

LOWER BOUNDS FOR THE ASYMPTOTIC BAYES RISK IN THE SCALE MODEL (WITH AN APPLICATION TO THE SECOND-ORDER MINIMAX ESTIMATION)¹

BY L. GAJEK AND M. KAŁUSZKA

Polish Academy of Sciences and Technical University of Łódź

The problem of Bayes estimation of the scale parameter is considered. Lower bounds for the asymptotic Bayes risk are given as the restricted parameter space increases to the positive half-line. The results are next applied to establish the second-order minimax estimator of the scale parameter. Surprisingly, the least favorable distribution coincides with that for the corresponding location parameter problem.

1. Introduction. Let Y be a random variable with Lebesgue density

$$f(y, \lambda) = \lambda^{-1} f_1(y/\lambda), \quad y > 0,$$

where $\lambda \in \Lambda$. This paper concerns Bayes and minimax estimation of λ^s , $s \neq 0$, under the quadratic loss normalized by λ^{-2s} , when the parameter space Λ is restricted to some interval (λ_1, λ_2) for $0 < \lambda_1 < \lambda_2 < \infty$. Roughly speaking, any restrictions on the parameter space Λ enable one to localize the unknown parameter λ with improved precision when compared with the model without restrictions $\Lambda_0 = (0, \infty)$. In particular,

$$V = 1 - a_s^2/a_{2s},$$

where $a_s = EX^s$, which is the minimax risk (i.e., the greatest Bayes risk) in the model without restrictions, is a good level to compare improvements. Before describing the main results of the paper, let us formulate the problem in an equivalent but more convenient form. Let us notice that after applying the transformation $X = \log Y$ and $\theta = \log \lambda$, we arrive at the equivalent location parameter family

$$f(x, \theta) = f_0(x - \theta), \quad x \in \mathbb{R},$$

where $f_0(u) = e^u f_1(e^u)$ and $\theta \in \Theta = \{\log \lambda, \lambda \in \Lambda\}$. Now, the corresponding estimation problem for the location parameter family is to estimate $e^{s\theta}$ under the loss

$$L(a, \theta) = (e^{s\theta} - a)^2 e^{-2s\theta}$$

for $\theta \in (\theta_1, \theta_2)$, where $\theta_1 = \log \lambda_1$ and $\theta_2 = \log \lambda_2$. Let $\delta = \delta(x)$ be an estimator of $e^{s\theta}$ and let $R(\delta, \theta) = E_\theta L(\delta(X), \theta)$. Let $g(\theta)$ be a prior density with respect to Lebesgue

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measure on Θ . The expected risk relative to g is $R(\delta, g) = \int R(\delta, \theta)g(\theta) d\theta$ and the Bayes risk for g is $R(g) = \inf_{\delta} R(\delta, g)$. Let δ_g denote the Bayes estimator corresponding to the prior g . Assume that $-\theta_1 = \theta_2 = m > 0$.

In Section 2 we give an upper bound for

$$\limsup_{m \rightarrow \infty} m^2 \{1 - a_s^2/a_{2s} - R(\delta_m, g_m)\},$$

where (g_m) is a sequence of priors on $(-m, m)$.

In Section 3 we construct a sequence of least favorable priors as well as the corresponding family of minimax estimators, which we call, according to Levit (1980), minimax of the second order. It turns out that the least favorable distributions, after rescaling to $(-1, 1)$, are converging weakly to a distribution with density $\cos^2(\pi x/2)$. This distribution was proven earlier to be the least favorable in the problem of second-order minimax estimation of θ , instead of $e^{s\theta}$, under the quadratic loss with a constant weight; see Levit (1980), Bickel (1981) and Melkman and Ritov (1987).

As pointed out by one of the referees the results of the last section can be alternatively proven using the approach of Melkman and Ritov although the regularity assumptions do not coincide exactly for both approaches.

2. Asymptotic bounds for the Bayes risk. In the sequel we shall use the following extension of the Van Trees inequality [see Van Trees (1968)]:

$$(1) \quad R(\delta, g_m) \geq \frac{[\int \int s e^{s\theta} f_0(x - \theta) h(x, \theta) g_m(\theta) dx d\theta]^2}{\int \int \left(\left\{ (\partial/\partial\theta) [f_0(x - \theta) h(x, \theta) g_m(\theta)] \right\}^2 / f_0(x - \theta) g_m(\theta) \right) e^{2s\theta} dx d\theta},$$

which follows from Cauchy's inequality if the following assumptions hold:

- A1: $f_0(\cdot)h(x, \cdot)g_m(\cdot)$ is absolutely continuous a.e.,
- A2: $\int \int |h(x, \theta)|f_0(x - \theta)e^{s\theta}g_m(\theta) dx d\theta < \infty$,
- A3: the function $\gamma_m(\theta) = g_m(\theta)$, $\theta \in \text{supp } g_m$, $= 0$, otherwise, is absolutely continuous, where from now on for any real function g , $\text{supp } g = \{x \in \mathbb{R}: g(x) \neq 0\}$.

One can also treat (1) as a variation of a global Cramér-Rao inequality of Bobrovsky, Mayer-Wolf and Zakai (1987) with the difference that $e^{s\theta}$, instead of θ , is estimated. As shown in Gajek and Kaluszka (1989), the equality in (1) also holds outside the exponential family of distributions, if h is chosen properly. For more information concerning related results we refer the reader to these authors as well as to Brown and Gajek (1990) and Bobrovsky, Mayer-Wolf and Zakai (1987).

From (1), it follows that

$$(2) \quad \limsup_{m \rightarrow \infty} m^2 \left[1 - \frac{a_s^2}{a_{2s}} - R(\delta, g_m) \right] \leq \limsup_{m \rightarrow \infty} m^2 \left[1 - \frac{a_s^2}{a_{2s}} - \frac{L^2}{M} \right],$$

where L^2 and M denote the numerator and the denominator of the right side of (1), respectively. So to bound the second-order behavior of the global risk of

any (possibly depending on m) estimator, it is enough to investigate the right side of (2).

Given a prior g_m with $\text{supp } g_m \subseteq [-m, m]$, we denote by g_m^* its version normalized to the interval $[-1, 1]$, that is, $g_m^*(v) = m^{-1}g_m(v/m)$. Let \mathcal{G} denote a class of family of priors $g_m \in C^1[-m, m]$ such that for all $g_m \in \mathcal{G}$ the Fisher information $I_m = \int_{-1}^1 \{[g_m^*(v)]'\}^2/g_m^*(v) dv$ is finite and there are functions φ_m such that

1. φ_m are two times continuously differentiable on $\text{supp } \varphi_m = (-1 - \beta_m, 1 + \beta_m)$, where $\beta_m \geq m^{-1}$,
2. $\lim_{m \rightarrow \infty} m^{-1} \sup_{x \in \text{supp } \varphi_m} |\varphi_m^{(i)}(x)| = 0$ for $i = 0, 1, 2$,
3. $\int_{-1}^1 [g_m^*(v)]' \varphi_m(v) dv = I_m + o(1)$,
4. $\int_{-1}^1 g_m^*(v) \varphi_m^2(v) dv = I_m + o(1)$,
5. $\int_{-1}^1 g_m^*(v) \varphi_m(v) dv = o(m^{-1})$.

REMARK 2.1. Conditions 3–5 above say that φ_m behaves asymptotically, as $m \rightarrow \infty$, like $(d/dx) \log g_m^*(x)$. In fact, to construct a proper φ_m , one can proceed in the following way. Let g_m be an even function differentiable on \mathbb{R} and three times differentiable on $(-m, m)$. If for some nonnegative reals α_m such that $\alpha_m \rightarrow 0$ and $\alpha_m m^{\gamma/(3\gamma+1)} \rightarrow \infty$, with $\gamma > 0$, the following conditions hold:

$$(3) \quad \lim_{m \rightarrow \infty} m^{-k/3} \sup_{|x| < 1 - \alpha_m} \frac{|g_m^{*(k)}(x)|}{g_m^*(x)} = 0 \quad \text{for } k = 1, 2, 3,$$

then one can define φ_m by

$$\varphi_m(x) = \begin{cases} \frac{d}{dx} \log g_m^* \left(\frac{1 - \alpha_m x}{1 + \beta_m} \right), & \text{when } |x| < 1 + \beta_m, \\ 0, & \text{otherwise,} \end{cases}$$

where β_m is any sequence satisfying $m^{-1} \leq \beta_m \leq \alpha_m$.

Easy examples are $g_m^*(x) = (1 - x^{2m})^2 c_m$, with c_m being a normalizing constant, and $g_m^*(x) = \cos^2(\pi x/2)$. The latter prior will turn out to be asymptotically least favorable.

THEOREM 2.2. Suppose f_0 satisfies the following conditions:

- (i) $\int (1 + e^{2su}) u^4 f_0(u) du < \infty$,
- (ii) $\int \left[\frac{F_{2s}(u)}{a_{2s}} - \frac{F_s(u)}{a_s} \right]^2 \frac{u^i e^{2ui}}{f_0(u)} du < \infty$ for $i = 0, 2$,
- (iii) $\int \left[\int_x^\infty \left(\frac{F_{2s}(u)}{a_{2s}} - \frac{F_s(u)}{a_s} - \frac{b}{a_{2s}} e^{2su} f_0(u) \right) du \right]^2 \frac{e^{-2sx}}{f_0(x)} dx < \infty$,

where $F_r(u) = \int_u^\infty e^{rx} f_0(x) dx$. If $g_m \in \mathcal{G}$, then any sequence (δ_m) of estimators of $e^{s\theta}$ satisfies the following inequality

$$\limsup_{m \rightarrow \infty} m^2 \left[1 - \frac{a_s^2}{a_{2s}} - R(\delta_m, g_m) \right] \leq b^2 \frac{a_s^2}{a_{2s}} \limsup_{m \rightarrow \infty} I_m,$$

where $b = a_{2s,1}/a_{2s} - a_{s,1}/a_s$ and $a_{s,r} = \int x^r e^{sx} f_0(x) dx$.

PROOF. Let us define $h_0(x, \theta) = h(x, \theta)e^{-s\theta}$. Then for any sequence of estimators (δ_m) , we have, from (1),

$$(4) \quad R(\delta_m, g_m) \geq \frac{L^2}{M_1 + M_2 + M_3},$$

where

$$\begin{aligned} L &= s \int g_m(\theta) \int h_0(x, \theta) f_0(x - \theta) dx d\theta, \\ M_1 &= \int g_m(\theta) \int \left(\frac{\partial}{\partial \theta} [h_0(x, \theta) e^{s(x-\theta)} f_0(x - \theta)] \right)^2 e^{-2s(x-\theta)} f_0^{-1}(x - \theta) dx d\theta, \\ M_2 &= 2 \int g'_m(\theta) \int \frac{\partial}{\partial \theta} [h_0(x, \theta) e^{s(x-\theta)} f_0(x - \theta)] h_0(x - \theta) e^{-s(x-\theta)} f_0(x - \theta) \\ &\quad \times e^{-2s(x-\theta)} f_0^{-1}(x - \theta) dx d\theta, \\ M_3 &= \int \frac{[g'_m(\theta)]^2}{g_m(\theta)} \int (h_0(x, \theta) e^{s(x-\theta)} f_0(x - \theta))^2 e^{-2s(x-\theta)} f_0^{-1}(x - \theta) dx d\theta. \end{aligned}$$

A crucial point of the proof is that if h_0 is chosen properly, then the equality in (4) holds. So assume that h_0 satisfies the following equation:

$$(5) \quad \begin{aligned} h_0(x, \theta) e^{s(x-\theta)} f_0(x - \theta) &= a_s [a_{2s}^{-1} F_{2s}(x - \theta) - a_s^{-1} F_s(x - \theta)] \\ &\quad + m^{-1} \varphi_m(x/m) B(x - \theta), \end{aligned}$$

with

$$(6) \quad B(x) = \int_x^\infty A(v) e^{sv} dv,$$

where

$$(7) \quad A(v) = e^{-sv} [ba_s f_0(v) e^{2sv} a_{2s}^{-1} - a_s a_{2s}^{-1} F_{2s}(v) + F_s(v)].$$

One can get (5) from necessary and sufficient conditions that the equality in (1) be satisfied after dropping all terms of order $o(m^{-2})$. Now we will find an

asymptotic expansion of M_1, M_2, M_3 and L when $m \rightarrow \infty$. Applying the change of variables $u = x - \theta$ and $v = \theta/m$, we get

$$M_1 = 1 - \frac{a_s^2}{a_{2s}} + 2m^{-1} \int g_m^*(v) \int \left(\frac{a_s}{a_{2s}} e^{su} - 1 \right) \varphi_m \left(\frac{u}{m} + v \right) A(u) du dv + m^{-2} \int g_m^*(v) \int \varphi_m^2 \left(\frac{u}{m} + v \right) A^2(u) f_0^{-1}(u) du dv.$$

Using Taylor's expansion of φ_m at v and integrating by parts, we get

$$(8) \quad M_1 = 1 - \frac{a_s^2}{a_{2s}} + 2m^{-1} \int g_m^*(v) \varphi_m(v) dv \int \left(\frac{a_s}{a_{2s}} e^{su} - 1 \right) A(u) du + m^{-2} \left[2 \int g_m^*(v) \varphi_m'(v) dv \int \left(\frac{a_s}{a_{2s}} e^{su} - 1 \right) u A(u) du + I_m \int A^2(u) f_0^{-1}(u) du \right] + o(m^{-2})$$

$$= 1 - \frac{a_s^2}{a_{2s}} + m^{-2} I_m \left[-2 \int \left(\frac{a_s}{a_{2s}} e^{su} - 1 \right) u A(u) du + \int A^2(u) f_0^{-1}(u) du \right] + o(m^{-2}),$$

where the fact that $g_m \in \mathcal{G}$ was taken into account. Applying now the same change of variables for M_2 , we arrive at

$$M_2 = 2m^{-2} \int g_m^*(v) \int \varphi_m \left(\frac{u}{m} + v \right) \times e^{-su} \left[a_s \left(\frac{F_{2s}(u)}{a_{2s}} - \frac{F_s(u)}{a_s} \right) A(u) f_0^{-1}(u) + \left(\frac{a_s}{a_{2s}} e^{su} - 1 \right) B(u) \right] du dv + 2m^{-3} \int g_m^*(v) \int A(u) B(u) e^{-su} f_0^{-1}(u) \varphi_m^2 \left(\frac{u}{m} + v \right) du dv.$$

Using Taylor's expansion of φ_m at v , we get

$$(9) \quad M_2 = m^{-2} I_m \int \left[a_s \left(\frac{F_{2s}(u)}{a_{2s}} - \frac{F_s(u)}{a_s} \right) \frac{A(u)}{f_0(u)} + \left(\frac{a_s}{a_{2s}} e^{su} - 1 \right) B(u) \right] e^{-su} du + o(m^{-2}),$$

where condition 4 of φ_m was taken into account. The same procedure applied to M_3 gives

$$(10) \quad M_3 = m^{-2} I_m \int \left[\frac{a_s}{a_{2s}} F_{2s}(u) - F_s(u) \right]^2 e^{-2su} f_0^{-1}(u) du + o(m^{-2}).$$

Finally, it is easy to see that

$$L = 1 - \frac{\alpha_s^2}{a_{2s}} + \frac{s}{m} \int g_m^*(v) \int \varphi_m\left(\frac{u}{m} + v\right) B(u) e^{-su} du dv.$$

Applying Taylor’s expansion of φ_m at v , we get

$$(11) \quad L = 1 - \frac{\alpha_s^2}{a_{2s}} + m^{-2} I_m s \int u B(u) e^{-su} du + o(m^{-2}),$$

since $\int g_m^*(v) \varphi_m'(v) dv = -\int [g_m^*(v)]' \varphi_m(v) dv + o(1)$ and condition 3 holds. Taking into account (8)–(11) gives

$$\begin{aligned} & 1 - \frac{\alpha_s^2}{a_{2s}} - \frac{L^2}{M_1 + M_2 + M_3} \\ &= m^{-2} \left\{ I_m b^2 \frac{\alpha_s^2}{a_{2s}} + 2 \int \left(\frac{\alpha_s}{a_{2s}} e^{su} - 1 \right) \right. \\ & \quad \left. \times [B(u) e^{-su} - uA(u)] du - 2s \int u B(u) e^{-su} du \right\} + o(m^{-2}). \end{aligned}$$

Observe that

$$s \int e^{-su} u B(u) du = \int e^{-su} [B(u) - e^{-su} A(u)] du$$

and

$$\int B(u) du = \int e^{su} u A(u) du,$$

which holds because of Lemma 2.3 below. Therefore

$$(12) \quad 1 - \frac{\alpha_s^2}{a_{2s}} - \frac{L^2}{M_1 + M_2 + M_3} = m^{-2} I_m b^2 \frac{\alpha_s^2}{a_{2s}} + o(m^{-2}).$$

Now the result follows from (4) and (12). \square

In the proof of Theorem 2.2 the following lemma is applied.

LEMMA 2.3. *Suppose f satisfies conditions (i)–(iii) of Theorem 2.2. Then the following integrals are finite:*

$$J_1 = \int u^2 |A(u)| du, \quad J_2 = \int A^2(u)/f_0(u) du, \quad J_3 = \int u^2 A^2(u)/f_0(u) du,$$

$$J_4 = \int u^2 e^{su} |A(u)| du, \quad J_5 = \int |uB(u)| du, \quad J_6 = \int u^2 e^{-su} |B(u)| du$$

and

$$J_7 = \int B^2(u) e^{-2su} / f_0(u) du,$$

where $s \neq 0$ and B and A are defined by (6) and (7).

PROOF. In each case the proof is straightforward. For instance, $J_6 < \infty$ because the Cauchy inequality gives

$$\begin{aligned} J_6^2 &= \left(\int |B(u)|e^{-su}f_0(u)^{-1/2}f_0(u)^{1/2}u^2 du \right)^2 \\ &\leq \int |B(u)|^2e^{-2su}f_0(u)^{-1} du \int u^4f_0(u) du, \end{aligned}$$

which is finite by assumptions (i) and (iii). \square

REMARK 2.4. In Gajek and Kałuszka (1989) a method of simplifying (ii) and (iii) was given. In fact, sufficient conditions for (ii) and (iii) to hold are

$$(ii') \quad \int e^{4su}u^2f_0(u) du < \infty,$$

and

$$(iii') \quad \sup_{x \geq 0} \frac{1 - F_0(x)}{f_0(x)} < \infty, \quad \sup_{x < 0} \frac{F_0(x)}{f_0(x)} < \infty,$$

which are sometimes easier to verify than (ii) and (iii).

3. Second-order minimax estimators. Since $R(\delta_m, g_m) \leq \sup R(\delta_m, \theta)$, it follows from Theorem 2.2 that

$$(13) \quad \limsup_{m \rightarrow \infty} m^2 \left[1 - a_s^2/a_{2s} - \sup_{\theta} R(\delta_m, \theta) \right] \leq b^2 a_s^2/a_{2s} \limsup_{m \rightarrow \infty} \int_{-1}^1 \frac{\{[g_m^*]'\}^2}{g_m^*}.$$

Observe that the left side of (13) does not depend on g_m^* , so to get the best bound for asymptotic minimax value, one should look for g_m^* such that

$$\limsup_{m \rightarrow \infty} \int_{-1}^1 \{[g_m^*(u)]'\}^2 / g_m^*(u) du = \min!$$

To solve the above minimization problem, it is enough to construct a sequence of priors from \mathcal{G} converging weakly to a distribution with density $\cos^2(\pi x/2)$ which produces the minimum Fisher information equal to π^2 [see Melkman and Ritov (1987)]. Thus we arrive at the following result.

PROPOSITION 3.1. *Suppose f_0 satisfies conditions (i)–(iii) of Theorem 2.2. Then*

$$(14) \quad \limsup_{m \rightarrow \infty} m^2 \left[1 - a_s^2/a_2 - \sup_{-m \leq \theta \leq m} R(\delta_m, \theta) \right] \leq \pi^2 b^2 a_s^2/a_{2s}.$$

Now, we will show that there is a sequence (δ_m^*) of estimators for which the equality in (14) is achieved. According to Levit (1980), this sequence of estimators is called minimax of the second order.

Let us define

$$(15) \quad \delta_m^*(x) = a_s e^{sx} \left[1 + bm^{-1} \varphi_m(x/m) \right] / a_{2s},$$

where φ_m corresponds to $g_m^*(x) = \cos^2(\pi x/2)$ via Remark 2.1. The sequence of estimators was heuristically constructed taking into account conditions under which the equality in (1) holds.

THEOREM 3.2. *If a_{2s} , $a_{s,2}$ and $a_{2s,2}$ are finite, then*

$$\limsup_{m \rightarrow \infty} m^2 \left[1 - a_s^2/a_{2s} - \sup_{-m \leq \theta \leq m} R(\delta_m^*, \theta) \right] = b^2 a_s^2 \pi^2 / a_{2s}.$$

If, in addition, conditions (i)–(iii) of Theorem 2.2 are satisfied, then (δ_m^) defined by (15) is minimax of the second order.*

PROOF. Since

$$\begin{aligned} R(\delta_m^*, \theta) &= \int \left[\frac{a_s}{a_{2s}} e^{su} - 1 + \frac{b}{m} \varphi_m \left(\frac{u+\theta}{m} \right) \frac{a_s}{a_{2s}} e^{su} \right]^2 f_0(u) du \\ &= 1 - \frac{a_s^2}{a_{2s}} + \frac{2}{m} \frac{ba_s}{a_{2s}} \int \left(\frac{a_s}{a_{2s}} e^{su} - 1 \right) e^{su} \varphi_m \left(\frac{u+\theta}{m} \right) f_0(u) du \\ &\quad + m^{-2} \left(\frac{ba_s}{a_{2s}} \right)^2 \int \varphi_m^2 \left(\frac{u+\theta}{m} \right) e^{2su} f_0(u) du, \end{aligned}$$

therefore

$$(16) \quad \begin{aligned} &1 - \frac{a_s^2}{a_{2s}} - \sup_{|\theta| \leq m} R(\delta_m^*, \theta) \\ &= - \sup_{|v| \leq 1} \left\{ \frac{2}{m} \frac{ba_s}{a_{2s}} \int \left(\frac{a_s}{a_{2s}} e^{su} - 1 \right) e^{su} \varphi_m(u/m+v) f_0(u) du \right. \\ &\quad \left. + m^{-2} \left(\frac{ba_s}{a_{2s}} \right)^2 \int \varphi_m^2(u/m+v) e^{2su} f_0(u) du \right\}. \end{aligned}$$

Now, using Taylor’s formula for $\varphi_m(u/m+v)$ at v , we obtain

$$(17) \quad \begin{aligned} &\left| \int \left(\frac{a_s}{a_{2s}} e^{su} - 1 \right) e^{su} \varphi_m \left(\frac{u}{m} + v \right) f_0(u) du + m^{-1} \int \left(\frac{a_s}{a_{2s}} e^{su} - 1 \right) \right. \\ &\quad \left. \times e^{su} \varphi_m'(v) u f_0(u) du \right| \\ &\leq m^{-2} \sup_{|x| \leq 1+\beta m} |\varphi_m''(x)| \int \left| \frac{a_s}{a_{2s}} e^{su} - 1 \right| e^{su} u^2 f_0(u) du, \end{aligned}$$

because $\int (a_s e^{su} / a_{2s} - 1) e^{su} f_0(u) du = 0$. In a similar way

$$\begin{aligned}
 & \left| \int \varphi_m^2(u/m + v) e^{2su} f_0(u) du - \varphi_m^2(v) a_{2s} \right| \\
 (18) \quad & \leq 2m^{-1} \sup |\varphi_m| \sup |\varphi'_m| \int |u| e^{2su} f_0(u) du \\
 & \quad + m^{-2} \sup |\varphi'_m|^2 \int u^2 e^{2su} f_0(u) du.
 \end{aligned}$$

Since $\sup |\varphi_m| \sup |\varphi'_m| / m = o(1)$ and $\sup |\varphi''_m| / m = o(1)$, (16)–(18) altogether give

$$\begin{aligned}
 1 - \frac{a_s^2}{a_{2s}} - \sup_{|\theta| \leq m} R(\delta_m^*, \theta) &= -m^{-2} \sup_{|v| \leq 1} \left[2\varphi'_m(v) b^2 \frac{a_s^2}{a_{2s}} + \varphi_m^2(v) b^2 \frac{a_s^2}{a_{2s}} \right] \\
 & \quad + o(m^{-2}).
 \end{aligned}$$

Since

$$\lim_{m \rightarrow \infty} \sup_{|v| \leq 1} [2\varphi'_m(v) + \varphi_m^2(v)] = -\pi^2,$$

the result holds. \square

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POLITECHNIKA ŁÓDZKA
 AL. POLITECHNIKI 11
 90-924 ŁÓDŹ
 POLAND