The Annals of Statistics
1996, Vol. 24, No. 2, 628–640

NONPARAMETRIC LIKELIHOOD RATIO CONFIDENCE BANDS FOR QUANTILE FUNCTIONS FROM INCOMPLETE SURVIVAL DATA

BY GANG LI,[1] MYLES HOLLANDER, IAN W. MCKEAGUE[2] AND JIE YANG

University of North Carolina, Charlotte, and Florida State University

The purpose of this paper is to derive confidence bands for quantile functions using a nonparametric likelihood ratio approach. The method is easy to implement and has several appealing properties. It applies to right-censored and left-truncated data, and it does not involve density estimation or even require the existence of a density. Previous approaches (e.g., bootstrap) have imposed smoothness conditions on the density. The performance of the proposed method is investigated in a Monte Carlo study, and an application to real data is given.

1. Introduction. Confidence bands and intervals for quantile functions provide an attractive and readily interpretable way of summarizing survival data. For example, Figure 1 gives confidence bands for patients treated for malignant melanoma. Such curves are useful to medical researchers for assessing the effectiveness of treatments.

Consider the right-censored survival data consisting of n i.i.d. pairs $(Z_1, \delta_1), \dots, (Z_n, \delta_n)$, where $Z_i = \min\{X_i, Y_i\}$, $\delta_i = I(X_i \le Y_i)$ and X_i and Y_i are independent positive random variables representing the survival time and the censoring time of the ith subject under study. Let F_0 and G_0 be the distribution functions of X_i and Y_i, respectively. We study the problem of constructing confidence bands for the quantile function $F_0^{-1}(p)$ on an interval $[p_1, p_2] \subset (0, 1)$, where, for any nondecreasing function x, the right-continuous inverse is defined by

$$x^{-1}(p) = \sup\{t: x(t) \le p\}.$$

Let F_n be the Kaplan–Meier (1958) estimator of F_0. It is well known [cf. Shorack and Wellner (1986), Section 18.4] that if F_0 has a positive and

Received January 1995; revised July 1995.
[1]Research partially supported by a UNC–Charlotte Faculty Research Grant.
[2]Research partially supported by NSF Grant ATM-94-17528.
AMS 1991 *subject classifications*. Primary 62G07; secondary 62G20.
Key words and phrases. Empirical likelihood, Hall–Wellner band, Kaplan–Meier estimator, multiplicative intensity model, censoring, truncation.

continuous derivative f_0 on $[0, F_0^{-1}(p_0) + \varepsilon)$ for some $0 < p_0 < 1$, $\varepsilon > 0$ and $G_0(F_0^{-1}(p_0)) < 1$, then

$$n^{1/2}\left(F_n^{-1} - F_0^{-1}\right) \to_d -\frac{W\left(F_0^{-1}\right)}{f_0\left(F_0^{-1}\right)} \quad \text{in } D[0, p_0], \tag{1.1}$$

where W is a Gaussian process with mean 0 and covariance function $\operatorname{cov}\{W(s), W(t)\} = (1 - F_0(s))(1 - F_0(t))\sigma^2(s)$ for $s < t$,

$$\sigma^2(s) = \int_0^s \frac{dF_0(u)}{[1 - F_0(u)][1 - F_0(u-)][1 - G_0(u-)]} \tag{1.2}$$

and $D(I)$ is the Skorohod space on the interval I. It is not easy, however, to construct confidence bands for F_0^{-1} on an interval $[p_1, p_2] \subset (0, p_0]$ from the result in (1.1), because the distribution of

$$\sup_{p_1 \le p \le p_2} \left| \frac{W\left(F_0^{-1}(p)\right)}{f_0\left(F_0^{-1}(p)\right)} \right|$$

involves the unknown F_0 and is intractable except in some special cases. One possible solution is to transform the weak convergence result in (1.1) to a Brownian bridge form, along the lines of Hall and Wellner's (1980) method of constructing confidence bands for the distribution function F_0. Details will be given in subsection 2.4. Here we only point out that the resulting band is still difficult to use in practice because it requires knowledge of the density quantile function $g_0 = f_0(F_0^{-1})$. As in density estimation, estimation of g_0 involves smoothing [cf. Xiang (1994)], and the choice of smoothing parameter is problematic. Another solution is to bootstrap the distribution of $n^{1/2}(F_n^{-1} - F_0^{-1})$; see Efron (1981). The bootstrap does not require estimation of g_0. In this context, its validity was established by Lo and Singh (1986), Theorem 2, under the condition that F_0 has a bounded second derivative. Bootstrap confidence bands for quantiles were also studied by Doss and Gill (1992). A rather different approach was taken by Aly, Csörgő and Horváth (1985), who used strong approximations. Keaney and Wei (1994) proposed a resampling method, different from that of Efron (1981), and which can be used to construct pointwise confidence intervals for quantiles without making strong assumptions. Further work, however, would be needed to extend their results to provide simultaneous bands for quantiles.

The purpose of this paper is to show that the nonparametric likelihood ratio approach provides a simple solution to the problem of constructing confidence bands for quantile functions. Let Θ be the space of all distribution functions on $[0, \infty)$. Let

$$L(F) = \prod_i [F(Z_i) - F(Z_i-)]^{\delta_i}[1 - F(Z_i)]^{1-\delta_i}$$

be the likelihood function based on the right-censored data described earlier. Here F is treated as a parameter taking values in Θ. For any $t \ge 0$ and $0 < p < 1$, define

$$R(p,t) = \frac{\sup\{L(F): F(t) = p \text{ and } F \in \Theta\}}{\sup\{L(F): F \in \Theta\}}, \tag{1.3}$$

and, for $0 \le r \le 1$,

$$C(p,r) = \{t: R(p,t) \ge r\}. \tag{1.4}$$

Clearly, a large value of $R(p,t)$ gives evidence in favor of the hypothesis H_0: $F_0(t) = p$. Therefore, $C(p,r)$ can be interpreted, for each fixed p, as the set of times t for which H_0 is not rejected by a test based on $R(p,t)$. This suggests that $C(p,r)$ be used as a confidence set for $F_0^{-1}(p)$. We show that $C(p,r)$ is always an interval and that an r_α can be determined so that $\{C(p,r_\alpha), p_1 \le p \le p_2\}$ gives an approximate $1-\alpha$ simultaneous confidence band for F_0^{-1} on the interval $[p_1, p_2]$. The band is easy to compute using a standard root-finding procedure.

Our approach has some appealing features. First, the method is quite general; it works not only for right-censored data, but also for other important missing data schemes including random truncation. In fact, without major changes in the arguments, the method can be extended to Aalen's (1978) multiplicative intensity counting process model, which is known to encompass a variety of models in survival analysis [cf. Andersen, Borgan, Gill and Keiding (1993), Chapters 2 and 3]. More details will be given in subsection 2.4. To the best of our knowledge, very little has been done concerning quantile function estimation beyond the standard right censorship model. Moreover, it appears that for such extensions our approach is more tractable than the bootstrap or strong approximation approaches. Second, the likelihood ratio confidence bands are valid under much weaker conditions. They do not require F_0 to be differentiable. In contrast, the methods based on weak convergence of F_n^{-1} or the bootstrap were derived under the strong condition that F_0 has a bounded second derivative, as mentioned earlier. Finally, our approach does not require estimation of the density quantile function g_0.

The nonparametric likelihood ratio approach was introduced by Thomas and Grunkemeier (1975) to derive confidence intervals for survival probabilities from right-censored data. Their simulation studies showed that the method has a better small-sample performance than that of normal approximation. Theoretical justification was given by Li (1995a) and Murphy (1995), and in the case of truncated data by Li (1995b). Likelihood ratio based confidence bands for survival functions have been derived by Hollander, McKeague and Yang (1995).

The theoretical development of nonparametric likelihood ratio based inference was initiated by Owen (1988, 1990), who used an empirical likelihood to construct confidence regions for the mean of a random vector and some of its smooth functions in the i.i.d. complete data setting. In recent years the nonparametric likelihood method has received much attention. It has many attractive properties; for instance, it only uses data to determine the shape of a confidence region. It respects the range of the parameter, which is appealing for estimating probabilities. Moreover, empirical likelihood is Bartlett-correctable, unlike the bootstrap; see DiCiccio, Hall and Romano (1991). Owen's (1988, 1990) results have been extended to more general models including linear regression, generalized linear models and projection pursuit. See Owen (1991, 1992, 1995), Kolaczyk (1992) and Qin and Lawless (1994) for further discussion and references in this area.

The paper is organized as follows. In Section 2 we derive our confidence bands and intervals for the quantile function and explain how they are computed. Extensions beyond the standard right censorship model and Hall–Wellner type confidence bands for F_0^{-1} are discussed in subsection 2.4. In Section 3 we illustrate the proposed procedure on a set of melanoma data and compare it with the bootstrap method. We also investigate its small-sample performance by simulation. Proofs are given in Section 4.

2. Main results.

2.1. *Preliminaries.* We assume throughout that F_0 is continuous. The distinct and ordered uncensored survival times are denoted $0 < T_1 < \cdots < T_k < \infty$. Let

$$r_j = \sum_{i=1}^{n} I(Z_i \geq T_j)$$

be the number of subjects that are "alive" just before time T_j. The Kaplan–Meier estimator of F_0 is

$$F_n(t) = 1 - \prod_{j:\, T_j \leq t} \left(1 - \frac{1}{r_j}\right), \tag{2.1}$$

which maximizes $L(F)$ in Θ. Its variance can be estimated by Greenwood's formula $(1 - F_n(t))^2 \hat{\sigma}^2(t)/n$, where

$$\hat{\sigma}^2(t) = n \sum_{j:\, T_j \leq t} \frac{1}{r_j(r_j - 1)}. \tag{2.2}$$

The function $R(p, t)$ given in (1.3) arises as the solution of an infinite-dimensional constrained maximization problem, but it can be reduced to a finite one as given in the following result of Li (1995a).

LEMMA 1. *For each $T_1 \leq t \leq T_k$,*

$$\begin{aligned} R(p, t) = {} & \left[\max\left\{ \prod_{j=1}^{k} h_j (1 - h_j)^{r_j - 1} : \mathbf{h} \in (0,1)^k \right.\right. \\ & \qquad \left.\left. \text{and} \prod_{j:\, T_j \leq t} (1 - h_j) = 1 - p \right\}\right] \\ & \times \left[\max\left\{ \prod_{j=1}^{k} h_j (1 - h_j)^{r_j - 1} : \mathbf{h} \in (0,1)^k \right\}\right], \end{aligned} \tag{2.3}$$

where $\mathbf{h} = (h_1, \ldots, h_k)$.

Note that any discrete distribution function F supported on $\{T_1, \dots, T_k\}$ can be written as $F(t) = 1 - \prod_{j:\, T_j \le t}(1 - h_j)$, $t > 0$, by writing $h_j = (F(T_j) - F(T_{j-1}))/(1 - F(T_{j-1}))$ for $j \ge 2$ and $h_1 = F(T_1)$. For such an F, $L(F) = \prod_{j=1}^{k} h_j (1 - h_j)^{r_j - 1}$. Therefore, Lemma 1 says that the nonparametric likelihood ratio $R(p, t)$ can be obtained by restricting Θ to be the subspace of all discrete distributions F supported on $T_1, \dots, T_k$.

Applying Lagrange's method, one can show from (2.3) that

$$\begin{aligned} &-2 \log R(p, t) \\ &\quad = -2 \sum_{j:\, T_j \le t} \left\{ (r_j - 1) \log\left(1 + \frac{\lambda_n(t)}{r_j - 1}\right) - r_j \log\left(1 + \frac{\lambda_n(t)}{r_j}\right) \right\}, \end{aligned} \tag{2.4}$$

where $\lambda_n(t) > -\min_{j:\, T_j \le t}\{r_j - 1\}$ is uniquely determined by

$$\prod_{j:\, T_j \le t} \left(1 - \frac{1}{r_j + \lambda_n(t)}\right) = 1 - p. \tag{2.5}$$

The last equation is easily solved for $\lambda_n(t)$ using a standard root-finding procedure (see Section 3). The expression (2.4) was first used by Thomas and Grunkemeier (1975) to construct confidence intervals for survival probabilities.

2.2. *Computing $C(p, r)$.* A quantile confidence set should not fall outside the range of the uncensored data, so we shall (implicitly) restrict $C(p, r)$ to be contained within $[T_1, T_k]$. This is done for notational simplicity and has no effect asymptotically. In view of (2.4), it is natural to write $C(p, r) = \{t\colon -2 \log R(p, t) \le -2 \log r\}$. Although $-2 \log R(p, t)$ is not a convex function of t, the following theorem shows that $C(p, r)$ is always an interval.

THEOREM 1. *For every $0 < p < 1$ and $0 < r < 1$, $C(p, r)$ is an interval.*

The proof is given in Section 4. This result enables one to compute $C(p, r)$ using a simple algorithm.

First note that the $\lambda_n(t)$ determined by (2.5) is a right-continuous step function of t on the interval (T_1, T_k) with positive jumps at $T_1, \dots, T_k$ only. This, together with (2.4), implies that $-2 \log R(p, t)$ is a right-continuous step function of t with nonzero jumps at $T_1, \dots, T_k$ only. This fact and Theorem 1 imply that the boundaries of $C(p, r)$ are uncensored survival times.

To compute $C(p, r)$, search through the uncensored survival times in the order $T_1, \dots, T_k$. Take the lower boundary of $C(p, r)$ to be the first T_j for which $-2 \log R(p, T_j) \le -2 \log r$, and the upper boundary to be the first subsequent T_j for which $-2 \log R(p, T_j) > -2 \log r$.

2.3. *LR confidence bands and intervals for F_0^{-1}.* We now state our main result and explain how it can be used to construct the confidence bands and intervals.

THEOREM 2. *Assume that* F_0 *is continuous and strictly increasing on* $[0, F_0^{-1}(p_0)]$ *for some fixed* $0 < p_0 < 1$ *and that* $G_0(F_0^{-1}(p_0)) < 1$. *Then, for every* $0 < r < 1$:

(a) *For* $0 < p_1 < p_2 \le p_0$,

$$\lim_{n\to\infty} P\{F_0^{-1}(p) \in C(p,r) \text{ for } p \in [p_1, p_2]\} = P\left\{ \sup_{t\in[t_1,t_2]} \left| \frac{B^0(t)}{\{t(1-t)\}^{1/2}} \right| \le \sqrt{-2\log r} \right\},$$

where B^0 *is a Brownian bridge on* $[0,1]$,

$$t_l = \frac{\sigma^2(F_0^{-1}(p_l))}{1+\sigma^2(F_0^{-1}(p_l))} \quad \text{for } l = 1, 2 \tag{2.6}$$

and $\sigma^2(t)$ *is defined in* (1.2).

(b) *For* $0 < p \le p_0$,

$$\lim_{n\to\infty} P(F_0^{-1}(p) \in C(p,r)) = P(\chi_1^2 \le -2\log r),$$

where χ_1^2 *is a chi-square random variable with* 1 *degree of freedom.*

Our LR confidence band is obtained by pasting together intervals of the form $C(p,r)$ with r chosen appropriately. Specifically, an asymptotic $1-\alpha$ confidence band for F_0^{-1} on the interval $[p_1, p_2]$ is given by

$$\{C(p, r_\alpha): p \in [p_1, p_2]\}, \tag{2.7}$$

where $r_\alpha = \exp\{-e_\alpha(\hat{t}_1, \hat{t}_2)/2\}$, the $\hat{t}_l$ is a consistent estimator of t_l obtained by replacing F_0 and $\sigma^2(\cdot)$ in (2.6) by their estimated versions (2.1) and (2.2), and $e_\alpha(t_1, t_2)$ is the upper α-quantile of the distribution of

$$W(t_1, t_2) \equiv \sup_{t\in[t_1,t_2]} \left| \frac{B^0(t)}{\{t(1-t)\}^{1/2}} \right|.$$

An asymptotic $1-\alpha$ confidence interval for the p-quantile of F_0 is $C(p, r_\alpha^*)$, where $r_\alpha^* = \exp\{-\chi_{1,\alpha}^2/2\}$ and $\chi_{1,\alpha}^2$ is the upper α-quantile of χ_1^2.

For any given $0 < t_1 < t_2 < 1$, the distribution of $W(t_1, t_2)$ was studied by Miller and Siegmund (1982). In particular, they showed that, as $w \to \infty$,

$$P(W(t_1,t_2) \ge w) = \frac{4\phi(w)}{w} + \phi(w)\left(w - \frac{1}{w}\right)\log\left(\frac{\tau_1}{\tau_2}\right) + o\{w^{-1}\phi(w)\},$$

where $\tau_l = t_l/(1-t_l)$ and $\phi(w)$ is the standard normal density function. This approximation can be used to find $e_\alpha(t_1, t_2)$. Some specific values of $e_\alpha(t_1, t_2)$ are given by Nair (1984).

2.4. *Remarks.* The idea and techniques used in this article also work for other important survival models. In fact, the likelihood ratio method can be extended to Aalen's (1978) multiplicative intensity counting process model by using an empirical version of the binomial type likelihood described by Andersen, Borgan, Gill and Keiding [(1993), (4.1.37)]; see also Murphy (1995). The multiplicative intensity model is known to encompass a number of models in survival analysis, including those with very general forms of censoring and truncation. Below we describe briefly how the likelihood ratio approach works for left-truncated data.

Confidence bands from randomly truncated data. Left-truncated data consists of n i.i.d. pairs $(X_1^*, Y_1^*), \dots, (X_n^*, Y_n^*)$ from the conditional distribution of (X, Y) given that $X > Y$. Here X and Y are independent positive random variables representing the survival time and the truncation time of a subject under study. Thus, in the left-truncation model, the pair (X, Y) is observable only when $X > Y$. See Woodroofe (1985), Wang, Jewell and Tsai (1986) and Keiding and Gill (1990) for further discussions. Let F_0 and G_0 denote the distribution functions of X and Y, respectively. Assume F_0 is continuous. We consider the problem of constructing confidence bands for F_0^{-1}.

Define, for any $t > 0$ and $0 < p < 1$,

$$R_c(p, t) = \frac{\sup\{L_c(F) \colon F(t) = p \text{ and } F \in \Theta\}}{\sup\{L_c(F) \colon F \in \Theta\}},$$

where $L_c(F) = \prod_{i=1}^n \{[F(X_i^*) - F(X_i^* -)]/[1 - F(Y_i^*)]\}$ is the conditional likelihood of F given $Y_1^*, \dots, Y_n^*$. Let $X_{(1)}^* < \cdots < X_{(n)}^*$ denote the order statistics of $X_1^*, \dots, X_n^*$. Then $-2 \log R_c(p, t)$ is given by (2.4), where r_j is now defined by $r_j = \sum_{i=1}^n I(Y_i^* < X_{(j)}^* \le X_i^*)$. Moreover, one can establish exact analogs of Theorems 1 and 2 for the left-truncation model along the same lines. This leads to confidence bands and intervals for F_0^{-1}.

Hall–Wellner type confidence band for F_0^{-1}. Hall and Wellner (1980) derived confidence bands for F_0 from censored data by transforming the weak convergence of the Kaplan–Meier estimator to a Brownian bridge form. Their idea can also be used (as we outline below) to obtain confidence bands for the quantile function F_0^{-1}. Unfortunately, such a band requires estimation of the density quantile function, which is a serious drawback, as discussed in the Introduction.

From (1.1), it can be shown that

$$\frac{n^{1/2}}{1-p}\left(\frac{(F_n^{-1} - F_0^{-1})g_n}{(1 + \hat{\sigma}^2 \circ F_n^{-1})}\right)(p) \to_d \left(B^0 \circ \left(\frac{\sigma^2}{1+\sigma^2}\right) \circ F_0^{-1}\right)(p) \quad \text{in } D[0, p_0],$$

where $\circ$ denotes functional composition, B^0 is a Brownian bridge process, $\hat{\sigma}^2$ is given by (2.2) and g_n is a uniformly consistent estimate of the density quantile function $g_0 = f_0 \circ F_0^{-1}$. This leads to the following asymptotic $1 - \alpha$ confidence band for F_0^{-1} on $[p_1, p_2]$

$$F_n^{-1}(p) \pm c(a_1, a_2)\frac{(1-p)\left(1 + \hat{\sigma}^2\left(F_n^{-1}(p)\right)\right)}{n^{1/2} g_n(p)},$$

where $a_l = \hat{\sigma}^2(F_n^{-1}(p_l))/\{1 + \hat{\sigma}^2(F_n^{-1}(p_l))\}$ for $l = 1, 2$, and $c(a_1, a_2)$ is determined by

$$P\left[\sup_{a_1 \le u \le a_2} |B^0(u)| \le c(a_1, a_2)\right] = 1 - \alpha.$$

See Hall and Wellner (1980) for the computation of $c(a_1, a_2)$.

3. Application and simulation study. In this section we apply our LR band (2.7) and compare it to a bootstrap band for a real data set. We also carry out a simulation study to assess the small-sample performance of the LR band.

We considered a data set consisting of survival times following treatment for malignant melanoma; see Andersen, Borgan, Gill and Keiding [(1993), pages 11 and 709]. The analysis was restricted to the 87 males under study, of whom 31 were observed to die from the disease and the remaining were censored observations. Figure 1 gives the 90% LR and bootstrap bands for the quantile function on the interval [0, 0.25]. The bootstrap band is the "method 1" band of Doss and Gill (1992). For this data set the LR band is considerably narrower than the bootstrap band.

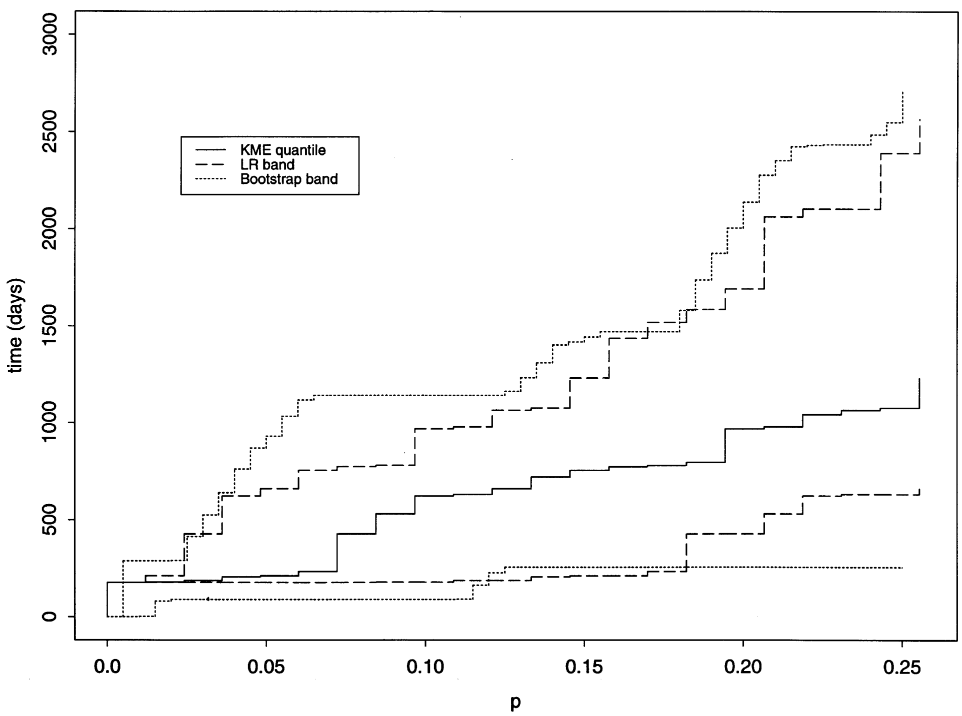

FIG. 1. *90% LR and bootstrap confidence bands for quantiles of survival time for men with malignant melanoma.*

In Table 1 we report the results of simulations to estimate the coverage probabilities of the LR band in three examples. The three pairs of survival and censoring distributions are: (a) F_0 = standard exponential and G_0 = uniform on $(0, b)$; (b) F_0 = standard exponential and G_0 = exponential; and (c) $F_0(t) = 1 - \exp(-\sqrt{2t})$ (Weibull) and $G_0(t)$ = Weibull with the same shape parameter as F_0. In each case, the censoring distribution is adjusted to give the prescribed censoring rate.

The bands were calculated on the interval [0.1, 0.9], and each had a nominal coverage probability of 0.95. Each entry in the table was based on 10,000 Monte Carlo samples that were simulated using the uniform random number generator in Press, Teukolsky, Vetterling and Flannery (1992). Values of $\lambda_n(t)$ solving (2.5) were computed using the Van Wijngaarden–Dekker–Brent root-finding procedure [Press, Teukolsky, Vetterling and Flannery (1992), page 359].

The coverage probabilities are seen to be close to their nominal value of 0.95, except under heavy censoring and small sample size ($n = 50$).

4. Proofs.

PROOF OF THEOREM 1. Let $0 < p < 1$ be fixed. Recall that $C(p, r) = \{t: R(p, t) \geq r\}$, where $R(p, t)$ has the form (2.3). Because the denominator on the right-hand side of (2.3) does not depend on t, it suffices to show that the set

$$(4.1)\quad I = \left\{t: \min\left\{g(\mathbf{h}): \mathbf{h} \in (0,1)^k \text{ and } \prod_{j: T_j \leq t} (1 - h_j) = 1 - p\right\} \leq c\right\}$$

is an interval, where $c > 0$ is a constant and

$$g(\mathbf{h}) \equiv -2\, log\left\{\prod_{j=1}^{k} h_j(1 - h_j)^{r_j - 1}\right\}.$$

So we only need to show that $[t_1, t_2] \subset I$ for any two points $t_1 < t_2$ in I.

Let $t_1, t_2 \in I$ and $t^* \in [t_1, t_2]$. Then there exists an $\mathbf{h}_l = (h_{l1}, \dots, h_{lk})$ that attains the minimum in (4.1) for $t = t_l$, $l = 1, 2$, and for which $g(\mathbf{h}_l) \leq c$.

TABLE 1
Observed coverage probabilities of nominal 95% *LR quantile bands for examples* (a)–(c)

	(a)			**(b)**			**(c)**		
Sample size	**Censoring rate**			**Censoring rate**			**Censoring rate**		
n	25%	50%	75%	33%	50%	66%	35%	50%	65%
50	0.9434	0.9587	0.9912	0.9434	0.9418	0.9584	0.9430	0.9538	0.9778
100	0.9518	0.9612	0.9618	0.9539	0.9456	0.9492	0.9518	0.9540	0.9716
200	0.9520	0.9570	0.9550	0.9576	0.9500	0.9455	0.9520	0.9546	0.9610

For $0 \le x \le 1$, let $\mathbf{q}(x) = (q_1(x), \dots, q_k(x)) = x\mathbf{h}_1 + (1-x)\mathbf{h}_2$. Define

$$f(x) = \prod_{j:\, T_j \le t^*} \big(1 - q_j(x)\big),$$

which is a continuous function of x on the interval $[0, 1]$. Moreover,

$$f(0) = \prod_{j:\, T_j \le t^*} (1 - h_{2j}) \ge \prod_{j:\, T_j \le t_2} (1 - h_{2j}) = 1 - p$$

and

$$f(1) = \prod_{j:\, T_j \le t^*} (1 - h_{1j}) \le \prod_{j:\, T_j \le t_1} (1 - h_{1j}) = 1 - p.$$

Thus, by the intermediate value theorem, there exists $x^* \in [0, 1]$ such that $f(x^*) = 1 - p$. Set $\mathbf{h}^* = (h_1^*, \dots, h_k^*) = \mathbf{q}(x^*)$. Then

$$\prod_{j:\, T_j \le t^*} (1 - h_j^*) = 1 - p. \tag{4.2}$$

Because g is a convex function, we have $g(\mathbf{h}^*) \le x^* g(\mathbf{h}_1) + (1 - x^*) \times g(\mathbf{h}_2) \le c$. This, together with (4.2) and the fact that $\mathbf{h}^* \in (0, 1)^k$, implies that $t^* \in I$. This proves the theorem. □

The following lemma, needed for proving Theorem 2, relates the asymptotic behavior of $\lambda_n(t)$ to the Kaplan–Meier estimator F_n.

LEMMA 2. *Let $\lambda_n(t)$ be the unique solution of equation* (2.5) *with $p = F_0(t)$. Under the assumptions of Theorem* 2, $\lambda_n(t) = O_p(n^{1/2})$ *and*

$$\lambda_n(t) = n\hat{\sigma}^{-2}(t)\big(\log[1 - F_n(t)] - \log[1 - F_0(t)] + O_p(n^{-1})\big), \tag{4.3}$$

uniformly in $t \in [0, F_0^{-1}(p_0)]$, where $\hat{\sigma}^2$ is defined by (2.2).

PROOF. The proof of the first part is very similar to that of Lemma 2.2 of Li (1995a) and is omitted. We only mention that it uses inequalities (2.12) and (2.13) of Li (1995a) and the weak convergence of the Nelson–Aalen estimator of the cumulative hazard function and the Kaplan–Meier estimator in $D[0, F_0^{-1}(p_0)]$. The proof of (4.3) is exactly the same as that of (2.15) of Li (1995a). □

PROOF OF THEOREM 2. We first show that

$$-2\log R\big(F_0^{-1}(p), p\big) \to_d \left[\frac{U\big(F_0^{-1}(p)\big)}{\sigma\big(F_0^{-1}(p)\big)}\right]^2 \quad \text{in } D[0, p_0], \tag{4.4}$$

where $U(t)$ is a Gaussian process with mean 0 and covariance function $\operatorname{cov}\{U(s), U(t)\} = \sigma^2(\min\{s, t\})$ and σ^2 is defined in (1.2).

By (2.4), we can write $-2\log R(F_0^{-1}(p), p) = \psi(\lambda_n(t))$, where

$$\psi(x) = -2 \sum_{j:\, T_j \le F_0^{-1}(p)} \left\{(r_j - 1)\log\left(1 + \frac{x}{r_j - 1}\right) - r_j \log\left(1 + \frac{x}{r_j}\right)\right\}$$

and $\lambda_n(t)$ is determined by (2.5) with $t = F_0^{-1}(p)$. It can be verified that $\psi(0) = \psi'(0) = 0$ and $\psi''(0) = 2\hat{\sigma}^2(F_0^{-1}(p))/n$. A Taylor series expansion of $\psi(t)$ about $t = 0$ then gives

$$-2\log R\big(F_0^{-1}(p), p\big) = \frac{\hat{\sigma}^2\big(F_0^{-1}(p)\big)\lambda_n(t)^2}{n} + \frac{\psi'''(\xi_n)}{6}\lambda_n(t)^3, \tag{4.5}$$

where $|\xi_n| \le |\lambda_n(t)| = O_p(n^{1/2})$.

By the Glivenko–Cantelli theorem, $\sup_{0\le t\le F_0^{-1}(p_0)}|n^{-1}\sum_{i=1}^n I(Z_i \ge t) - P(Z_1 \ge t)| \to 0$ a.s. as $n \to \infty$. Because $r_j = \sum_{i=1}^n I(Z_i \ge T_j)$ and $P(Z_1 \ge t) = [1 - F_0(t)][1 - G(t)] \ge [1 - p_0][1 - G(F_0^{-1}(p_0))] > 0$ for all $t \in [0, F_0^{-1}(p_0)]$, we have $r_j = O_p(n)$ and $1/r_j = O_p(n^{-1})$ uniformly in $j \in \{j\colon T_j \le F_0^{-1}(p_0)\}$. This, together with $\xi_n = O_p(n^{1/2})$, implies

$$\begin{aligned}
&\left|\frac{\psi'''(\xi_n)}{6}\lambda_n(t)^3\right| \\
&\quad\le \frac{2|\lambda_n(t)|^3}{3}\sum_{j\colon T_j\le F_0^{-1}(p_0)}\left|\frac{(r_j-1)(r_j+\xi_n)^3 - r_j(r_j-1+\xi_n)^3}{(r_j-1+\xi_n)^3(r_j+\xi_n)^3}\right| \\
&\quad= O_p(n^{3/2})\sum_{j\colon T_j\le F_0^{-1}(p_0)} O_p(n^{-3}) \\
&\quad= O_p(n^{-1/2}).
\end{aligned}\tag{4.6}$$

It follows from (4.3), (4.5), (4.6) and since $\hat{\sigma}^2(t)$ is uniformly consistent for $\sigma^2(t)$ on $[0, F_0^{-1}(p_0)]$, that

$$\begin{aligned}
&-2\log R\big(F_0^{-1}(p), p\big) \\
&\quad= \left(\frac{n^{1/2}\{\log[1 - F_n(F_0^{-1}(p))] - \log(1-p)\}}{\hat{\sigma}\big(F_0^{-1}(p)\big)} + O_p(n^{-1/2})\right)^2 \\
&\qquad+ O_p(n^{-1/2}).
\end{aligned}$$

Moreover, by Theorem 4.2.2 of Gill (1980) and the functional δ-method [see Gill (1989)],

$$\frac{n^{1/2}\{\log[1 - F_n(t)] - \log[1 - F_0(t)]\}}{\hat{\sigma}(t)} \to_d \frac{U(t)}{\sigma(t)} \quad \text{in } D\big[0, F_0^{-1}(p_0)\big],$$

leading immediately to (4.4).

Now we prove part (a). Note that

$$\frac{U}{\sigma} =_d (\beta B^0)\circ\left(\frac{\sigma^2}{1+\sigma^2}\right), \tag{4.7}$$

where $\beta(x) = 1/\{x(1-x)\}^{1/2}$, since both sides are Gaussian processes with the same mean and covariance function. Therefore,

$$\begin{aligned}
&\lim_{n\to\infty} P\big(F_0^{-1}(p) \in C(p,r) \text{ for all } p \in [p_1, p_2]\big)\\
&\quad = \lim_{n\to\infty} P\Big(\sup_{p\in[p_1,p_2]} \big[-2\log R\big(F_0^{-1}(p), p\big)\big] \le -2\log r\Big)\\
&\quad = P\left(\sup_{p\in[p_1,p_2]} \left|\frac{U\big(F_0^{-1}(p)\big)}{\sigma\big(F_0^{-1}(p)\big)}\right| \le \sqrt{-2\log r}\right)\\
&\quad = P\left(\sup_{t\in[t_1,t_2]} \left|\frac{B^0(t)}{\{t(1-t)\}^{1/2}}\right| \le \sqrt{-2\log r}\right),
\end{aligned}$$

as required, where the second equality is from (4.4), the last equality is from (4.7) and t_1 and t_2 are defined by (2.6).

To prove part (b), we only need to note that

$$\begin{aligned}
\lim_{n\to\infty} P\big(F_0^{-1}(p) \in C(p,r)\big) &= \lim_{n\to\infty} P\big(-2\log R\big(F_0^{-1}(p), p\big) \le -2\log r\big)\\
&= P\left(\left[\frac{U\big(F_0^{-1}(p)\big)}{\sigma\big(F_0^{-1}(p)\big)}\right]^2 \le -2\log r\right)\\
&= P\big(\chi_1^2 \le -2\log r\big),
\end{aligned}$$

where the second step is from (4.4) and the last equality follows from $U(t)/\sigma(t)$ being standard normal for $t > 0$. □

Acknowledgments. The authors thank an Associate Editor and a referee for their comments and suggestions, which led to several improvements in the paper. The authors also thank Hani Doss for providing bootstrap output from Doss and Gill (1992).

REFERENCES

AALEN, O. O. (1978). Nonparametric inference for a family of counting processes. *Ann. Statist.* **6** 701–726.

ALY, E.-E. A. A., CSÖRGŐ, M. and HORVÁTH, L. (1985). Strong approximation of the quantile process of the product-limit estimator. *J. Multivariate Anal.* **16** 185–210.

ANDERSEN, P. K., BORGAN, O., GILL, R. D. and KEIDING, N. (1993). *Statistical Models Based on Counting Processes*. Springer, New York.

DICICCIO, T. J., HALL, P. and ROMANO, J. (1991). Empirical likelihood is Bartlett-correctable. *Ann. Statist.* **19** 1053–1061.

DOSS, H. and GILL, R. D. (1992). An elementary approach to weak convergence for quantile processes, with applications to censored survival data. *J. Amer. Statist. Assoc.* **87** 869–877.

EFRON, B. (1981). Censored data and the bootstrap. *J. Amer. Statist. Assoc.* **76** 312–319.

GILL, R. D. (1980). *Censoring and Stochastic Integrals. Math. Centre Tract* **124** Centrum Wisk. Inform. Amsterdam.

GILL, R. D. (1989). Non- and semi-parametric maximum likelihood estimators and the von Mises method. I. *Scand. J. Statist.* **16** 97–128.

HALL, W. J. and WELLNER, J. A. (1980). Confidence bands for a survival curve from censored data. *Biometrika* **67** 133–143.

HOLLANDER, M., MCKEAGUE, I. W. and YANG, J. (1995). Likelihood ratio based confidence bands for survival functions. Technical Report 901, Dept. Statistics, Florida State Univ.

KAPLAN, E. and MEIER, P. (1958). Nonparametric estimation from incomplete observations. *J. Amer. Statist. Assoc.* **53** 457–481.

KEANEY, K. M. and WEI, L. J. (1994). Interim analyses based on median survival times. *Biometrika* **81** 279–286.

KEIDING, N. and GILL, R. (1990). Random truncation models and Markov processes. *Ann. Statist.* **18** 582–602.

KOLACZYK, E. D. (1994). Empirical likelihood for generalized linear models. *Statist. Sinica* **4** 199–218.

LI, G. (1995a). On nonparametric likelihood ratio estimation of survival probabilities for censored data. *Statist. Probab. Lett.* **25** 95–104.

LI, G. (1995b). Nonparametric likelihood ratio estimation of probabilities for truncated data. *J. Amer. Statist. Assoc.* **90** 997–1003.

LO, S. H. and SINGH, K. (1986). The product-limit estimator and the bootstrap: some asymptotic representations. *Probab. Theory Related Fields* **71** 455–465.

MILLER, R. G. and SIEGMUND, D. (1982). Maximally selected chi-square statistics. *Biometrics* **38** 1011–1016.

MURPHY, S. A. (1995). Likelihood ratio-based confidence intervals in survival analysis. *J. Amer. Statist. Assoc.* **90** 1399–1405.

NAIR, V. N. (1984). Confidence bands for survival functions with censored data. *Technometrics* **26** 265–275.

OWEN, A. (1988). Empirical likelihood ratio confidence intervals for a single functional. *Biometrika* **75** 237–249.

OWEN, A. (1990). Empirical likelihood ratio confidence regions. *Ann. Statist.* **18** 90–120.

OWEN, A. (1991). Empirical likelihood for linear models. *Ann. Statist.* **19** 1725–1747.

OWEN, A. (1992). Empirical likelihood and generalized projection pursuit. Technical Report 393, Dept. Statistics, Stanford Univ.

OWEN, A. (1995). Nonparametric likelihood confidence bands for a distribution function. *J. Amer. Statist. Assoc.* **90** 516–521.

PRESS, W. H., TEUKOLSKY, S. A., VETTERLING, W. T. and FLANNERY, B. P. (1992). *Numerical Recipes in C*, 2nd ed. Cambridge Univ. Press.

QIN, J. and LAWLESS, J. (1994). Empirical likelihood and general estimating equations. *Ann. Statist.* **22** 300–325.

SHORACK, G. R. and WELLNER, J. A. (1986). *Empirical Processes with Applications to Statistics*. Wiley, New York.

THOMAS D. R. and GRUNKEMEIER, G. L. (1975). Confidence interval estimation of survival probabilities for censored data. *J. Amer. Statist. Assoc.* **70** 865–871.

WANG, M.-C., JEWELL, N. P. and TSAI, W.-Y. (1986). Asymptotic properties of the product limit estimate under random truncation. *Ann. Statist.* **14** 1597–1605.

WOODROOFE, M. (1985). Estimating a distribution function with truncated data. *Ann. Statist.* **13** 163–177.

XIANG, X. J. (1994). A law of the logarithm for kernel quantile density estimators. *Ann. Probab.* **22** 1078–1091.

G. LI
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NORTH CAROLINA
CHARLOTTE, NORTH CAROLINA 28223

M. HOLLANDER
I. W. MCKEAGUE
J. YANG
DEPARTMENT OF STATISTICS
FLORIDA STATE UNIVERSITY
TALLAHASSEE, FLORIDA 32306