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# CONSTRUCTIONS FOR SOME CLASSES OF NEIGHBOR DESIGNS

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Rees [2] introduced the concept and name of "neighbor" designs. The problem can be described as that of arranging $v$ kinds of objects on $b$ plates each containing $k$ objects in a loop such that every object on a plate has two neighbors. The requirements are that each object appears $r$ times (but not necessarily on $r$ different plates) and is a neighbor of every other object exactly $\lambda$ times. This paper constructs neighbor designs with parameters as follows:

(i) $k > 2$: $v = 2k + 1, \quad \lambda = 1$
(ii) $k \equiv 0 \pmod 2 > 2$: $v = 2^i k + 1, \quad i = 1, 2, \cdots, \lambda = 1$
(iii) $k \equiv 0 \pmod 4$: $v = 2mk + 1, \quad m = 1, 2, \cdots, \quad \lambda = 1$.

**1. Introduction.** Rees [2] introduced the concept and name of "neighbor designs" for use in serology. He wrote, "A technique used in virus research requires the arrangement in circles of samples from a number of virus preparations in such a way that over the whole set a sample from each virus preparation appears next to a sample from every other virus preparation."

Figure 1 shows such an arrangement of a set of antigens (virus preparations) around an antiserum on a plate. Hence, on the plate, every antigen has as neighbors two other antigens.

In general, there are $v$ kinds of antigens to be arranged on $b$ plates each

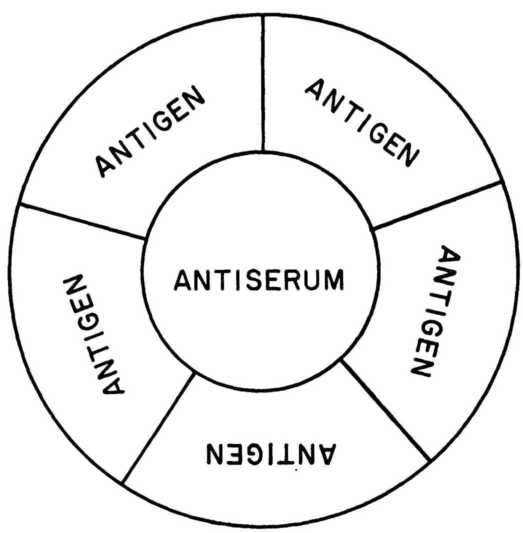


FIG. 1.

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containing $k$ antigens. Each antigen appears $r$ times (but not necessarily on $r$ different plates) and is a neighbor of every other antigen exactly $\lambda$ times. The following is an example for $v = 9$, $b = 9$, $k = 4$, $r = 4$, $\lambda = 1$:

$$P_1 = (5, 6, 4, 1)\,, \quad P_2 = (6, 7, 5, 2)\,, \quad P_3 = (7, 8, 6, 3)\,,$$
$$P_4 = (8, 9, 7, 4)\,, \quad P_5 = (9, 1, 8, 5)\,, \quad P_6 = (1, 2, 9, 6)\,,$$
$$P_7 = (2, 3, 1, 7)\,, \quad P_8 = (3, 4, 2, 8)\,, \quad P_9 = (4, 5, 3, 9)\,.$$

If the plate should also be a factor in the design, then one would like to add the requirements that all antigens on a plate are distinct as well as some balance in the number of times each pair of antigens appears on a plate. This seems to be the case Lawless [1] considered when he suggested that a series of Sprott's BIB designs [3] are neighbor designs too.

For the complete block case, i.e., $v = k$, Rees constructed neighbor designs for every odd $v$. For the incomplete block case, i.e., $v > k$, Rees constructed neighbor designs for every $v$ up to $v = 41$ whenever $k$ is not greater than 10 and $\lambda = 1$, some by using Galois field theory (namely, when $v = 3 \pmod 4$ is a prime power), but most others just by trial and error.

In this paper we construct some infinite classes of neighbor designs with parameters as follows:

$$\begin{array}{lll} \text{(i)} \quad k > 2: & v = 2k + 1\,, \quad \lambda = 1\,, & \\ \text{(ii)} \quad k \equiv 0 \pmod 2: & v = 2^i k + 1\,, \quad i = 1, 2, \cdots, & \lambda = 1\,, \\ \text{(iii)} \quad k \equiv 0 \pmod 4: & v = 2mk + 1\,, \quad m = 1, 2, \cdots, & \lambda = 1\,. \end{array}$$

Note that by repeating the $\lambda = 1$ designs $t$ times, we obtain corresponding designs for $\lambda = t$.

**2. Rees' cyclic method.** For $v = 2mk + 1$, Rees suggested deriving $m$ basic blocks and from each to derive $2mk$ others by cyclic addition to obtain a neighbor design in $m(2mk + 1)$ blocks of $k$ (as in the example above, in which $m = 1$). He noted, "The success of the method depends on arranging the numbers in the initial blocks in such a way that the combined set of forward and backward differences between neighboring elements takes on all the values 1 to $2mk$ (mod $v$) once." He also noted:

> "... All that is needed is to construct a basic block to satisfy the following conditions, (I) all the differences, forward and backward, must be distinct; (II) the sum of the forward differences must be zero (mod $v$)."

For example, the sequence of forward differences for the basic block (5 6 4 1), mod 9, is (6–5, 4–6, 1–4, 5–1); and the sequence of backward differences is (5–6, 6–4, 4–1, 1–5).

Note that condition (II) does not rule out the possibility that the partial sum over some subsequence of the forward differences can be zero. For example, Rees gave this sequence of forward differences $(1, -2, 3, -4, -5, 6, -7, 8)$ for the design $v = 33$, $k = 8$. Note that the partial sum $3 + (-4) + (-5) + 6 = 0$, implying that the third antigen and the seventh antigen are the same. Similar examples can be found for designs $v = 41$, $k = 10$ and $v = 27$, $k = 9$ given by Rees.

**3.** $k > 2$. We now construct neighbor designs by constructing basic blocks satisfying conditions (I) and (II). First consider $v = 2k + 1$, $k > 2$ and $\lambda = 1$.

Let $F_k(1) = (f_1, f_2, \cdots, f_k)$ denote the sequence of forward differences in the basic block and "$C$" a constant. Let $F_k(1) \circ C$ denote the sequence $(f_1 \circ C, f_2 \circ C, \cdots, f_k \circ C)$ where

$$\begin{aligned} f_i \circ C &= f_i + C \qquad \text{if } f_i \geqq 0 \\ &= f_i - C \qquad \text{if } f_i < 0 \end{aligned} \qquad \pmod{v}.$$

Note that the sequence of backward differences is $B_k(1) = -F_k(1) = (-f_1, -f_2, \cdots, -f_k) \pmod{v}$.

For $k = 3$, let $F_3(1) = (1, 2, -3)$, for $k = 4$, let $F_4(1) = (1, -2, -3, 4)$. For $k = 5$, let $F_5(1) = (1, -2, 3, 4, -6)$. For $k = 6$, let $F_6(1) = (1, -2, 3, -4, -5, 7)$. Then it is easy to verify that the sequence $F_k(1)$ sums to zero and takes on each value from 1 to $k$ (disregarding signs) once except that in $F_5(1)$ and $F_6(1)$, the value $k$ is replaced by $k + 1$. Therefore for $k = 3, 4, 5, 6$, $F_k(1)$ and $B_k(1) = -F_k(1)$ together, include each value $1, 2, \cdots, 2k \pmod{v}$ exactly once.

In general, supposing $F_k(1)$ is given for $k = 3, 4, \cdots, K - 1$ where $K \geqq 7$, we construct $F_K(1)$ by defining

$$\begin{aligned} F_K(1) &= (1, -2, -3, 4, F_{K-4}(1) \circ 4) \qquad && \text{if } K \text{ is even}, \\ F_K(1) &= (1, 2, -3, F_{K-3}(1) \circ 3) && \text{if } K \text{ is odd}. \end{aligned}$$

Then each sequence $F_K(1)$ sums to zero since for $x$ even there are equal numbers of positive and negative terms in $F_x(1)$. Furthermore, by induction, $F_K(1)$ takes on each value from 1 to $K$ (disregarding signs) exactly once except when $K = 1, 2 \pmod 4$, then the value $K$ is replaced by $K + 1$. Thus in all cases the sequences $F_K(1)$ and $B_K(1)$ together include every value from 1 to $2K \pmod{v}$ exactly once.

EXAMPLE. For $v = 15$, $k = 7$, $\lambda = 1$, then $F_7(1) = (1, 2, -3, 4, -5, -6, 7)$ and $P_1 = (1, 2, 4, 1, 5, 15, 9)$ is a basic block with $F_7(1)$ as its forward differences sequence. The whole design will be

$P_1 = (1, 2, 4, 1, 5, 15, 9)$, $P_2 = (2, 3, 5, 2, 6, 1, 10)$,
$P_3 = (3, 4, 6, 3, 7, 2, 11)$, $P_4 = (4, 5, 7, 4, 8, 3, 12)$,
$P_5 = (5, 6, 8, 5, 9, 4, 13)$, $P_6 = (6, 7, 9, 6, 10, 5, 14)$,
$P_7 = (7, 8, 10, 7, 11, 6, 15)$, $P_8 = (8, 9, 11, 8, 12, 7, 1)$,
$P_9 = (9, 10, 12, 9, 13, 8, 2)$, $P_{10} = (10, 11, 13, 10, 14, 9, 3)$,
$P_{11} = (11, 12, 14, 11, 15, 10, 4)$, $P_{12} = (12, 13, 15, 12, 1, 11, 5)$,
$P_{13} = (13, 14, 1, 13, 2, 12, 6)$, $P_{14} = (14, 15, 2, 14, 3, 13, 7)$,
$P_{15} = (15, 1, 3, 15, 4, 14, 8)$.

**4.** $k \equiv 0 \pmod 2 > 2$. For $v = 2mk + 1$ and $\lambda = 1$, we have $m$ basic blocks to start the cyclic developments. Let

$$G_k(m) = \{F_k(1), F_k(2), \cdots, F_k(m)\}$$

where $F_k(i)$ denotes the $i$th basic block. The requirements for a neighbor design are: (I) each $F_k(i)$ sums to zero and (II) $G_k(m)$ and $B_k(m) = -G_k(m)$ together take on every value from 1 to $2mk$ once.

For $k = 0 \pmod 2 > 2$, then neighbor designs for $v = 2^j k + 1$ ($j$ an arbitrary integer) and $\lambda = 1$ can be constructed by specifying

$$F_k(i) = F_k(1) \circ 2(i-1)k\,, \qquad i = 1, 2, 3, \cdots 2^{j-1}\,.$$

In Section 3, we have shown that $F_k(1)$ sums to zero and has equal numbers of positive terms and negative terms. Hence $F_k(i)$ sums to zero for each $i$ and requirement (I) is met. Furthermore, from the definition of $F_k(i)$, we obtain the recursion relation

$$G_k(2^l) = \{G_k(2^{l-1})\,, \qquad G_k(2^{l-1}) \circ 2^{l-1}k\}\,, \qquad \text{for } l = 1, 2, \cdots, j-1\,.$$

We prove that $G_k(2^{j-1})$ and $-G_k(2^{j-1})$ together take on every value from 1 to $2^j k$ once by induction. In Section 3, we have shown that $G_k(1)(= F_k(1))$ and $-G_k(1)$ together take on every value from 1 to $2k \pmod{2k}$, or equivalently, take on every value in $\{\pm\, i : i = 1, 2, \cdots, k\}$ once. Assuming

$$\{G_k(2^{j-2}), -G_k(2^{j-2})\} = \{\pm\, i : i = 1, 2, \cdots, 2^{j-2}k\}\,,$$

then,

$$\{G_k(2^{j-2}) \circ 2^{j-2}k,\ -G_k(2^{j-2}) \circ 2^{j-2}k\} = \{\pm\, i : i = 2^{j-2}k + 1, 2^{j-2}k + 2, \cdots, 2^{j-1}k\}\,.$$

Hence

$$\{G_k(2^{j-1}), -G_k(2^{j-1})\} = \{\pm\, i : i = 1, 2, \cdots, 2^{j-1}k\}$$

and requirement (II) is met.

EXAMPLE. For $v = 49$, $k = 6$, $\lambda = 1$ (hence $j = 3$), then

$$F_6(1) = (1, -2, 3, -4, -5, 7)\,,$$
$$F_6(2) = (13, -14, 15, -16, -17, 19)\,,$$
$$F_6(3) = (25, -26, 27, -28, -29, 31)\,,$$
$$F_6(4) = (37, -38, 39, -40, -41, 43)\,.$$

**5.** $k \equiv 0 \pmod 4$. For $k \equiv 0 \pmod 4$, then neighbor designs for $v = 2mk + 1$ ($m$ an arbitrary integer) and $\lambda = 1$ can be constructed by specifying

$$F_k(i) = \{\varepsilon_y(ki - k + y) : y = 1, 2, \cdots, k\}\,, \qquad i = 1, 2, \cdots, m\,,$$

where

$$\begin{aligned} \varepsilon_y &= \phantom{-}1, \quad &&\text{if } y \equiv 0, 1 \pmod 4\,, \\ &= -1, &&\text{if } y \equiv 2, 3 \pmod 4\,. \end{aligned}$$

Disregarding the signs, then $G_k(m)$ is just the set of numbers $\{1, 2, 3, \cdots, mk\}$, hence requirement (II) is met. Now the signs are repeatedly of the pattern "$+ - - +$", therefore every four consecutive numbers counting from the start sums to zero. Since $k \equiv 0 \pmod 4$, $F_k(i)$ sums to zero for each $i$ and requirement (I) is met.

EXAMPLE. For $v = 17$, $k = 4$, $\lambda = 1$ (hence $m = 2$), then

$$F_4(1) = (1, -2, -3, 4), \qquad F_4(2) = (5, -6, -7, 8).$$

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