 2.1.

3. Berry's conjecture. We are now in a position to prove Berry's conjecture. It is necessary, however, to prove a wider result which includes the conjecture as a special case.

In the statement of the conjecture in Section 1, $\Delta_n(R, L)$ denotes the expected advantage when the first of n observations is taken on the right arm. We shall use the symbol $\Delta_n(r, r'; l, l')$ to denote the expected advantage in taking the first of n observations on the right arm, when the probability distribution of ρ is given by (4), and that of λ by

$$dF_1(\lambda) = K_1 \lambda^l (1 - \lambda)^{l'} \, d\mu . \tag{55}$$

As in (3.2)* for any r, r', l, l' we put $\Delta_n^-(r, r'; l, l') = \min \{0, \Delta_n(r, r'; l, l')\}$ and $\Delta_n^+(r, r'; l, l') = \max \{0, \Delta_n(r, r'; l, l')\}$.

THEOREM 3.1. *The distributions of ρ and λ being as given in* (1i) *and* (1ii), *if*

(m_1, n_1), (m_2, n_2) *are arbitrary pairs of nonnegative real numbers and if the following relations hold viz.*,

$$E(\rho \mid m_1 + r_0, n_1 + r_0') \geqq E(\rho \mid m_2 + r_0, n_2 + r_0') \geqq E(\rho \mid r_0, r_0'), \tag{56 i}$$

$$E(\rho \mid r_0, r_0') \geqq E(\rho \mid l_0, l_0'), \tag{56 ii}$$

and

$$r_0 + r_0' \leqq l_0 + l_0', \tag{56 iii}$$

then for all integers n,

$$\Delta_n(r_0 + m_1, r_0' + n_1; l_0 + m_2, l_0' + n_2) + \Delta_n^-(r_0 + m_2, r_0' + n_2; l_0 + m_1; l_0' + n_1) \geqq 0. \tag{57}$$

COROLLARY 3.1. *Under the conditions of Theorem* 3.1

$$\Delta_n(r_0 + m_1, r_0' + n_1, l_0 + m_2, l_0' + n_2) \geqq 0 \qquad \textit{for all} \quad n. \tag{58}$$

PROOF. The proof is by induction. Let $n = 1$. By (3.8)* we have

$$\begin{aligned} \Delta_1(r_0 + m_1, r_0' + n_1, l_0 + m_2, l_0' + n_2) &= E(\rho \mid r_0 + m_1, r_0' + n_1) - E(\rho \mid l_0 + m_2, l_0' + n_2) \\ &\geqq 0, \end{aligned} \tag{59}$$

by (56i), and (31) in Theorem 2.1. Hence (57) holds for $n = 1$. Now suppose that (57) and consequently (58) hold for all $n \leqq j$. In (3.7)* put $n = j + 1$, and substitute respectively $(r_0 + m_1, r_0' + n_1)$ for R, $(l_0 + m_2, l_0 + n_2)$ for L. Note that R in [1] means the distribution specified by the pair $(r_0 + m_1 + 1, r_0' + n_1)$ and φR that by the pair $(r_0 + m_1, r_0' + n_1 + 1)$. We thus obtain from (3.7)*

$$\begin{aligned} &\Delta_{j+1}(r_0 + m_1, r_0' + n_1; l_0 + m_2, l_0' + n_2) \\ &\quad = E(\rho \mid r_0 + m_1, r_0' + n_1)\Delta_j^+(r_0 + m_1 + 1, r_0' + n_1; l_0 + m_2, l_0' + n_2) \\ &\qquad + [1 - E(\rho \mid r_0 + m_1, r_0' + n)] \\ &\qquad \times \Delta_j^+(r_0 + m_1, r_0' + n_1 + 1; l_0 + m_2, l_0' + n_2) \\ &\qquad + E(\rho \mid l_0 + m_2, l_0' + n_2) \\ &\qquad \times \Delta_j^-(r_0 + m_1, r_0' + n_1; l_0 + m_2 + 1, l_0' + n_2) \\ &\qquad + [1 - E(\rho \mid l_0 + m_2, l_0' + n_2)] \\ &\qquad \times \Delta_j^-(r_0 + m_1, r_0' + n_1; l_0 + m_2, l_0' + n_2 + 1). \end{aligned} \tag{60}$$

By (4.16)* and using $\Delta_n(r, r'; l, l') = -\Delta_n(l, l'; r, r')$, we have

$$\begin{aligned} &\Delta_j(r_0 + m_1, r_0' + n_1; l_0 + m_2, l_0' + n_2 + 1) \\ &\qquad \geqq \Delta_j(r_0 + m_1, r_0' + n_1; l_0 + m_2, l_0' + n_2) \geqq 0 \end{aligned} \tag{61}$$

by (58) which holds for $n = j$ by the inductive hypothesis. Hence by (61)

$$\Delta_j^-(r_0 + m_1, r_0' + n_1; l_0 + m_2, l_0' + n_2 + 1) = 0. \tag{62}$$

By (56i) the condition (32) of Corollary 2.1 is satisfied with $(m, n) = (m_2, n_2)$.

Hence by that corollary

$$E(\rho \mid r_0 + m_2, r_0' + n_2) \geqq E(\rho \mid l_0 + m_2, l_0' + n_2)\,. \tag{63}$$

(63) combined with (56 i) yields

$$E(\rho \mid r_0 + m_1, r_0' + n_1) \geqq E(\rho \mid l_0 + m_2, l_0' + n_2)\,. \tag{64}$$

Using (62) and (64), we obtain from (60) that

$$\begin{aligned} &\Delta_{j+1}(r_0 + m_1, r_0' + n_1, l_0 + m_2, l_0' + n_2) \\ &\quad \geqq E(\rho \mid l_0 + m_2, l_0' + n_2)\{\Delta_j^+(r_0 + m_1 + 1, r_0' + n_1; l_0 + m_2, l_0' + n_2) \\ &\qquad + \Delta_j^-(r_0 + m_1, r_0' + n_1; l_0 + m_2 + 1, l_0' + n_2)\}\,. \end{aligned} \tag{65}$$

By (4.16)*

$$\begin{aligned} &\Delta_j(r_0 + m_1 + 1, r_0' + n_1; l_0 + m_2, l_0' + n_2) \\ &\qquad \geqq \Delta_j(r_0 + m_1, r_0' + n_1; l_0 + m_2, l_0' + n_2) \geqq 0 \end{aligned} \tag{66}$$

where the last step follows from (58) and the inductive hypothesis for $n = j$. Hence,

$$\begin{aligned} &\Delta_j^+(r_0 + m_1 + 1, r_0' + n_1; l_0 + m_2, l_0' + n_2) \\ &\qquad = \Delta_j(r_0 + m_1 + 1, r_0' + n_1; l_0 + m_2, l_0' + n_2)\,. \end{aligned} \tag{67}$$

We substitute by (67) in the right-hand side of (65) and then apply (57) by the inductive hypothesis for $n = j$. Note that the inductive hypothesis in the theorem for $n \leqq j$ is assumed to be true not only for the particular pairs of nonnegative numbers (m_1, n_1), (m_2, n_2) and the particular distributions (r_0, r_0'), (l_0, l_0'), but for arbitrary pairs of nonnegative numbers (m_1', n_1'), (m_2', n_2') and for distributions (r, r'), (l, l'), where (r, r') and (l, l') are any points in the possibility region of μ, as defined in the remark below (3). This point is of crucial importance in the further argument.

In (57) putting

$$(m_1', n_1') = (1, 0)\,, \qquad (m_2', n_2') = (0, 0)\,,$$
$$(r, r') = (r_0 + m_1, r_0' + n_1)\,, \qquad (l, l') = (l_0 + m_2, l_0' + n_2)$$

we obtain that the right-hand side of (65) $\geqq 0$. Hence (58) holds for $n = j + 1$.

We next show that (57) holds for $n = j + 1$. By rearranging the terms of (60) and using (62), we obtain

$$\begin{aligned} &\Delta_{j+1}(r_0 + m_1, r_0' + n_1; l_0' + m_2, l_0' + n_2) \\ &\quad = E(\rho \mid r_0 + m_1, r_0' + n_1)\{\Delta_j^+(r_0 + m_1 + 1, r_0' + n_1; l_0 + m_2, l_0' + n_2) \\ &\qquad - \Delta_j^+(r_0 + m_1, r_0' + n_1 + 1; l_0 + m_2, l_0' + n_2)\} \\ &\qquad + \Delta_j^+(r_0 + m_1, r_0' + n_1 + 1; l_0 + m_2, l_0' + n_2) \\ &\qquad + E(\rho \mid l_0 + m_2, l_0' + n_2) \\ &\qquad \times \Delta_j^-(r_0 + m_1, r_0' + n_1; l_0 + m_2 + 1, l_0' + n_2)\,. \end{aligned} \tag{68}$$

In view of (56 ii) and Corollary 2.1,

$$E(\rho \mid r_0 + m_1, r_0' + n_1) \geqq E(\rho \mid l_0 + m_1, l_0' + n_1) . \tag{69}$$

Also by (4.16)*,

$$\begin{aligned} &\Delta_j(r_0 + m_1 + 1, r_0' + n_1; l_0 + m_2, l_0' + n_2) \\ &\qquad \geqq \Delta_j(r_0 + m_1, r_0' + n_1; l_0 + m_2, l_0' + n_2) \end{aligned}$$

which is nonnegative by (58) and the inductive hypothesis. Hence

$$\begin{aligned} &\Delta_j^+(r_0 + m_1 + 1, r_0' + n_1; l_0 + m_2, l_0' + n_2) \\ &\qquad = \Delta_j(r_0 + m_1 + 1, r_0' + n_1; l_0 + m_2, l_0' + n_2) . \end{aligned} \tag{70}$$

Since the term in braces in (68) is nonnegative by (4.16)*, by (69) and (70) the first term in the right-hand side of (68) is

$$\begin{aligned} &\geqq E(\rho \mid l_0 + m_1, l_0' + n_1)\{\Delta_j(r_0 + m_1 + 1, r_0' + n_1; l_0 + m_1, l_0' + n_1) \\ &\qquad - \Delta_j^+(r_0 + m_1, r_0' + n_1 + 1; l_0 + m_1, l_0' + n_1)\} . \end{aligned}$$

Hence after a slight rearrangement of the terms in (68),

$$\begin{aligned} &\Delta_{j+1}(r_0 + m_1, r_0' + n_1; l_0 + m_2, l_0' + n_2) \\ &\qquad \geqq E(\rho \mid l_0 + m_1, l_0' + n_1)\Delta_j(r_0 + m_1 + 1, r_0' + n_1; l_0 + m_2, l_0 + n_2) \\ &\qquad\quad + [1 - E(\rho \mid l_0 + m_1, l_0' + n_1)] \\ &\qquad\quad \times \Delta_j^+(r_0 + m_1, r_0' + n_1 + 1; l_0 + m_2, l_0' + n_2) \\ &\qquad\quad + E(\rho \mid l_0 + m_2, l_0' + n_2) \\ &\qquad\quad \times \Delta_j^-(r_0 + n_1, r_0' + m_1; l_0 + m_2 + 1, l_0' + n_2) . \end{aligned} \tag{71}$$

Again applying (3.7)*,

$$\begin{aligned} &\Delta_{j+1}(r_0 + m_2, r_0' + n_2; l_0 + m_1, l_0' + n_1) \\ &\qquad = E(\rho \mid r_0 + m_2, r_0' + n_2)\Delta_j^+(r_0 + m_2 + 1, r_0' + n_2; l_0 + m_1, l_0' + n_1) \\ &\qquad\quad + [1 - E(\rho \mid r_0 + m_2, r_0' + n_2)] \\ &\qquad\quad \times \Delta_j^+(r_0 + m_2, r_0' + n_2 + 1; l_0 + m_1, l_0' + n_1) \\ &\qquad\quad + E(\rho \mid l_0 + m_1, l_0' + n_1) \\ &\qquad\quad \times \Delta_j^-(r_0 + m_2, r_0' + n_2; l_0 + m_1 + 1, l_0' + n_1) \\ &\qquad\quad + [1 - E(\rho \mid l_0 + m_1, l_0' + n_1)] \\ &\qquad\quad \times \Delta_j^-(r_0 + m_2, r_0' + n_2; l_0 + m_1, l_0' + n_1 + 1) . \end{aligned} \tag{72}$$

Using (63) in the first term in the right-hand side of (72) and noting that its second term is nonnegative we obtain that

$$\begin{aligned} &\Delta_{j+1}(r_0 + m_2, r_0' + n_2; l_0 + m_1, l_0' + n_1) \\ &\qquad \geqq E(\rho \mid l_0 + m_2, l_0' + n_2)\Delta_j^+(r_0 + m_2 + 1, r_0' + n_2; l_0 + m_1, l_0' + n_1) \\ &\qquad\quad + E(\rho \mid l_0 + m_1, l_0' + n_1) \\ &\qquad\quad \times \Delta_j^-(r_0 + m_2, r_0' + n_2; l_0 + m_1 + 1, l_0' + n_1) \\ &\qquad\quad + [1 - E(\rho \mid l_0 + m_1, l_0' + n_1)] \\ &\qquad\quad \times \Delta_j^-(r_0 + m_2, r_0 + n_2; l_0 + m_1, l_0' + n_1 + 1) . \end{aligned} \tag{73}$$

Combining (71) and (73), we get

$$
\begin{aligned}
&\Delta_{j+1}(r_0 + m_1, r_0' + n_1; l_0 + m_2, l_0' + n_2) \\
&\quad + \Delta_{j+1}(r_0 + m_2, r_0' + n_2; l_0 + m_1, l_0' + n_1) \\
&\qquad \geqq E(\rho \mid l_0 + m_1, l_0' + n_1)\{\Delta_j(r_0 + m_1 + 1, r_0' + n_1; l_0 + m_2, l_0' + n_2) \\
&\qquad\quad + \Delta_j^-(r_0 + m_2, r_0' + n_2; l_0 + m_1 + 1, l_0' + n_1)\} \\
&\qquad\quad + [1 - E(\rho \mid l_0 + m_1, l_0' + n_1)] \\
&\qquad\quad \times \{\Delta_j^+(r_0 + m_1, r_0' + n_1 + 1; l_0 + m_2, l_0' + n_2) \\
&\qquad\quad + \Delta_j^-(r_0 + m_2, r_0' + n_2; l_0 + m_1, l_0' + n_1 + 1)\} \\
&\qquad\quad + E(\rho \mid l_0 + m_2, l_0' + n_2) \\
&\qquad\quad \times \{\Delta_j^+(r_0 + m_2 + 1, r_0' + n_2; l_0 + m_1, l_0' + n_2) \\
&\qquad\quad + \Delta_j^-(r_0 + m_1, r_0' + n_1; l_0 + m_2 + 1, l_0' + n_2)\}\,.
\end{aligned}
\tag{74}
$$

We next show that each expression in braces in the right-hand side of (74) is nonnegative.

Recalling the note below (67), in the right-hand side of (74) in the expression in braces in the first term, say T_1, put

$$
(m_1', n_1') = (m_1 + 1, n_1)\,, \qquad (m_2', n_2') = (m_2, n_2)
$$
$$
(r, r') = (r_0', r_0')\,, \qquad (l, l') = (l_0, l')\,.
$$

Since $E(\rho \mid r_0 + m_1 + 1, r_0' + n_1) \geqq E(\rho \mid r_0 + m_1, r_0' + n_1)$, (m_1', n_1'), (m_2', n_2'), (r, r'), (l, l') satisfy the conditions of the theorem. Hence by the inductive hypothesis for $n = j$ (57) holds. Hence $T_1 \geqq 0$.

Let T_2 denote the second expression within braces in the right-hand side of (74). Here two alternatives are possible viz.,

$$
E(\rho \mid r_0 + m_2, r_0' + n_2) \geqq E(\rho \mid l_0 + m_1, l_0' + n_1 + 1)
\tag{75}
$$

or not. Suppose (75) holds. Putting

$$
(r, r') = (r_0 + m_2, r_0' + n_2)\,, \qquad (l, l') = (l_0 + m_1, l_0' + n_1 + 1)
$$
$$
m_1' = n_1' = m_2' = n_2' = 0\,,
$$

the conditions of the theorem are satisfied for the distributions (r, r'), (l, l') and nonnegative numbers (m_1', n_1'), (m_2', n_2') and hence by the inductive hypothesis we have for $n = j$ by (58)

$$
\Delta_j(r_0 + m_2, r_0' + n_2; l_0 + m_1, l_0' + n_1 + 1) \geqq 0
$$

so that

$$
\Delta_j^-(r_0 + m_2, r_0' + n_2; l_0 + m_1, l_0' + n_1 + 1) = 0\,.
\tag{76}
$$

Hence

$$
T_2 \geqq 0\,.
$$

Alternatively suppose (75) does not hold. Then by (75) and (56i)

$$
E(\rho \mid l_0 + m_1, l_0' + n_1 + 1) > E(\rho \mid r_0 + m_2, r_0' + n_2) \geqq E(\rho \mid r_0, r_0')\,.
\tag{77}
$$

Thus the condition (30) in Theorem 2.1 is satisfied and we have by that theorem

$$E(\rho \mid r_0 + m_1, r_0' + n_1 + 1) \geqq E(\rho \mid l_0 + m_1, l_0' + n_1 + 1) \qquad \text{by (31).}$$

Putting

$$(m_1', n_1') = (m_1, n_1 + 1), \qquad (m_2', n_2') = (m_2, n_2)$$
$$(r, r') = (r_0, r_0'), \qquad (l, l') = (l_0, l_0'),$$

the conditions of the Theorem 3.1 are satisfied for (r, r') etc., and hence by the inductive hypothesis, (57) holds for $n = j$, so that $T_2 \geqq 0$. Thus under either of the alternatives, i.e., whether (75) holds or not, $T_2 \geqq 0$.

By an exactly analogous argument it follows that the expression in braces in the third term in the right-hand side of (74), T_3 say, is nònnegative. As in (74), T_1, T_2, T_3 are all nonnegative; we obtain that

(78) $$\text{left-hand side of (74)} \geqq 0 .$$

The second term in the left-hand side of (74) may be either (a) non-positive or (b) positive. If it is non-positive,

(79) $$\Delta_{j+1}(r_0 + m_2, r_0' + n_2; l_0 + m_1, l_0' + n_1) = \Delta_{j+1}^{-}(r_0 + m_2, r_0' + n_2; l_0 + m_1, l_0' + n_1) .$$

Substituting by (79) in (78), we obtain that (57) holds for $n = j + 1$. If alternative (b) holds the

(80) $$\Delta_{j+1}^{-}(r_0 + m_2, r_0' + n_2; l_0 + m_1, l_0' + n_1) = 0 .$$

It has already been shown from (67) that (58) holds for $n = j + 1$, i.e.,

(81) $$\Delta_{j+1}(r_0 + m_1, r_0' + n_1; l_0 + m_1, l_0' + n_1) \geqq 0 .$$

(80) and (81) combined give that (57) holds for $n = j + 1$. Thus under both alternatives (a) and (b), if (57) holds for $n = j$, it holds for $n = j + 1$. Since it holds for $n = 1$, it holds for all n. This completes the proof of Theorem 3.1. Corollary 3.1 follows as a trivial consequence.

4. Concluding remarks. The proof of Berry's conjecture is obtained as a particular case of Corollary 3.1 by putting $m_1 = n_1 = m_2 = n_2 = 0$. This completes the proof of the conjecture.

Under the assumption

(82) $$E(\rho \mid r_0, r_0') \geqq E(\rho \mid l_0, l_0'),$$

among other assumptions, we have shown that

(83) $$\Delta_n(r_0, r_0'; l_0, l_0') \geqq 0 \qquad \text{for all } n .$$

By modifying the proofs slightly, it is seen that the stronger result

(84) $$\Delta_n(r_0, r_0'; l_0, l_0') > 0$$

holds if either

$$E(\rho \mid r_0, r_0') > E(\rho \mid l_0, l_0') \tag{85 i}$$

or

$$E(\rho \mid r_0, r_0') = E(\rho \mid l_0, l_0'), \quad r_0 + r_0' < l_0 + l_0', \tag{85 ii}$$

and μ is not a one- or two-point distribution. If (85 i) holds then the argument from (34) to (35) shows that in this case, in Theorem 2.1, we obtain in place of

$$E(\rho \mid r_0 + m, r_0' + n) \geqq E(\rho \mid l_0 + m, l_0' + n)$$

the stronger result

$$E(\rho \mid r_0 + m_1, r_0' + n) > E(\rho \mid l_0 + m_1, l_0' + n). \tag{86}$$

If (85 ii) holds then we again obtain (86) by using the fact that strict inequality holds in (26) by (4.9)*. Then using the strict inequality in (86) and the arguments of Theorems 2.1, 3.1, we obtain the stronger result in (84).

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