The Annals of Statistics
1973, Vol. 1, No. 6, 1096–1104

A NEW NONPARAMETRIC ESTIMATOR OF THE CENTER OF A SYMMETRIC DISTRIBUTION

BY E. F. SCHUSTER AND J. A. NARVARTE

The University of Texas at El Paso

Let $F_n(x)$ be the empirical distribution function based on a random sample of size n from a continuous symmetric distribution with center θ. As a nonparametric estimator of θ, we propose a^* where a^* is chosen so as to minimize the function h where $h(a) = \max_x |F_n(x) + F_n((2a - x)^-) - 1|$. In this paper we present an algorithm for constructing the interval of all a which minimize h. We show that if a^* is chosen as the center of this interval then a^* is an unbiased estimator of θ which converges to θ with probability one at a rate of $n^{1/2-\delta}$ for $\delta > 0$. We then use the large or small sample distribution of $h(\theta)$ given by Butler (1969) to construct confidence intervals for θ and show how one can test for symmetry when the center is not specified under the null hypothesis.

1. Introduction. Let $F_n(x)$ be the usual empirical distribution function based on a random sample $x_1, \cdots, x_n$ from a continuous symmetric distribution function F with center θ. Since F is symmetric with center θ, $F(x) + F(2\theta - x) - 1 = 0$ for all x. It is well known that $F_n(x)$ converges uniformly to $F(x)$ with probability one (w.p. 1) and hence $F_n(x) + F_n((2\theta - x)^-) - 1$ will converge uniformly to 0 (w.p. 1).

In this paper we consider the nonparametric estimation of θ based on the information contained in $x_1, \cdots, x_n$. As an estimator of θ, we propose $a^* = a^*(x_1, \cdots, x_n)$ where a^* is chosen so as to minimize the function h where

$$h(a) = \max_x |F_n(x) + F_n((2a - x)^-) - 1| . \tag{1.1}$$

We present an algorithm for constructing the interval of all a which minimize (1.1). We show that if a^* is chosen as the center of this interval then a^* is an unbiased estimator of θ which converges to θ with probability one at a rate of $n^{1/2-\delta}$ for $\delta > 0$. We then use the large or small sample distribution of $h(\theta)$ given by Butler (1969) to construct confidence intervals for θ and show how one can test for symmetry when the center θ is not specified under the null hypothesis.

The main properties of the estimator a^* are given in Section 2. The proofs of the theorems are deferred to Section 3.

2. Main properties of a^*. Let $x_1, \cdots, x_n$ be the order statistics of a random sample of size n from a continuous distribution function F and let $F_n(x)$ be the empirical distribution function based on $x_1, \cdots, x_n$; i.e., $nF_n(x)$ equals the number of $x_i \leqq x$ where $1 \leqq i \leqq n$. Let $[x]$ denote the greatest integer less than or equal

Received June 1972; revised January 1973.
AMS 1970 *subject classification*. Primary 62G05.
Key words and phrases. Nonparametric estimator, center of symmetry, symmetric distribution, empirical distribution function.

to x and for each pair (i, j) with $1 \leqq i \leqq j \leqq n$, let $x_{ij} = (x_i + x_j)/2$. Then for each $k = 0, 1, \cdots, n-1$ we define

$$s(k) = \{x_{ij} : 1 \leqq i \leqq [(n-k+1)/2], j = n-k+1-i\},$$
$$S(k) = \{x_{ij} : k+1 \leqq i \leqq [(n+k+1)/2], j = n+k+1-i\},$$
$$m(k) = \max\{x \in s(k)\},$$

and

$$M(k) = \min\{x \in S(k)\}.$$

The following Theorem 1 indicates that the set of all a which minimize (1.1) (the set of minimax solutions) is an interval. The theorem gives an algorithm for finding this interval and indicates the corresponding value of $h(a)$ for any minimax solution a.

THEOREM 1. *The set $A = \{k : m(k) \leqq M(k)$, where $0 \leqq k \leqq n-1\}$ is nonempty and if $k^* = \min\{k \in A\}$, then the following are equivalent*:

(i) *a minimizes* (1.1).
(ii) $m(k^*) \leqq a \leqq M(k^*)$.
(iii) $nh(a) = k^*$.

REMARK. The referee has indicated that a^* can be obtained by a modification of the graphical procedure given by Jaeckel (1969, page 44). Concerning the method which is employed to define our estimate, he has pointed out three further references of interest, Wolfowitz (1957), Knüssel (1969) and Fine (1966).

Let k^* be as in Theorem 1 and let us assume that the sample $x_1, \cdots, x_n$ is from a continuous symmetric distribution function F with center θ. As an estimator of θ we propose the statistic a^* where $a^* = (m(k^*) + M(k^*))/2$. We then obtain

THEOREM 2. *The distribution function of a^* is symmetric with center θ.*

It now follows that a^* is unbiased for θ whenever $E(a^*)$ exists. In Lemma 6 of Section 3 we show that for $n \geqq 3$, $x_{[n/3]} \leqq a^* \leqq x_{n-[n/3]+1}$ and hence $E(a^*) = \theta$ whenever $E(x_{[n/3]})$ exists. Theorem 3 now formulates this result in terms of the natural necessary and sufficient condition for $E(x_{[n/3]}) < \infty$, for n sufficiently large, given by Bickel (1967). In this theorem we assume that the distribution F has a density f which is continuous on $\{x : 0 < F(x) < 1\}$.

THEOREM 3. *If $\lim_{|x|\to\infty} |x|^\alpha F(x)(1 - F(x)) = 0$ for some $\alpha > 0$ then $E(a^*) = \theta$ for n sufficiently large.*

If we are sampling from a Cauchy distribution with center θ, then the expected values of x_2 through x_{n-1} exist. So the above indicates that if $n \geqq 6$ then $E(a^*) = \theta$, so that a^* is unbiased for θ for all the well-known symmetric distributions.

The following Theorem 4 indicates the rate at which a^* converges to θ.

THEOREM 4. *If* $r_n = o(n^{\frac{1}{2}-\delta})$ *for some* $\delta > 0$, *then* $r_n(a^* - \theta)$ *converges to zero with probability one.*

The small and large sample distributions of $n^{\frac{1}{2}}h(\theta)$ are given by Butler (1969). In Theorem 5 we show how this distribution can be used to set confidence limits for the unknown θ.

THEOREM 5. (Confidence limits for θ.) *Let* d_α *be such that*

$$\lim_{n\to\infty} \Pr\{n^{\frac{1}{2}}h(\theta) \geqq d_\alpha\} = \alpha .$$

If $k = [n^{\frac{1}{2}}d_\alpha]$ *then*

$$\lim_{n\to\infty} \Pr\{m(k) \leqq \theta \leqq M(k)\} \geqq 1 - \alpha .$$

REMARK 1. Using Butler's result (1969) (it contains a misprint i.e., $\exp-((2n+1)^2\pi^2(8x^2))$ should be $\exp-((2n+1)^2\pi^2/(8x^2))$) we see that $\lim_{n\to\infty} \Pr\{n^{\frac{1}{2}}h(\theta) \geqq 2\} \simeq 0.910$. Hence $k = [2n^{\frac{1}{2}}]$ in Theorem 5 yields a (conservative) 91% confidence interval for θ. This also says that 91% of the time $k^* \leqq 2n^{\frac{1}{2}}$. In practice we have found k^* and a^* rather easy to obtain both manually for moderate sample sizes and on a computer for large sample sizes, and in the examples we have studied the estimator a^* performs quite well.

REMARK 2. Butler has noted that the statistic $h(\theta) = \max_x |F_n(x) + F_n((2\theta - x)^-) - 1|$ can be used in the usual manner to test the hypothesis of symmetry

$$H_0\colon F(x) + F(2\theta - x) - 1 = 0 , \qquad \text{all } x \tag{2.2}$$

against

$$H_1\colon F(x) + F(2\theta - x) - 1 \neq 0 , \qquad \text{some } x ,$$

when θ is completely specified under the null hypothesis H_0. He rejects H_0 at level α if the computed value of $n^{\frac{1}{2}}h(\theta) \geqq d_\alpha$, where $\Pr\{n^{\frac{1}{2}}h(\theta) \geqq d_\alpha\} = \alpha$.

If we want to test the hypothesis of symmetry in (2.2) when θ is not specified under H_0 then we propose the statistic

$$\begin{aligned} h(a^*) &= \max_x |F_n(x) + F_n((2a^* - x)^-) - 1| \\ &= \min_a \max_x |F_n(x) + F_n((2a - x)^-) - 1| . \end{aligned}$$

Since $h(a^*) \leqq h(\theta)$, we can use the above d_α as a conservative test, i.e., we reject H_0 at level α if the computed value of $n^{\frac{1}{2}}h(a^*) \geqq d_\alpha$ where $\Pr\{n^{\frac{1}{2}}h(\theta) \geqq d_\alpha\} = \alpha$. If the hypothesis of symmetry is accepted, then we use a^* as our estimate of θ.

REMARK 3. One result which we have not been able to obtain is the asymptotic distribution of $n^{\frac{1}{2}}(a^* - \theta)$. Hence a comparison of a^* with other existing estimates is not yet possible on this basis. P. J. Bickel has pointed out that one can argue as follows. Take $\theta = 0$, then,

$$\begin{aligned} &n^{\frac{1}{2}}(F_n(x) + F_n((2a - x)^-) - 1) \\ &\quad = n^{\frac{1}{2}}\{(F_n(x) - F(x)) + F_n((2a - x)^-) - F(2a - x)\} \\ &\qquad + 2n^{\frac{1}{2}}af(x) + o_p(1) \end{aligned}$$

at least in $n^{-\frac{1}{2}}$ neighborhoods of $a = 0$ ($F'(x) = f(x)$). The right-hand side behaves like

$$W^0(F(x)) + W^0(F(2a - x)^-) + 2n^{\frac{1}{2}}af(x) \approx 2^{\frac{1}{2}}W(F(x)) + 2n^{\frac{1}{2}}af(x)$$

where W, W^0 are used generically for Wiener and Brownian bridge processes. Hence Professor Bickel has conjectured that $n^{\frac{1}{2}}a^*$ has the same distribution as θ^* where

$$|2^{\frac{1}{2}}(F(x(\theta^*))) + 2\theta^* f(x(\theta^*))| = \min_\theta \sup_{x \leqq 0} |2^{\frac{1}{2}}W(F(x)) + 2\theta f(x)| .$$

So far we have been unable to prove this conjecture.

3. Proofs of theorems. Let $x_1, \cdots, x_n$ be the order statistics of a random sample of size n from a continuous distribution function F, $F_n(x)$ the empirical distribution function based on $x_1, \cdots, x_n$; and let $g(x, a) = F_n(x) - G_n(x)$, where $G_n(x) = 1 - F_n((2a - x)^-)$ as before. We proceed to the proofs of Theorems 1 and 2 via the following four lemmas.

LEMMA 1. *If* $0 \leqq k \leqq l \leqq n - 1$, *then* $m(k) \geqq m(l)$ *and* $M(k) \leqq M(l)$.

PROOF. It suffices to prove the result for $l = k + 1$. If $x \in s(k + 1)$, then $x = x_{ij}$ for some i and j satisfying $1 \leqq i \leqq [(n - k)/2]$ and $j = n - k - i$. But then $j \leqq n - 1$ so that $x_j \leqq x_{j+1}$, which implies that $x_{ij} \leqq x_{i,j+1}$. From this it follows that $m(k + 1) \leqq m(k)$ once we observe that $x_{i,j+1} \in s(k)$. The proof that $M(k) \leqq M(k + 1)$ follows by a similar argument by observing that if $x_{ij} \in S(k + 1)$, then $x_{ij} \geqq x_{i-1,j}$ where $x_{i-1,j} \in S(k)$.

LEMMA 2. *For each* $i = 1, 2, \cdots, n$, *let* $x_i' = 2a - x_{n-i+1}$, *and for each* $k = 0, 1, \cdots, n - 1$, *let* $s'(k)$, $S'(k)$, $m'(k)$ *and* $M'(k)$ *be defined for* $x_1', \cdots, x_n'$ *as in Section* 2. *Then* $m'(k) = 2a - M(k)$ *and* $M'(k) = 2a - m(k)$.

PROOF. Let $x'_{ij} = (x_i' + x_j')/2 \in s'(k)$. Then $1 \leqq i \leqq [(n - k + 1)/2]$ and $j = (n - k + 1) - i$, and

$$x'_{ij} = 2a - (x_{n-i+1} + x_{n-j+1})/2 = 2a - (x_{n-j+1} + x_{n-i+1})/2 = 2a - x_{i'j'} ,$$

where $i' = n - j + 1$ and $j' = n - i + 1$. Then since $j = (n - k + 1) - i$, it follows that $i' = k + i$ and $j' = (n + k + 1) - i'$, which, together with $1 \leqq i \leqq [(n - k + 1)/2]$, imply that $k + 1 \leqq i' \leqq [(n + k + 1)/2]$ and $j' = (n + k + 1) - i'$, so that $x_{i'j'} = 2a - x'_{ij} \in S(k)$. Similarly, if $x_{ij} \in S(k)$, it follows that $x_{ij} = 2a - x'_{i'j'}$, where $i' = n - j + 1$ and $j' = n - i + 1$, from which $1 \leqq i' \leqq [(n - k + 1)/2]$ and $j' = (n - k + 1) - i'$, so that $x'_{i'j'} = 2a - x_{ij} \in s(k)$. Hence it follows that $m'(k) = \max\{x \in s'(k)\} = \max\{2a - x : x \in S(k)\} = 2a - \min\{x \in S(k)\} = 2a - M(k)$. The proof that $M'(k) = 2a - m(k)$ follows in a similar fashion.

LEMMA 3. *If* $m(k) \leqq M(k)$, *then* $m(k) \leqq a \leqq M(k)$ *implies that* $nh(a) \leqq k$.

PROOF. Recalling that $g(x; a) = F_n(x) - G_n(x)$ is a step function whose jumps occur at the x_i's and y_i's, then $h(a) = \max_x |g(x; a)|$ is realized at either an x_i

or $y_j = 2a - x_j$. Here we can assume that $x_i \neq a$ for some i, for if otherwise, then $nh(a) = 0$ and the result follows. In fact $h(a)$ is realized at an x_i or a y_i different from a, since $x_i = a$ if and only if $y_i = 2a - x_i = a$, in which case one jump cancels the other. Suppose now that $h(a)$ is not realized to the left of a. We note that if $h(a)$ is realized at x_i then $g(x_i; a) \geqq 0$ and if $h(a)$ is realized at y_i then $g(y_i; a) \leqq 0$. If $h(a) = |g(x_{i_0}; a)| = g(x_{i_0}; a)$ for $x_{i_0} > a$, choose $\bar{x} = x_{i_0} + \varepsilon$, where $\varepsilon > 0$ is small enough to insure that $\bar{x}$ is strictly between x_{i_0} and any x_i and y_i to the right of x_{i_0}. Since $g(2a - x; a) = F_n(2a - x) - G_n(2a - x) = F_n(x^-) - G_n(x^-) = g(x^-; a)$, $g(2a - \bar{x}; a) = g(\bar{x}; a)$, and since $h(a) = g(x_{i_0}; a) = g(\bar{x}; a)$, then $h(a)$ is realized at $2a - \bar{x}$ which is a contradiction, since $a < x_{i_0} < \bar{x}$ places $2a - \bar{x}$ to the left of a. If $h(a) = |g(y_{i_0}; a)| = -g(y_{i_0}; a)$, then by choosing $\bar{y} = y_{i_0} + \varepsilon$, a similar argument shows that $h(a) = |g(2a - \bar{y}; a)|$, with $2a - \bar{y}$ to the left of a. Hence $h(a)$ is realized at an x_i or a y_i which is strictly to the left of a.

Suppose now that $h(a) = g(x_{i_0}; a)$ with $x_{i_0} < a$, and suppose further that $nh(a) > k$. Since $ng(x_{i_0}; a)$ equals the number of $x_i \leqq x_{i_0}$ less the number of $y_i \leqq x_{i_0}$, then we must have $i_0 \geqq k + 1$. Furthermore, to account for possible ties at x_{i_0}, we require that $i_0 = \max\{i : x_i = x_{i_0}\}$. Then by the original hypothesis, $a \leqq M(k) = \min\{x \in S(k)\}$, so that in particular $x_{i_0} < a \leqq x_{ij}$, where $i = [(n + k + 1)/2]$ and $j = (n + k + 1) - [(n + k + 1)/2] = [(n + k)/2] + 1$, and since $x_{ij} \leqq x_j$, it follows that $k + 1 \leqq i_0 \leqq [(n + k)/2]$. But if $j_0 = (n + k + 1) - i_0$, $x_{i_0 j_0} \in S(k)$ and since $a \leqq M(k) \leqq x_{i_0 j_0}$ we then have $y_{j_0} = 2a - x_{j_0} \leqq x_{i_0}$. Thus the number of $y_i \leqq x_{i_0}$ must be at least $n - j_0 + 1$, which equals $i_0 - k$, since $j_0 = (n + k + 1) - i_0$. Moreover, by the choice of i_0, the number of $x_i \leqq x_{i_0}$ is precisely i_0, so that $ng(x_{i_0}; a) \leqq i_0 - (i_0 - k) = k$. But this says that $nh(a) \leqq k$, which contradicts the supposition that $nh(a) > k$, so that the result follows providing that $h(a)$, which is always realized at an x_i or a y_i strictly to the left of a, is in fact realized at an x_i.

Suppose then that $h(a)$ is realized at some $y_{i_0} < a$. Then letting $x_i' = y_{n-i+1}$ and $y_i' = x_{n-i+1}$ for all $i = 1, 2, \cdots, n$ as in Lemma 2, we observe that $g(x; a)$, which is the number of $x_i \leqq x$ minus the number of $y_i \leqq x$, is exactly equal to the number of $y_i' \leqq x$ minus the number of $x_i' \leqq x$. Thus $g(x; a) = -g'(x; a)$, where $g'(x; a)$ is the corresponding step function for $x_1', \cdots, x_n'$, so that $h(a) = |g(y_{i_0}; a)| = -g(y_{i_0}; a) = g'(x'_{n-i_0+1}; a)$ is also realized at an $x_i' < a$. Moreover by Lemma 2, we have that $m'(k) = 2a - M(k)$ and $M'(k) = 2a - m(k)$, so that if $m(k) \leqq a \leqq M(k)$, it follows that $m'(k) \leqq a \leqq M'(k)$. But here we can appeal to the first part of the present proof to conclude that $nh(a) \leqq k$ which completes the proof of the lemma.

LEMMA 4. *If $a < m(k)$ or $M(k) < a$, then $nh(a) > k$.*

PROOF. Suppose $a < m(k)$. Then there exists an $x_{ij} \in s(k)$ for which $a < x_{ij}$, with $1 \leqq i \leqq [(n - k + 1)/2]$ and $j = (n - k + 1) - i$. But then, $y_j = 2a - x_j < 2x_{ij} - x_j = x_i$, from which it follows that $nG_n(x_i^-) \geqq n - j + 1 = i + k$.

But this says that $ng(x_i^-; a) = nF_n(x_i^-) - nG_n(x_i^-) \leqq (i-1) - (i+k) = -(k+1)$, so that $nh(a) = n \cdot \max_x |g(x; a)| = n \cdot \max_x |g(x^-; a)| \geqq n|g(x_i^-; a)| \geqq k+1$. If $M(k) < a$, then there exists an $x_{ij} \in S(k)$ for which $x_{ij} < a$, with $k+1 \leqq i \leqq [(n+k+1)/2]$ and $j = (n+k+1) - i$. But again, $y_j = 2a - x_j > 2x_{ij} - x_j = x_i$, so that $n - nG_n(x_i) \geqq j$, from which it follows that $nG_n(x_i) \leqq n - j = i - k - 1$, and thus

$$nh(a) = n \cdot \max_x |F_n(x) - G_n(x)| \geqq nF_n(x_i) - nG_n(x_i)$$
$$\geqq i - (i - k - 1) = k + 1 .$$

REMARK. Using the above lemmas one can easily show that if $m(k) \leqq M(k)$ (i.e., $k \geqq k^*$), then $nh(a) = k+1$ if and only if $m(k+1) \leqq a < m(k)$ or $M(k) < a \leqq M(k+1)$. This result could be used to compute $h(\theta)$.

With these lemmas we can now formulate the proof of

THEOREM 1. *The set* $A = \{k : m(k) \leqq M(k), 0 \leqq k \leqq n-1\}$ *is nonempty and if* $k^* = \min\{k \in A\}$, *then the following are equivalent;*

(i) a *minimizes* (1.1).
(ii) $m(k^*) \leqq a \leqq M(k^*)$.
(iii) $nh(a) = k^*$.

PROOF. The fact that A is nonempty follows by observing that $m(n-1) = x_1$ and $M(n-1) = x_n$ and since $x_1 \leqq x_n$, then $(n-1) \in A$. Suppose that a minimizes (1.1), and let $nh(a) = k$ for some k satisfying $0 \leqq k \leqq n-1$. Since $k^* = \min\{k \in A\} \in A$, then $m(k^*) \leqq M(k^*)$ and hence by Lemma 3, $nh(m(k^*)) \leqq k^*$, so that $k \leqq k^*$. Thus by Lemma 4, $nh(a) = k \leqq k^*$ implies that $m(k^*) \leqq a \leqq M(k^*)$ so that (i) implies (ii). Now suppose that $m(k^*) \leqq a \leqq M(k^*)$. Then by Lemma 3, $nh(a) = k \leqq k^*$, and by Lemma 4, $nh(a) = k$ implies that $m(k) \leqq a \leqq M(k)$ from which it follows that $k \in A$. But $k^* = \min\{k \in A\}$ so that $nh(a) = k = k^*$, and hence (ii) implies (iii). Finally, suppose $nh(a) = k^*$. If a does not minimize (1.1), then there is an a^* for which $nh(a^*) = k < k^*$, which implies by Lemma 4 that $m(k) \leqq a^* \leqq M(k)$, so that $k \in A$, which is a contradiction. Therefore a minimizes (1.1), so that (iii) implies (i) and the proof of the Theorem is complete.

In the remainder of the paper we will assume that the ordered sample $x_1, \cdots, x_n$ is from a continuous symmetric distribution function F with center θ, and that k^* is as in Theorem 1. As an estimator of θ we have proposed the statistic $a^* = a^*(x_1, \cdots, x_n)$ where $a^* = (m(k^*) + M(k^*))/2$. We now proceed to the properties of a^*, the first of which is Theorem 2. This theorem follows immediately from Lemma 2 and we omit its proof. Theorem 3 can then be obtained using Lemmas 5, 6, and Theorem 2.1 (a) of Bickel (1967).

LEMMA 5. *For* $n \geqq 2$, $nh(a^*) = k^* \leqq [(n+1)/3]$.

PROOF. Let $n \geqq 2$ and let $k = [(n+1)/3]$ (observe that $k < n$). It suffices to show that $m(k) \leqq M(k)$, for then $k^* = \min\{k : m(k) \leqq M(k)\} \leqq [(n+1)/3]$.

This would follow if $x \in s(k)$ implies $x \leqq y$ for any y in $S(k)$ (recall $m(k) = \max\{x \in s(k)\}$ and $M(k) = \min\{x \in S(k)\}$). Suppose then that $k = [(n+1)/3]$, $x \in s(k)$, and $y \in S(k)$. Then $x = x_{ij}$ for some i, j with $1 \leqq i \leqq [(n-k+1)/2]$, $j = n-k+1-i$ and $y = x_{i'j'}$ for some i', j' with $k+1 \leqq i' \leqq [(n+k+1)/2]$, $j' = n+k+1-i'$. Now $k = [(n+1)/3] \geqq (n-1)/3$. Then $2k+2 \geqq n-k+1$, so that $k+1 \geqq (n-k+1)/2 \geqq [(n-k+1)/2]$. Hence $i \leqq [(n-k+1)/2] \leqq k+1 \leqq i'$. Also $k \geqq (n-1)/3$ implies $4k+2 \geqq n+k+1$, so that $2k+1 \geqq (n+k+1)/2 \geqq [(n+k+1)/2]$. But then $j \leqq n-k \leqq (n+k+1) - [(n+k+1)/2] \leqq j'$. Since $i \leqq i'$ and $j \leqq j'$ it follows that $x_{ij} \leqq x_{i'j'}$ and the proof is complete.

LEMMA 6. *For* $n \geqq 3$, $x_{[n/3]} \leqq a^* \leqq x_{n-[n/3]+1}$.

PROOF. Suppose $a^* < x_{[n/3]}$. Then $n - nF_n(a^*) - nF_n(a^*)^- > n - 2[n/3] \geqq [(n+1)/3]$. But this contradicts Lemma 6 and hence $a^* \geqq x_{[n/3]}$. Suppose then that $a^* > x_{n-[n/3]+1}$. Then $nF_n(a^*) + nF_n(a^*)^- - n \geqq 2n - 2[n/3] + 2 - n = n - 2[n/3] + 2 \geqq [(n+1)/3] + 2 > [(n+1)/3]$. But this also contradicts Lemma 6. Hence $a^* \leqq x_{n-[n/3]+1}$ and the desired conclusion follows.

COROLLARY. $E(a^*) = \theta$ *whenever* $E(x_{[n/3]})$ *exists.*

Our Lemma 7 is needed in the proof of Theorem 4.

LEMMA 7. *Let* $C_1 > 0$ *and let* $b_n = C_1 n^k$ *where* $k > -\frac{1}{2}$. *If* $\varepsilon_n = \sup_x (F(b_n + x) - F(x))$ *where F is a continuous distribution function then* $\sum_{n=1}^{\infty} \exp(-C_2 n \varepsilon_n^2)$ *converges for all* $C_2 > 0$.

PROOF. If $k = 0$ then $b_n = C_1$. In this case ε_n equals some positive constant $a = a(C_1, F)$. When $k > 0$, $b_n \to \infty$ and $\varepsilon_n \to 1$. Hence there exists an $N = N(C_1, k, F)$ such that for $n \geqq N$, $\varepsilon_n \geqq \min(\frac{1}{2}, a) = b > 0$. The desired conclusion now follows for $k \geqq 0$ by observing that for $n \geqq N$,

$$\exp(-C_2 n \varepsilon_n^2) \leqq \exp(-C_2 n b^2)$$

and

$$\sum_{n=1}^{\infty} \exp(-Cn) \qquad \text{converges for all positive} \quad C\,.$$

Let us then consider the case when $-\frac{1}{2} < k < 0$. We claim there exists an $a = a(C_1, k, F) > 0$ and an $N = N(C_2, k, F)$ such that $\varepsilon_n / b_n \geqq a$ for $n \geqq N$. Suppose not, then there exists a subsequence $\{\varepsilon_{n_k}\}$ (with increasing indices) such that $\varepsilon_{n_k}/b_{n_k} \to 0^+$. But this says that the derivate (D^+) of the function $(-F)$ is everywhere nonnegative on each finite interval $[c, d]$. It then follows from an exercise in Royden (1963, page 84) that $F(d) \leqq F(c)$, which means that $F(d) = F(c)$. Since c and d are arbitrary F must be constant, but this is not possible since $F(-\infty) = 0$ and $F(+\infty) = 1$.

Hence there exist a and N such that $\varepsilon_n / b_n \geqq a$ for $n \geqq N$, so that

$$\begin{aligned} \exp(-C_2 n \varepsilon_n^2) &= \exp(-C_2 n b_n^2 \varepsilon_n^2 / b_n^2) \\ &\leqq \exp(-C_2 n b_n^2 a^2)\,. \end{aligned}$$

If $\delta = 1 + 2k$, then $nb_n^2 = nC_1^2n^{2k} = C_1^2n^{1+2k} = C_1^2n^\delta$. Since $-\frac{1}{2} < k < 0$, $0 < \delta < 1$. The proof of the theorem is then complete if one notes that $\exp(-C_2n\varepsilon_n^2) \leqq \exp(-C_2C_1^2a^2n^\delta)$ for $n \geqq N$ and that $\sum_{n=1}^\infty \exp(-Cn^\delta)$ converges for any $C > 0$.

We now prove

THEOREM 4. *If* $r_n = o(n^{\frac{1}{2}-\delta})$ *for some* $\delta > 0$, *then* $r_n(a^* - \theta)$ *converges to zero* (w.p. 1).

PROOF. By the Borel-Cantelli Lemma it suffices to show that

$$\sum_{n=1}^\infty \Pr\{r_n|a^* - \theta| > \varepsilon\}$$

converges for each $\varepsilon > 0$.

Let $b_n = \varepsilon/r_n$. Then since the distribution function of a^* is symmetric about θ (Theorem 2)

$$\Pr\{r_n|a^* - \theta| \geqq \varepsilon\} = \Pr\{|a^* - \theta| \geqq b_n\} = 2\Pr\{a^* - \theta \geqq b_n\}.$$

Let $\varepsilon_n = \sup_x (F(2b_n + 2\theta - x) + F(x) - 1)$. Then

$$\begin{aligned}\Pr\{a^* - \theta \geqq b_n\} &= \Pr\{2a^* - x \geqq 2b_n + 2\theta - x, \text{ all } x\}\\ &= \Pr\{F(2a^* - x) \geqq F(2b_n + 2\theta - x), \text{ all } x\}\\ &= \Pr\{F(2a^* - x) + F(x) - 1 \geqq F(2b_n + 2\theta - x)\\ &\qquad + F(x) - 1, \text{ all } x\}\\ &\leqq \Pr\{\sup_x |F(2a^* - x) + F(x) - 1| \geqq \varepsilon_n\}.\end{aligned}$$

Since a^* is a minimax solution (it minimizes (1.1)) and $F(x) + F(2\theta - x) = 1$, it is easy to see that

$$\begin{aligned}&\sup|F(2a^* - x) + F(x) - 1|\\ &\quad\leqq \sup|F_n(x) + F_n(2a^* - x) - 1| + \sup|F(2a^* - x) - F_n(2a^* - x)|\\ &\qquad + \sup|F_n(x) - F(x)|\\ &\quad\leqq \sup|F_n(x) + F_n(2\theta - x) - 1| + 2\sup|F_n(x) - F(x)|\\ &\quad\leqq 4\sup|F_n(x) - F(x)|.\end{aligned}$$

It now follows from Dvoretzky, Kiefer and Wolfowitz (1956) that there exists a universal constant C such that

$$\Pr\{a^* - \theta \geqq b_n\} \leqq \Pr\{\sup|F_n(x) - F(x)| > \varepsilon_n/4\} \leqq C\exp(-\varepsilon_n^2/8),$$

where $\varepsilon_n = \sup_x (F(2b_n + 2\theta - x) + F(x) - 1) = \sup_x (F(2b_n + x) - F(x))$. If we let $k = \delta - \frac{1}{2}$ and $c = 2\varepsilon$ then $2b_n = cn^k$ where $k > -\frac{1}{2}$. An application of Lemma 7 completes the proof.

Finally, Theorem 5 follows directly from Lemma 4.

REFERENCES

[1] BICKEL, P. J. (1967). Some contributions to the theory of order statistics. *Fifth Berkeley Symp. Math. Statist. Prob.* Univ. of California Press. 575–591.

[2] BUTLER, C. C. (1969). A test for symmetry using the sample distribution function. *Ann. Math. Statist.* **40** 2209–2210.

[3] DVORETZKY, A., KIEFER, J. and WOLFOWITZ, J. (1956). Asymptotic minimax character of the sample distribution function and of the classical multinomial estimator. *Ann. Math. Statist.* **27** 642–669.

[4] FINE, T. (1966). On the Hodges–Lehmann shift estimator in the two sample problem. *Ann. Math. Statist.* **37** 1814–1818.

[5] JAECKEL, L. A. (1969). Robust estimates of location. Unpublished Ph. D. thesis, Univ. of California, Berkeley.

[6] KNÜSSEL, L. F. (1969). Über minimum distance Schätzungen. Unpublished Ph. D. thesis Nr. 4380, ETH Zürich, Switzerland.

[7] ROYDEN, H. (1963). *Real Analysis*. Macmillan, New York.

[8] WOLFOWITZ, J. (1957). The minimum distance method. *Ann. Math. Statist.* **28** 75–88.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TEXAS
EL PASO, TEXAS

Note added in proof. R. C. Littell and P. V. Rao have recently informed us that the estimator a^* was considered as a one sample application of their work on the estimation of shift in the two sample shift problem which appears in "Robust estimation of shift parameters based on Kolmogorov–Smirnov statistics," Technical Report Number 30, 1971, Department of Statistics, University of Florida. The main overlap in our work and theirs is in the computing algorithm. Although their discussion in this direction (page 4) appears to contain a minor error (the procedure they outline estimates $-\theta$ instead of θ), the computing algorithm of our Section 2 is implicit in their subsequent two sample work. However, our work is of independent interest in that we do not make the assumption that our data points are distinct. Hence the (practical, not theoretical) problem of ties need not be considered separately. In addition, our algorithm exhibits the computational simplification inherent in the one sample case. Littell and Rao have also established some results concerning bounds on the asymptotic length of the confidence intervals and Schuster (Abstract 138–36, *IMS Bulletin*, May 1973) has proved Bickel's conjecture for all the well-known symmetric distributions.