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ESTIMATING NONQUADRATIC FUNCTIONALS OF A DENSITY USING HAAR WAVELETS

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To the memory of our friend Claude Kipnis

Consider the problem of estimating $\int \Phi(f)$, where Φ is a smooth function and f is a density with given order of regularity s. Special attention is paid to the case $\Phi(t) = t^3$. It has been shown that for low values of s the $n^{-1/2}$ rate of convergence is not achievable uniformly over the class of objects of regularity s. In fact, a lower bound for this rate is $n^{-4s/(1+4s)}$ for $0 < s \le 1/4$. As for the upper bound, using a Taylor expansion, it can be seen that it is enough to provide an estimate for the case $\Phi(x) = x^3$. That is the aim of this paper. Our method makes intensive use of special algebraic and wavelet properties of the Haar basis.

1. Introduction. Let $X_1, \dots, X_n$ be n independent identically distributed random variables according to a distribution P. We assume that P is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$ with density f. One aim of this paper is to study the problem of estimating $\int f^3$, assuming a priori that f lies in a class of low order of regularity, namely, $s \le 1/4$.

The problem of estimating nonlinear functionals of the density has now widely been studied. Some approaches can be found in Levit (1978), Hasminskii and Ibragimov (1978), Hall and Marron (1987), Bickel and Ritov (1988), Ritov and Bickel (1990), Donoho and Nussbaum (1990), Goldstein and Messer (1992) and Birgé and Massart (1995). A common feature is that for certain classes of regularity the parametric rate $n^{-1/2}$ is achievable. On the other hand, for low regularity situations, the rate typically becomes nonparametric.

The present work is principally based on the papers of Bickel and Ritov on estimating $\int f^2$ and the paper of Birgé and Massart extending their results to estimate integrals of general functionals of the density, $\int \phi(f)$. In Bickel and Ritov, it is shown that the rate $n^{-1/2}$ is achievable for regularity $s \ge 1/4$, whereas for $s < 1/4$ the optimal rate is $n^{-4s/(1+4s)}$. Birgé and Massart, in the more general context, have proven that $n^{-4s/(1+4s)}$ was a lower-bound rate. Moreover, they provide an $n^{-1/2}$ consistent estimate for the case $s \ge 1/4$. Furthermore, Laurent (1996) has built efficient estimates for $s > 1/4$.

Birgé and Massart noted that the case of low regularity, $0 < s < 1/4$, was not completely solved. This gap may be explained as follows: the method of

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Birgé and Massart is based (using a Taylor expansion of the function Φ) on an "initial" estimate corrected by estimates of functionals of order 1 (linear) and 2 (quadratic). Their method failed to go up to the third order as there was still the problem of producing an estimate of the cubic-order terms which become important in the low-regularity case, $0 < s < 1/4$.

In this paper, we provide a new estimator of $\int f^3$ exactly attaining the lower-rate bound in the low-regularity case and closing the problem.

Our approach has several interesting features:

1. Our estimator is constructed directly, not as a "one step" corrected estimate. To clarify this point, we give, as a starting point, a very simple construction of a direct estimator of $\int f^2$.
2. We make essential use of the Haar basis here. Two properties of the Haar basis are useful: first, the multiresolution properties, which provide an orthogonal partitioning of objects into coarse resolution terms on the one hand and details on the other, permitting us to estimate separately each of them; second, the more specific martingale-type properties of the Haar basis and more generally very special algebraic properties.
3. Although our approach is based on wavelets, the issues are different from some other work building wavelet estimators in other problems. We emphasize that we do not use here any thresholding methods [as we have used with profit in Donoho, Johnstone, Kerkyacharian and Picard (1995)].
4. We make essential use of Besov spaces here. Relying on their properties, we need not impose limitations on the support of the density, as other authors have done.

The paper is organized as follows. In Section 2 we introduce the properties of the Haar basis and Besov spaces used in the sequel. Section 3 is devoted to the special case of estimating $\int f^2$. In Section 4 we present our estimate of $\int f^3$. The evaluations of the behaviors of the different components of the estimate is postponed until the Appendix. In a final remark, we explain how to extend the previous results in order to complete the results of Birgé and Massart for the general case of estimating $\int \phi(f)$.

2. Haar basis and Besov spaces.

2.1. *Algebraic properties of the Haar basis.* Let $\phi(x) = 1_{[0,1]}(x)$ and let V_j be the subspace spanned by all the ϕ_{jk} for $k \in \mathbf{Z}$ [where $\phi_{jk}(x) = 2^{j/2}\phi(2^j x - k)$]. Similarly, let $\psi(x) = 1_{[0,1/2[}(x) - 1_{[1/2,1]}(x)$, $\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k)$ and W_j be the subspace spanned by all the ψ_{jk} for $k \in \mathbf{Z}$. Let $\alpha_{jk} = \langle f, \psi_{j,k}\rangle$, $\beta_{jk} = \langle f, \psi_{j,k}\rangle$ and $E_j f = \sum_{k\in\mathbf{Z}} \alpha_{jk}\phi_{j,k}$. The following (very simple) properties are classical and will be very useful in the sequel:

(i) $v_j v_{j'} \in V_{\max(j,j')}$ if $v_j \in V_j$, $v_{j'} \in V_{j'}$.
(ii) $\langle w_j, v_{j'}\rangle = 0$ if $w_j \in W_j$, $v_{j'} \in V_{j'}$, $j' \le j$.
(iii) $w_j^2 \in V_j$ if $w_j \in W_j$.
(iv) $w_j w_{j'} \in W_{j'}$ if $w_j \in W_j$, $w_{j'} \in W_{j'}$, $j' > j$.
(v) $w_{j'} v_j \in W_{j'}$ if $v_j \in V_j$, $w_{j'} \in W_{j'}$, $j' \ge j$.

2.2. *Haar basis and Besov spaces.* Besov spaces, like Sobolev or Lipschitz spaces, give the opportunity of restricting a nonparametric problem using regularity conditions. Those spaces, however, happen to lead to optimality properties in the minimax framework [cf. Kerkyacharian and Picard (1993)] and to be particularly adapted to the "wavelet" decompositions [cf. Johnstone, Kerkyacharian and Picard (1992), Kerkyacharian and Picard (1992a, b) and Donoho, Johnstone, Kerkyacharian and Picard (1995)]. Here, they will not be used in complete generality, but as can easily be seen, they will allow us to get rid of the condition that f be compactly supported. Let us only mention the following definition [see Bergh and Löfström (1976) and Peetre (1975)]. Let $\tau_h f(x) = f(x-h)$. For $0 < s < 1$ and $1 \le p < +\infty$, set

$$\|f\|_{spq} = \left(\int_0^1 \left(\|\tau_h f - f\|_p h^{-s} \right)^q \frac{dh}{h} \right)^{1/q},$$

while, for $q = \infty$, set

$$\|f\|_{sp\infty} = \sup_{h \in (0,1]} \|\tau_h f - f\|_p h^{-s}$$

and say that $f \in B^s_{pq}$ if and only if $\|f\|_{spq} < \infty$.

We need the following inequalities [see, e.g., Kerkyacharian and Picard (1992b)]: for all $0 < s < 1$, $1 \le p \le +\infty$,

$$(1) \qquad \|E_j f - f\|_p \le C \|f\|_{sp\infty} 2^{-js} \qquad \text{if } f \in B^s_{p\infty},$$

$$(2) \qquad |\beta_{jk}| \le C \|f\|_{s\infty\infty} 2^{-j(s+1/2)} \qquad \text{if } f \in B^s_{\infty\infty}.$$

For $J_* > J$, set $D_{J,J_*} f = E_{J_*} f - E_J f = \sum_{j=J}^{J_*-1} \sum_k \beta_{jk} \psi_{jk}$. From (1) we deduce

$$(3) \qquad \|D_{J,J_*} f\|_p = \left\| \sum_J^{J_*-1} \sum_k \beta_{jk} \psi_{jk} \right\|_p \le C \|f\|_{sp\infty} 2^{-Js} \quad \text{if } f \in B^s_{p\infty}.$$

Let $\|\alpha_{j.}\|_{l_p}$ denote the l_p norm of the sequence $(\alpha_{jk})_{k \in \mathbf{Z}}$:

$$(4) \qquad \forall j \ge 0, \qquad \|E_j f\|_p = 2^{j(1/2 - 1/p)} \|\alpha_{j.}\|_{l_p} \le \|f\|_p \quad \text{if } f \in B^s_{p\infty}.$$

REMARKS.

1. The constants we denote C appearing here and throughout the paper are generic, typically denoting different quantities from one occurrence to the other. However, they always denote quantities that are independent of both f and j. It would not be difficult to obtain explicit bounds for the constants occurring in this paper, but as it is of no apparent importance to our results, we have not tried to do so.
2. Of course, f itself may be estimated crudely, using the ϕ_{jk} by $\hat{f}(x) = (1/n)\sum_i G(X_i, x)$, where $G(x,y) = \sum_k \phi_{jk}(x)\phi_{jk}(y)$. It is well known that, for $0 < s < 1$, $E\|\hat{f} - f\|_2^2 \le C\{2^j/n + 2^{-2js}\}$ as soon as $f \in B^s_{2\infty}$. In this paper, we give an extension of such ideas to the problem of estimating the integral of higher powers of f.

3. Estimation of a quadratic functional. As a warmup for our technique we give here an estimator for the square of a density. Of course, the result we give, Theorem 1, is not new, since already in Bickel and Ritov (1988), there appears an estimator with the right rate of convergence even for small regularity assumptions. Other estimates also can be found in Laurent (1996). Nevertheless, in the Besov framework, it gives a primary understanding of our technique, since in the sequel we will only elaborate this basic construction to cover the more difficult case of the cube of the density.

We start from the basic identity

$$\int f^2 = \int (f - E_j f)^2 + \int (E_j f)^2, \qquad E_j f = \sum_k \alpha_{jk}\phi_{j,k}, \qquad \alpha_{jk} = \langle f, \phi_{j,k}\rangle.$$

We have used here the orthogonality of $(E_j f)$ with $(f - E_j f)$.

We have obviously the following results, using (1) and (4):

$$\int (f - E_j f)^2 \le C 2^{-2js}\|f\|^2_{s2\infty} \quad \text{and} \quad \int (E_j f)^2 = \sum_k \alpha^2_{jk}. \tag{5}$$

Introduce now the kernel $G(x, y) = \sum_k \phi_{jk}(x)\phi_{jk}(y)$.

Definition 1. By *quadratic wavelet estimator* we mean the U-estimator constructed from the kernel G using the n-sample:

$$B^2(j) = \frac{1}{C_n^2} \sum_{(i_1, i_2) \in I} G(X_{i_1}, X_{i_2}).$$

Here I is the set of all strictly ordered pairs (i_1, i_2) with values in $\{1, \dots, n\}$, and, of course, card $I = C_n^2 = n(n-1)/2$.

Now $B^2(j)$ is an unbiased estimator of $\int (E_j f)^2$; below, we will evaluate its variance. As a result we will obtain the following result on its mean-squared error.

Theorem 1. *Suppose f is such that $\|f\|_{s2\infty} \le M$, $\|f\|_3 \le M$. There exists a constant C depending on M such that*

$$E\left(\hat{T} - \int (f)^2\right)^2 \le C n^{-((8s/(1+4s)) \wedge 1)},$$

where $\hat{T} = B^2(j_1)$, $2^{j_1} \sim n$ if $s \ge 1/4$ and $2^{j_1} \sim n^{2/(1+4s)}$ if $s \le 1/4$.

Remarks. If f belongs to a bounded subset of $B^s_{\infty\infty}$ and is compactly supported, then it belongs to a bounded subset of $B^s_{2\infty} \cap B^s_{\infty\infty}$.

By the Sobolev injection theorem [see, e.g., Bergh and Löfström (1976)], we have $B^s_{2\infty} \subset B^{s'}_{3\infty}$ if $s - 1/2 = s' - 1/3$. But we also have $B^{s'}_{3\infty} \subset L_3$ if $s' > 0$.

Hence, if $s > 1/6$, we only need to impose that f belong to a bounded set of $B^s_{2\infty}$.

3.1. *Proof of Theorem* 1.

$$E\left(B^2(j) - \int (E_j f)^2\right)^2 = E\big(B^2(j)\big)^2 - \left(\int (E_j f)^2\right)^2. \tag{6}$$

But

$$E\big(B^2(j)\big)^2 = \big(C_n^2\big)^{-2} \sum_{(i_1, i_2)\in I} \sum_{(i'_1, i'_2)\in I} EG(X_{i_1}, X_{i_2})G(X_{i'_1}, X_{i'_2}).$$

We can split $I \times I$ into three subsets in the following way:

$$I_0 = \{((i_1, i_2), (i'_1, i'_2)) \in I \times I : \{i_1, i_2\} \cap \{i'_1, i'_2\} = \varnothing\},$$
$$\operatorname{card} I_0 = C_n^2 C_{n-2}^2,$$
$$I_1 = \{((i_1, i_2), (i'_1, i'_2)) \in I \times I : \operatorname{card}\{i_1, i_2\} \cap \{i'_1, i'_2\} = 1\},$$
$$\operatorname{card} I_1 = n(n-1)(n-2),$$
$$I_2 = \{((i_1, i_2), (i'_1, i'_2)) \in I \times I : \operatorname{card}\{i_1, i_2\} \cap \{i'_1, i'_2\} = 2\},$$
$$\operatorname{card} I_2 = C_n^2.$$

Thus, we can decompose (6) as $e_0 + e_1 + e_2$, where

$$e_0 = \big(C_n^2\big)^{-2} \sum_{I_0} EG(X_{i_1}, X_{i_2})G(X_{i'_1}, i_{i'_2}) - \left(\int (E_j f)^2\right)^2$$

and the other e_j are given by

$$e_j = \big(C_n^2\big)^{-2} \sum_{I_j} EG(X_{i_1}, X_{i_2})G(X_{i'_1}, X_{i'_2}).$$

Below, we let X, Y, Z, T denote independent variables with density f.

1. Bound for e_0:

$$e_0 = \frac{C_{n-2}^2}{C_n^2}\big(E(GX, Y)\big)^2 - \left(\int (E_j f)^2\right)^2 = \left(\frac{C_{n-2}^2}{C_n^2} - 1\right)\left(\int (E_j f)^2\right)^2.$$

Hence, by (4), $|e_0| \le C\|f\|_2^4/n$.

2. Bound for $e_1 = [(n^2 - n)(n-2)/(C_n^2)^2] EG(X, Y)G(X, T)$:

$$e_1 = \frac{(n^2 - n)(n-2)}{\big(C_n^2\big)^2} \sum_{kk'} \int f\phi_{jk}\,\phi_{jk'}(\alpha_{jk'}\alpha_{jk})$$
$$= \frac{(n^2 - n)(n-2)}{\big(C_n^2\big)^2} \sum_{k} 2^{j/2}\alpha_{jk}^3.$$

Thus, using (4), $|e_1| \le C\|f\|_3^3/n$.

3. Bound for e_2:

$$e_2 = \frac{1}{C_n^2} EG(X,Y)G(X,Y) = \frac{1}{C_n^2} \sum_k \left(\int \phi_{jk}^2 f \right)^2 = \frac{1}{C_n^2} 2^j \sum_k \alpha_{jk}^2.$$

Thus, using (4), $|e_2| \leq C \|f\|_2^2 2^j/(n^2)$.

Putting together all the above bounds, we obtain that, as soon as the density f is such that $\|f\|_3 \leq M$, there exists a constant C depending on M such that

$$E\left(B^2(j) - \int (E_j f)^2 \right)^2 \leq C\left\{ \frac{1}{n} + \frac{2^j}{n^2} \right\}. \tag{7}$$

REMARK. When $n \leq 2^j$, we obtain

$$E\left(B^2(j) - \int (E_j f)^2 \right)^2 \leq C\left\{ \frac{2^j}{n^2} \right\}. \tag{8}$$

Now suppose that f is such that $\|f\|_{s2\infty} \leq M$, $\|f\|_3 \leq M$, and use the inequality

$$E\left(B^2(j_1) - \int (f)^2 \right)^2 \leq 2\left\{ E\left(B^2(j_1) - \int (E_{j_1} f)^2 \right)^2 + \left(\int (f - E_{j_1} f)^2 \right)^2 \right\}.$$

Using the preceding bounds (5) and (8), we complete the proof. □

4. Estimation of a cubic functional. Let us introduce the following quantities:

Low-resolution term. Consider $G_j^B(x, y, z) = \sum_k 2^{j/2} \phi_{jk}(x)\phi_{jk}(y)\phi_{jk}(z)$ and the associated U-estimator constructed on the n-sample:

$$B^3(j) = \frac{1}{C_n^3} \sum_{(i_1, i_2, i_3) \in I} G_j^B(X_{i_1}, X_{i_2}, X_{i_3});$$

I is the set of all strictly ordered triples (i_1, i_2, i_3) with values in $\{1, \dots, n\}$.

Cross term. For $J < J^*$ define $A_k = \{(j', k'): J \leq j' < J_*, k/2^J \leq k'/2^{j'} < (k+1)/2^J\}$,

$$G_{JJ^*}^M(x, y, z) = 2^{J/2} \sum_k \phi_{Jk}(x) \sum_{A_k} \psi_{j'k'}(y) \psi_{j'k'}(z)$$

and the associated U-estimator constructed on the n-sample:

$$M^3(J, J_*) = \frac{1}{A_n^3} \sum_S G_{JJ^*}^M(X_{i_1}, X_{i_2}, X_{i_3}).$$

S is the set of all triples (i_1, i_2, i_3) with no common value in $\{1, \dots, n\}$, and $A_n^3 = n(n-1)(n-2)$.

Cubic term. Consider

$$G^T_{JJ^*}(x,y,z) = 3\sum_{j,j'=J}^{J^*-1}\sum_{k,k'}\psi_{jk}(x)\psi_{j'k'}(y)\psi_{j'k'}(z)\langle\psi_{jk}, 2^{j'/2}\phi_{j'k'}\rangle$$

and the associated U-estimator constructed on the n-sample:

$$T^3(J, J_*) = \frac{1}{A_n^3}\sum_S G^T_{JJ^*}(X_{i_1}, X_{i_2}, X_{i_3}).$$

DEFINITION 2. By *cubic wavelet estimator* we mean the statistic

$$\hat{T} = B^3(j_1) + 3M^3(j_1, j_3)1_{0<s<1/4} + T^3(j_1, j_2)1_{0<s<1/12},$$

where $2^{j_1} \sim n$, $2^{j_3} \sim n^{2/(1+4s)}$, $n^{(3/2-2s)/(1+4s)} \le 2^{j_2} \le n^{(3/2+2s)/(1+4s)}$.

THEOREM 2. *If f is such that $\|f\|_{s2\infty} \le M$, $\|f\|_{s\infty\infty} \le M$, then there exists a constant C depending on M so that the cubic Haar U-estimator obeys the bound*

$$E\left(\hat{T} - \int(f)^3\right)^2 \le Cn^{-(8s/(1+4s))\wedge 1}.$$

We prove Theorem 2 by considering three different cases of regularity for f.

4.1. $s \ge 1/4$. This case can be treated as in the previous section. We recall that, using the orthogonality of $(E_jf)^2$ with $(f - E_jf)$, we have $\int f^3 = \int(f - E_jf)^3 + 3\int(f - E_jf)^2(E_jf) + \int(E_jf)^3$. However, we have, using (1) and (4),

$$\int(f - E_jf)^3 \le C2^{-3js}\|f\|^3_{s3\infty}, \qquad \int(f - E_jf)^2(E_jf) \le C2^{-2js}\|f\|^2_{s2\infty}\|f\|_\infty.$$

If we suppose now that f belongs to a bounded subset of $B^s_{2\infty} \cap B^s_{\infty\infty}$, we can estimate $\int f^3$ at the rate $1/n$, by taking simply $\hat{T} = B^3(j_1)$: Indeed $(B^3(j_1) - \int(f)^3)^2 \le C\{E(B^3(j_1) - \int(E_{j_1}f)^3)^2 + (\int(f - E_{j_1}f)^3)^2 + (\int(f - E_{j_1}f)^2 \times (E_{j_1}f))^2\}$.

Using a bound on the first term of the sum on the r.h.s. (to be developed in Lemma 3 in the Appendix), we get

$$E\left(B^3(j_1) - \int(f)^3\right)^2 \le C\left\{\frac{1}{n} + n^{-4s}\right\} \le \frac{C}{n}.$$

4.2. $1/12 \le s \le 1/4$. Notice, to begin, that

$$\int f^3 = \int(f - E_{j_3}f)^3 + 3\int(f - E_{j_3}f)^2(E_{j_3}f) + \int(E_{j_3}f)^3$$

and

$$\int(E_{j_3}f)^3 = \int(D_{j_1,j_3}f)^3 + 3\int(D_{j_1,j_3}f)^2E_{j_1}f + \int(E_{j_1}f)^3$$

[because $\int (D_{j_1, j_3} f)(E_{j_1} f)^2 = 0$]. Just as before, we get an inequality (see Lemma 3)

$$(9) \qquad E\left(B^3(j_1) - \int (E_{j_1} f)^3 \right)^2 \le \left\{ \frac{C}{n} \right\}.$$

It remains to construct an estimator of the "cross term" $\int (D_{j_1, j_3} f)^2 E_{j_1} f$.

We do this using $M^3(j_1, j_3)$. Its behavior is investigated in the Appendix. Particularly, we obtain (cf. Lemma 2)

$$(10) \quad E\left(M^3(j_1, j_3) - \int (D_{j_1, j_3} f)^2 E_{j_1} f \right)^2 \le C\left\{ \frac{2^{j_1 + j_3}}{n^3} \right\} \le C n^{-8s/(1+4s)}.$$

Using (1),

$$\int (f - E_{j_3} f)^3 \le C 2^{-3sj_3} \|f\|^3_{s3\infty} \le C n^{-6s/(1+4s)}.$$

Using (1) and (4),

$$\int (f - E_{j_3} f)^2 (E_{j_3} f) \le C 2^{-2sj_3} \|f\|^2_{s2\infty} \|f\|_\infty \le C n^{-4s/(1+4s)}.$$

Using (3),

$$\int (D_{j_1, j_3} f)^3 \le C 2^{-3sj_1} \|f\|^3_{s3\infty} \le C n^{-3s}.$$

Hence, we have

$$\begin{aligned} &E\left(3M^3(j_1, j_3) + B^3(j_1) - \int f^3 \right)^2 \\ &\quad \le C\left\{ E\left(3M^3(j_1, j_3) - \int 3 (D_{j_1, j_3} f)^2 E_{j_1} f \right)^2 + E\left(B^3(j_1) - \int (E_{j_1} f)^3 \right)^2 \right. \\ &\qquad \left. + \left(\int (D_{j_1, j_3} f)^3 \right)^2 + \left(\int (f - E_{j_3} f)^3 \right)^2 + \left(\int (f - E_{j_3} f)^2 (E_{j_3} f) \right)^2 \right\}. \end{aligned}$$

As $6 \ge 8/(1 + 4s)$ we obtain the rate of convergence indicated in the theorem.

4.3. $1/12 \ge s$. In the case of very low regularity, the preceding estimator fails to obtain the right rate of convergence due to the term $\int (D_{j_1, j_3} f)^3$. But we have $\int (D_{j_1, j_3} f)^3 = \int (D_{j_2, j_3} f)^3 + 3 \int (D_{j_2, j_3} f)^2 (D_{j_1, j_2} f) + \int (D_{j_1, j_2} f)^3$. Using (3), we have the following bounds:

1. $\int |(D_{j_2, j_3} f)|^3 \le C 2^{-3sj_2} \|f\|^3_{s3\infty}$. If $2^{j_2} \ge n^{4/3(1+4s)}$, then the preceding quantity is bounded by $C n^{-4s/(1+4s)}$.
2. $\int |(D_{j_2, j_3} f)^2 (D_{j_1, j_2} f)| \le C 2^{-2sj_2} 2^{-j_1 s} \|f\|^2_{s2\infty} \|f\|_{s\infty\infty}$. If $2^{j_2} \ge n^{(3/2 - 2s)/(1+4s)}$, then the preceding quantity is bounded by $C n^{-4s/(1+4s)}$.

It remains now to estimate the "cubic term", $\int (D_{j_1, j_2} f)^3$. Using Lemma 1 (see the Appendix), we get $E(T^3(j_1, j_2) - \int (D_{j_1, j_2} f)^3)^2 \le C\{2^{2j_2}/n^3\}$.

It may be worthwhile to notice the similarity in the rates of convergence between this estimate and that of the "cross term" (cf. Lemma 2). If $2^{j_2} \le n^{(3/2+2s)/(1+4s)}$, then the preceding quantity is controlled by $Cn^{-8s/(1+4s)}$. As $4/3 \le 3/2 - 2s$, by choosing $n^{(3/2-2s)/(1+4s)} \le 2^{j_2} \le n^{(3/2+2s)/(1+4s)}$, we obtain the result.

5. Extensions. Our technique may certainly be extended to produce estimates for other integer powers. Unfortunately, the computations may become rather cumbersome. It seems more fruitful, at this stage, to extend the construction of Birgé and Massart (1995) since, in addition, it applies to general functionals. We will not perform the extension in detail here; instead we give the principal ideas. Looking at the proof given by Birgé and Massart, one easily sees that the important point is to give a bound for $\int f^3 g$, where g is a known function, where the bound has the same rate of convergence as our previous estimator of $\int f^3$.

We modify slightly the definitions of the three components B^3, M^3, T^3 to achieve this aim. Suppose that g belongs to $B^{1/2}_{\infty\infty}$ and let

$$E_j g = \sum_{k_0 \in \mathbf{Z}} \tilde{\alpha}_{jk_0} \phi_{j,k_0}, \qquad \tilde{\alpha}_{jk_0} = \langle g, \phi_{j,k_0} \rangle.$$

Instead of B^3, M^3, T^3, take now $\tilde{B}^3$, $\tilde{M}^3$, $\tilde{T}^3$ with the following definitions: let $\tilde{B}^3(j)$ be the U-estimator associated with the function $\tilde{G}$ (in place of G), where

$$\tilde{G}^B_j(x, y, z) = \sum_k 2^j \tilde{\alpha}_{jk} \phi_{jk}(x) \phi_{jk}(y) \phi_{jk}(z).$$

$\tilde{M}^3(J, J_*)$ is the U-estimator associated with the function $\tilde{G}$ (in place of G), where

$$\tilde{G}^M_{JJ_*}(x, y, z) = 2^J \sum_k \phi_{Jk}(x) \tilde{\alpha}_{Jk} \sum_{A_k} \psi_{j'k'}(y) \psi_{j'k'}(z).$$

$\tilde{T}^3(J, J_*)$ is the U-estimator associated with the function $\tilde{G}$ (in place of G), where

$$\begin{aligned}\tilde{G}^T_{JJ_*}&(x, y, z)\\ &= 3 \sum_{k_0} 2^{J/2} \tilde{\alpha}_{Jk_0} \sum_{j,k \in A_{k_0}} \sum_{j',k' \in A_{k_0}} \psi_{jk}(x) \psi_{j'k'}(y) \psi_{j'k'}(z) \langle \psi_{jk}, 2^{j'/2} \phi_{j'k'} \rangle\end{aligned}$$

(A_k was defined in Section 4). Apart from these modifications, the construction remains the same.

APPENDIX

We present here the details of the calculations for the variances of the three estimates T, M and B. We have chosen to present them in decreasing order of difficulty so that the easiest case is given in less detail.

A.1. Bounds on the cubic term. Consider the term $\int (D_{J,J_*} f)^3$. Of course, we have the a priori control of this integral, as soon as f is in $B^2_{3\infty}$, by $C\|f\|^3_{s,3,\infty} 2^{-3Js}$. We will now investigate the rate of convergence of an estimator of this quantity. For simplicity of the expressions, from now on we suppress the indices J and $J_* - 1$ in the summations as often as possible. Let $I_j = \sum_k \beta_{jk} \psi_{j,k}$ be the projection of f on W_j. We have

$$\int \Big(\sum I_j\Big)^3 = \sum_{j_1} \sum_{j_2} \sum_{j_3} \int I_{j_1} I_{j_2} I_{j_3}.$$

It is easy to see that if the three indices are all different the integral of the product has no contribution. In this case, as a matter of fact, by a special property of the Haar functions ψ_{jk} this product belongs to $W_{\max(j_1, j_2, j_3)}$. Hence,

$$\int (D_{J,J_*} f)^3 = 3 \sum_{j=J}^{J_*-1} \sum_{j'=J}^{J_*-1} \int I_j (I_{j'})^2 = 3 \sum_{j,k} \sum_{j',k'} \beta_{jk} (\beta_{j'k'})^2 \langle \psi_{jk}, 2^{j'/2} \phi_{j'k'} \rangle.$$

Hence $T^3(J, J_*)$ is an unbiased estimator of $\int (D_{J,J_*} f)^3$. Let us estimate its variance:

$$E\Big(T^3(J, J_*) - \int (D_{J,J_*} f)^3\Big)^2 = E\big(T^3(J, J_*)\big)^2 - \Big(\int (D_{J,J_*} f)^3\Big)^2. \tag{11}$$

In the remainder of this section, we will omit from G any mention of the indices T, J, J^*:

$$E\big(T^3(J, J_*)\big)^2 = \frac{1}{(A_n^3)^2} \sum_{S \times S} E G(X_{i_1}, X_{i_2}, X_{i_3}) G(X_{i'_1}, X_{i'_2}, X_{i'_3}).$$

We have to split $S \times S$ into different subsets in the following way:

$$S_0 = \{((i_1, i_2, i_3), (i'_1, i'_2, i'_3)) \in S \times S : \{i_1, i_2, i_3\} \cap \{i'_1, i'_2, i'_3\} = \emptyset\},$$
$$S_1 = \{((i_1, i_2, i_3), (i'_1, i'_2, i'_3)) \in S \times S : \operatorname{card}\{i_1, i_2, i_3\} \cap \{i'_1, i'_2, i'_3\} = 1\},$$
$$S_2 = \{((i_1, i_2, i_3), (i'_1, i'_2, i'_3)) \in S \times S : \operatorname{card}\{i_1, i_2, i_3\} \cap \{i'_1, i'_2, i'_3\} = 2\},$$
$$S_3 = \{((i_1, i_2, i_3), (i'_1, i'_2, i'_3)) \in S \times S : \operatorname{card}\{i_1, i_2, i_3\} \cap \{i'_1, i'_2, i'_3\} = 3\}.$$

Thus, we can decompose (11) in $e_0 + e_1 + e_2 + e_3$, where

$$e_0 = \frac{1}{(A_n^3)^2} \sum_{S_0} E G(X_{i_1}, X_{i_2}, X_{i_3}) G(X_{i'_1}, X_{i'_2}, X_{i'_3}) - \Big(\int (D_{J,J_*} f)^3\Big)^2,$$

and the other e_j:

$$e_j = \frac{1}{(A_n^3)^2} \sum_{S_j} E G(X_{i_1}, X_{i_2}, X_{i_3}) G(X_{i'_1}, X_{i'_2}, X_{i'_3}).$$

A.1.1. *Bound for* e_0.

$$e_0 = \frac{A_n^6}{(A_n^3)^2}(EG(X,Y,Z))^2 - \left(\int (D_{J,J_*}f)^3\right)^2$$
$$= \left(\frac{A_n^6}{(A_n^3)^2} - 1\right)\left(\int (D_{J,J_*}f)^3\right)^2.$$

Using (3), we get

$$|e_0| \le C2^{-6Js}\|f\|_{s3\infty}^6/n. \tag{12}$$

A.1.2. *Bound for* e_1. As we will have to calculate terms like $EG(X_{i_1}, X_{i_2}, X_{i_3})G(X_{i'_1}, X_{i'_2}, X_{i'_3})$, it will be worthwhile to split again S_1 into subsets reflecting the setting of the two indices which are equal. There obviously are nine such subsets, each of cardinality A_n^5. Fortunately, because of the symmetry of the function G with respect to the two last arguments, we have just to consider three such subsets, as the other ones may be reduced to one of these. In fact, we only have to evaluate

$$e_{1,1} = \frac{A_n^5}{(A_n^3)^2}EG(X,Y,Z)G(X,T,U),$$
$$e_{1,3} = \frac{A_n^5}{(A_n^3)^2}EG(X,Y,Z)G(T,X,U).$$

(i) Bound for $e_{1,1}$: For the sake of simplicity we shall now denote in the long formulas the multiindex (jk) by L, with the obvious extensions, $(j'k') = L'$, $(j_1k_1) = L_1$, $(j'_1k'_1) = L'_1$:

$$e_{1,1} = \frac{9A_n^5}{(A_n^3)^2}\sum_{LL'}\sum_{L_1L'_1}\int f\psi_L\psi_{L_1}(\beta_{L'}\beta_{L'_1})^2\langle\psi_L, 2^{j'/2}\phi_{L'}\rangle\langle\psi_{L_1}, 2^{j'_1/2}\phi_{L'_1}\rangle$$
$$= 9\frac{A_n^5}{(A_n^3)^2}\int f\left(\sum_L\psi_L\left\langle\psi_L, \sum_{L'}2^{j'/2}\phi_{L'}(\beta_{L'})^2\right\rangle\right)^2$$
$$\le \frac{C}{n}\|f\|_\infty\sum_L\left\{\left\langle\sum_{j'}I_{j'}^2, \psi_L\right\rangle\right\}^2.$$

We have used the Parseval identity and

$$I_{j'}^2 = \sum_{k'}(\beta_{L'})^2 2^{j'/2}\phi_{L'}. \tag{13}$$

Moreover, $\langle \sum_{j'} I_{j'}^2, \psi_L\rangle = \langle \sum_{j'=j+1}^{J_*-1} I_{j'}^2, \psi_L\rangle = \langle (\sum_{j'=j+1}^{J_*-1} I_{j'})^2, \psi_L\rangle$. We used here the special algebraic properties of the Haar functions recalled in the Introduction. Thus,

$$
\begin{aligned}
|e_{1,1}| &\le \frac{C}{n}\|f\|_\infty \sum_L \left\{\left\langle \left(\sum_{j'=j+1}^{J_*-1} I_{j'}\right)^2, \psi_L\right\rangle\right\}^2 \\
&\le \frac{C}{n}\|f\|_\infty \sum_j \int \left(\sum_{j'=j+1}^{J_*-1} I_{j'}\right)^4 \qquad (14)\\
&\le \frac{C}{n}\|f\|_\infty \|f\|_{s4\infty}^4 2^{-4Js}.
\end{aligned}
$$

Here we used inequality (3). For later use, we remark that

$$
\begin{aligned}
&\left\| \sum_L \psi_L \left\langle \psi_L, \sum_{L'} 2^{j'/2}\phi_{L'}(\beta_{L'})^2 \right\rangle \right\|_2^2 \qquad (15)\\
&\qquad = \left\| D_{JJ_*}\left(\sum I_j^2\right)\right\|_2^2 \le C 2^{-4Js}\|f\|_{s4\infty}^4.
\end{aligned}
$$

(ii) Bound for $e_{1,2}$:

$$
\begin{aligned}
e_{1,2} &= \frac{9A_n^5}{(A_n^3)^2} \sum_{LL'} \sum_{L_1 L_1'} \beta_L \beta_{L'} \beta_{L_1} \beta_{L_1'} \int f \psi_{L'} \psi_{L_1'} \langle \psi_L, 2^{j'/2}\phi_{L'}\rangle \langle \psi_{L_1}, 2^{j_1'/2}\phi_{L_1'}\rangle \\
&= 9\frac{A_n^5}{(A_n^3)^2} \int f\left(\sum_{LL'} \beta_L \beta_{L'} \langle \psi_L, 2^{j'/2}\phi_{L'}\rangle \psi_{L'}\right)^2 \\
&\le 9C\|f\|_\infty \frac{A_n^5}{(A_n^3)^2} \sum_{L'} 2^{j'}(\beta_{L'})^2 \left(\langle D_{J,J_*} f, \phi_{L'}\rangle\right)^2,
\end{aligned}
$$

using the Parseval identity. Now, by (2) and (3),

$$
\begin{aligned}
|e_{1,2}| &\le \|f\|_\infty \|f\|_{s\infty\infty}^2 \frac{C}{n} \sum_{j'} 2^{-2j's} \sum_{k'} \left(\langle D_{J,J_*} f, \phi_{L'}\rangle\right)^2 \\
&\le \|f\|_\infty \|f\|_{s\infty\infty}^2 \frac{C}{n} \sum_{j'} 2^{-2j's} \int (D_{J,J_*} f)^2 \\
&\le \|f\|_\infty \|f\|_{s\infty\infty}^2 \|f\|_{s2\infty}^2 \frac{C}{n} \sum_{j'} 2^{-2j's} 2^{-2Js} \\
&\le \|f\|_\infty \|f\|_{s\infty\infty}^2 \|f\|_{s2\infty}^2 \frac{C}{n} 2^{-4Js}.
\end{aligned}
$$

Let us also remark that we have in fact proven that

$$\sum_{L'} 2^{j'}(\beta_{L'})^2 \big(\langle D_{J,J_*} f, \phi_{L'}\rangle\big)^2 \le C\|f\|_{s\infty\infty}^2 \|f\|_{s2\infty}^2 2^{-4Js}. \tag{16}$$

(iii) Bound for $e_{1,3}$:

$$\begin{aligned} e_{1,3} &= 9\frac{A_n^5}{(A_n^3)^2} \sum_{LL'} \sum_{L_1 L_1'} (\beta_{L'})^2 \beta_{L_1} \beta_{L_1'} \int f\psi_L \psi_{L_1'} \langle \psi_L, 2^{j'/2}\phi_{L'}\rangle \langle \psi_{L_1}, 2^{j_1'/2}\phi_{L_1'}\rangle \\ &= 9\frac{A_n^5}{(A_n^3)^2} \sum_{L_1'} \langle D_{J,J_*} f, 2^{j_1'/2}\phi_{L_1'}\rangle \beta_{L_1'} \int f\psi_{L_1'} D_{JJ_*}\Big(\sum I_j^2\Big). \end{aligned}$$

Using the Schwarz inequality, we have

$$\begin{aligned} |e_{1,3}| \le 9\frac{A_n^5}{(A_n^3)^2} &\left\{ \sum_{L'} 2^{j'}(\beta_{L'})^2 \big(\langle D_{J,J_*} f, \phi_{L'}\rangle\big)^2 \right\}^{1/2} \\ &\times \left(\sum_{L_1'} \left(\int f\psi_{L_1'} D_{JJ_*}\Big(\sum I_j^2\Big)\right)^2 \right)^{1/2}. \end{aligned}$$

Using the Bessel inequality, (15) and then (16), we have

$$\left(\sum_{L_1'} \left(\int f\psi_{L_1'} D_{JJ_*}\Big(\sum I_j^2\Big)\right)^2 \right)^{1/2} \le \Big\| f D_{JJ_*}\Big(\sum I_j^2\Big)\Big\|_2 \le C 2^{-2Js} \|f\|_{s4\infty}^2 \|f\|_\infty,$$

$$|e_{1,3}| \le \frac{C}{n} \|f\|_{s4\infty}^2 \|f\|_\infty \|f\|_{s\infty\infty} \|f\|_{s2\infty} 2^{-4Js}.$$

A.1.3. *Bound for* e_2. As for e_1 we have to split again S_2 into subsets reflecting the position of the two pairs of indices which are equal. There are 18 such subsets, each of them of cardinality A_n^4. Fortunately, again because of the symmetry of the function G with respect to the two last variables, we have just to consider four such subsets, as the other ones may be reduced to one of these. In fact, we have to evaluate

$$e_{2,1} = \frac{A_n^4}{(A_n^3)^2} EG(X,Y,Z)G(T,Y,Z),$$

$$e_{2,2} = \frac{A_n^4}{(A_n^3)^2} EG(X,Y,Z)G(X,T,Z),$$

$$e_{2,3} = \frac{A_n^4}{(A_n^3)^2} EG(X,Y,Z)G(Y,T,Z),$$

$$e_{2,4} = \frac{A_n^4}{(A_n^3)^2} EG(X,Y,Z)G(Z,T,X).$$

(i) Bound for $e_{2,1}$:

$$e_{2,1} = 9\frac{A_n^4}{(A_n^3)^2}\sum_{LL'}\sum_{L_1L'_1}\beta_L\beta_{L_1}\Big(\int f\psi_{L'}\psi_{L'_1}\Big)^2\langle\psi_L, 2^{j'/2}\phi_{L'}\rangle\langle\psi_{L_1}, 2^{j'_1/2}\phi_{L'_1}\rangle$$

$$= 9\frac{A_n^4}{(A_n^3)^2}\sum_{L'}\sum_{L'_1}\Big(\int f\psi_{L'}\psi_{L'_1}\Big)^2\langle D_{J,J_*}f, 2^{j'/2}\phi_{L'}\rangle\langle D_{J,J_*}f, 2^{j'_1/2}\phi_{L'_1}\rangle,$$

$$(17)\qquad |\langle D_{J,J_*}f, 2^{j'_1/2}\phi_{L'_1}\rangle| \le \|D_{J,J_*}f\|_\infty\|2^{j'/2}\phi_{L'}\|_1 \le C2^{-Js}\|f\|_{s\infty\infty},$$

$$(18)\qquad \sum_{L'L'_1}\Big(\int f\psi_{L'}\psi_{L'_1}\Big)^2 \le \sum_{L'}\int(f\psi_{L'})^2 = \|f\|_2^2\sum_{j'}2^{j'} \le 2^{J_*}\|f\|_2^2$$

and thus, $|e_{2,1}| \le \|f\|_{s\infty\infty}^2\|f\|_2^2 2^{-2Js}2^{J_*}C/n^2$.

(ii) Bound for $e_{2,2}$:

$$e_{2,2} = 9\frac{A_n^4}{(A_n^3)^2}\sum_{LL'}\sum_{L_1L'_1}\beta_{L'}\beta_{L'_1}\int f\psi_L\psi_{L_1}\int f\psi_{L'}\psi_{L'_1}\langle\psi_L, 2^{j'/2}\phi_{L'}\rangle\langle\psi_{L_1}, 2^{j'_1/2}\phi_{L'_1}\rangle$$

$$= 9\frac{A_n^4}{(A_n^3)^2}\sum_{L'L'_1}\int f\beta_{L'}\beta_{L'_1}\psi_{L'}\psi_{L'_1}\int fD_{J,J_*}\big(2^{j'_1/2}\phi_{L'_1}\big)D_{J,J_*}\big(2^{j'/2}\phi_{L'}\big)$$

$$= 9\frac{A_n^4}{(A_n^3)^2}\int\int f(x)f(y)\Big(\sum_{L'}\beta_{L'}\psi_{L'}(x)D_{J,J_*}\big(2^{j'/2}\phi_{L'}\big)(y)\Big)^2\,dx\,dy.$$

Applying the Parseval identity and the fact that D_{J,J_*} is a projector:

$$|e_{2,2}| \le \|f\|_\infty^2\frac{C}{n^2}\sum_{L'}(\beta_{L'})^2 2^{j'}\int\big(D_{J,J_*}(\phi_{L'})\big)^2 \le \|f\|_\infty^2\frac{C}{n^2}2^{J_*}2^{-2Js}\|f\|_{s2\infty}^2.$$

(iii) Bound for $e_{2,3}$:

$$e_{2,3} = 9\frac{A_n^4}{(A_n^3)^2}\sum_{LL'}\sum_{L_1L'_1}\beta_{L'}\beta_{L_1}\int f\psi_L\psi_{L'_1}\int f\psi_{L'}\psi_{L'_1}\langle\psi_L, 2^{j'/2}\phi_{L'}\rangle\langle\psi_{L_1}, 2^{j'_1/2}\phi_{L'_1}\rangle$$

$$= 9\frac{A_n^4}{(A_n^3)^2}\sum_{L'L'_1}\int f\psi_{L'_1}D_{J,J_*}\big(2^{j'/2}\phi_{L'}\big)\int fD_{JJ_*}(f)2^{j'_1/2}\phi_{L'_1}\beta_{L'}\int f\psi_{L'}\psi_{L'_1}.$$

Using (17) and the Bessel inequality,

$$|e_{2,3}| \le 9\frac{A_n^4}{(A_n^3)^2}C2^{-Js}\|f\|_{s\infty\infty}\sum_{L'}|\beta_{L'}|\sum_{L'_1}\Big|\int f\psi_{L'_1}D_{J,J_*}\big(2^{j'/2}\phi_{L'}\big)\Big|\,\Big|\int f\psi_{L'}\psi_{L'_1}\Big|$$

$$\le 9\frac{A_n^4}{(A_n^3)^2}C2^{-Js}\|f\|_{s\infty\infty}\sum_{L'}|\beta_{L'}|\,\|fD_{J,J_*}\big(2^{j'/2}\phi_{L'}\big)\|_2\|f\psi_{L'}\|_2$$

$$\le 9\frac{A_n^4}{(A_n^3)^2}C2^{-Js}\|f\|_{s\infty\infty}2^{J_*/2}\|f\|_\infty\sum_{L'}|\beta_{L'}|\,\|f\psi_{L'}\|_2.$$

By the Schwarz inequality,

$$|e_{2,3}| \le 9\frac{A_n^4}{(A_n^3)^2} C2^{-Js}\|f\|_{s\infty\infty} 2^{J_*/2}\|f\|_\infty \Big(\sum_{L'}|\beta_{L'}|^2\Big)^{1/2}\Big(\sum_{L'}\|f\psi_{L'}\|_2^2\Big)^{1/2}$$

$$\le 9\frac{A_n^4}{(A_n^3)^2} C2^{-2Js}\|f\|_{s\infty\infty}\|f\|_{s2\infty} 2^{J_*/2}\|f\|_\infty\Big(\sum_{j'}\int f^2 2^{j'}\Big)^{1/2}$$

$$\le \frac{C}{n^2}2^{-2Js}\|f\|_{s\infty\infty}\|f\|_{s2\infty}2^{J^*}\|f\|_\infty\|f\|_2.$$

(iv) Bound for $e_{2,4}$:

$$e_{2,4} = 9\frac{A_n^4}{(A_n^3)^2}\sum_{LL'}\sum_{L_1L'_1}\beta_{L'}\beta_{L'_1}\int f\psi_L\psi_{L'_1}\int f\psi_{L'}\psi_{L_1}\langle\psi_L, 2^{j'/2}\phi_{L'}\rangle\langle\psi_{L_1}, 2^{j'_1/2}\phi_{L'_1}\rangle$$

$$= 9\frac{A_n^4}{(A_n^3)^2}\sum_{L'}\sum_{L'_1}\beta_{L'}\beta_{L'_1}\int f\psi_{L'_1}D_{J,J_*}(2^{j'/2}\phi_{L'})\int f\psi_{L'}D_{J,J_*}(2^{j'_1/2}\phi_{L'_1}).$$

By the Schwarz and Bessel inequalities,

$$|e_{2,4}| \le 9\frac{A_n^4}{(A_n^3)^2}\sum_{L'}\sum_{L'_1}\Big(\beta_{L'}\int f\psi_{L'_1}D_{J,J_*}(2^{j'/2}\phi_{L'})\Big)^2$$

$$\le 9\frac{A_n^4}{(A_n^3)^2}\sum_{L'}\beta_{L'}^2\int f^2|D_{J,J_*}(2^{j'/2}\phi_{L'})|^2$$

$$\le 9\frac{A_n^4}{(A_n^3)^2}\sum_{L'}\beta_{L'}^2 2^{j'}\|f\|_\infty^2$$

$$\le \|f\|_\infty^2\frac{C}{n^2}2^{J_*}2^{-2Js}\|f\|_{s2\infty}^2.$$

A.1.4. *Bound for* e_3. We now have to split S_3 in six sets, each of them of cardinality A_n^3. In fact, we just have to evaluate two cases:

$$e_{3,1} = \frac{A_n^3}{(A_n^3)^2}E(G(X,Y,Z))^2, \qquad e_{3,2} = \frac{A_n^3}{(A_n^3)^2}EG(X,Y,Z)G(Y,X,Z).$$

(i) Bound for $e_{3,1}$:

$$e_{3,1} = \frac{9}{(A_n^3)}\sum_{LL'}\sum_{L_1L'_1}\int f\psi_L\psi_{L_1}\Big(\int f\psi_{L'}\psi_{L'_1}\Big)^2\langle\psi_L, 2^{j'/2}\phi_{L'}\rangle\langle\psi_{L_1}, 2^{j'_1/2}\phi_{L'_1}\rangle$$

$$= \frac{9}{(A_n^3)}\sum_{L'}\sum_{L'_1}\Big(\int f\psi_{L'}\psi_{L'_1}\Big)^2\int fD_{J,J_*}(2^{j'_1/2}\phi_{L'_1})D_{J,J_*}(2^{j'/2}\phi_{L'})$$

$$\le \|f\|_\infty\frac{C}{n^3}2^{2J_*}\|f\|_2^2.$$

We used here inequality (18) and the following inequality:

$$\int f D_{J,J_*}\left(2^{j_1'/2}\phi_{L_1'}\right) D_{J,J_*}\left(2^{j'/2}\phi_{L'}\right) \le \|f\|_\infty 2^{(j'+j_1')/2} \le \|f\|_\infty 2^{J_*}.$$

(ii) Bound for $e_{3,2}$:

$$e_{3,2} = \frac{9}{(A_n^3)} \sum_{LL'L_1L_1'} \int f\psi_L\psi_{L_1'} \int f\psi_{L'}\psi_{L_1} \int f\psi_{L'}\psi_{L_1'} \langle \psi_L, 2^{j'/2}\phi_{L'}\rangle \langle \psi_{L_1}, 2^{j_1'/2}\phi_{L_1'}\rangle$$

$$= \frac{9}{(A_n^3)} \sum_{L'L_1'} \int f\psi_{L'}\psi_{L_1'} \int f D_{J,J_*}\left(2^{j_1'/2}\phi_{L_1'}\right)\psi_{L'} \int f D_{J,J_*}\left(2^{j'/2}\phi_{L'}\right)\psi_{L_1'}.$$

By the Schwarz inequality and $|\int f\psi_{L'}\psi_{L_1'}| \le \|f\|_\infty \int |\psi_{L'}\psi_{L_1'}| \le \|f\|_\infty$, we get

$$|e_{3,2}| \le \frac{9}{(A_n^3)} \|f\|_\infty \sum_{L'L_1'} \left\{ \int f D_{J,J_*}\left(2^{j_1'/2}\phi_{L_1'}\right)\psi_{L'} \right\}^2$$

$$= \frac{9}{(A_n^3)} \|f\|_\infty \sum_{L'L_1'} \left\{ \int 2^{j_1'/2}\phi_{L_1'} D_{J,J_*}(f\psi_{L'}) \right\}^2$$

$$\le \frac{9}{(A_n^3)} \|f\|_\infty \sum_{L'j_1'} 2^{j_1'} \int \left(D_{J,J_*}(f\psi_{L'})\right)^2$$

$$\le \frac{9}{(A_n^3)} \|f\|_\infty \sum_{L'j_1'} 2^{j_1'} \int (f\psi_{L'})^2$$

$$\le \frac{9}{(A_n^3)} \|f\|_\infty \sum_{j_1'} 2^{j_1'} \sum_{j'} \int f^2 2^{j'}$$

$$\le \|f\|_\infty \frac{C}{n^3} 2^{2J_*} \|f\|_2^2.$$

Hence, putting together all the previous evaluations, we obtain the following result.

LEMMA 1. *As soon as f is such that $\|f\|_{s2\infty} \le M$, $\|f\|_{s\infty\infty} \le M$, then there exists a constant C depending on M so that*

$$E\left(T^3(J,J_*) - \int \left(D_{J,J_*}f\right)^3\right)^2 \le C\left\{ \frac{2^{-4Js}}{n} + \frac{2^{J_*}2^{-2Js}}{n^2} + \frac{2^{2J_*}}{n^3} \right\}.$$

If $n \le 2^{J_*}$,

$$E\left(T^3(J,J_*) - \int \left(D_{J,J_*}f\right)^3\right)^2 \le C\left\{ \frac{2^{2J_*}}{n^3} \right\}.$$

A.2. Estimation of the cross term. Let us evaluate $\int (D_{J,J_*}f)^2 E_J f$. We have the a priori control of this integral, as soon as f is in $B^s_{2\infty} \cap B^s_{\infty\infty}$ by $C\|f\|^2_{s2\infty}\|f\|_\infty 2^{-2Js}$, and we are going to investigate the rate of convergence of an estimator of this quantity:

$$\int \left(D_{J,J_*}f\right)^2 E_J f = \sum_k \alpha_{Jk} \sum_{A_k} (\beta_{L'})^2 2^{J/2}.$$

Hence $M^3(J, J_*)$ is an unbiased estimator of $\int(D_{J,J_*} f)^2 E_J f$. As previously, let us estimate its variance

$$E\big(M^3(J, J_*)\big)^2 = \frac{1}{\big(A_n^3\big)^2} \sum_{S\times S} EG\big(X_{i_1}, X_{i_2}, X_{i_3}\big)G\big(X_{i'_1}, X_{i'_2}, X_{i'_3}\big).$$

(As previously, we omit the indices J, J^*, M in G.) We have the same splitting of $S \times S$ as in the previous section into the same four subsets. And we can decompose this variance into $e_0 + e_1 + e_2 + e_3$, where

$$e_0 = \frac{1}{\big(A_n^3\big)^2} \sum_{S_0} EG\big(X_{i_1}, X_{i_2}, X_{i_3}\big)G\big(X_{i'_1}, X_{i'_2}, X_{i'_3}\big) - \left(\int \big(D_{J,J_*} f\big)^2 E_J f\right)^2.$$

And the other e_j:

$$e_j = \frac{1}{\big(A_n^3\big)^2} \sum_{S_j} EG\big(X_{i_1}, X_{i_2}, X_{i_3}\big)G\big(X_{i'_1}, X_{i'_2}, X_{i'_3}\big).$$

A.2.1. *Bound for* $e_0 = [A_n^6/(A_n^3)^2]E(G(X,Y,Z))^2 - (\int(D_{J,J_*} f)^2 E_J f)^2$.

$$e_0 = \left(\frac{A_n^6}{\big(A_n^3\big)^2} - 1\right)\left(\int \big(D_{J,J_*} f\big)^2 E_J f\right)^2.$$

So, by (3) and (4),

$$|e_0| \le \frac{C}{n} 2^{-4Js}\big(\|f\|_{s2\infty}^2 \|f\|_\infty\big)^2. \tag{19}$$

A.2.2. *Bound for* e_1. As in the previous section, we have to evaluate three terms:

(i) Bound for $e_{1,1} = [A_n^5/(A_n^3)^2]EG(X,Y,Z)G(X,T,U)$:

$$e_{1,1} = \frac{A_n^5}{\big(A_n^3\big)^2} 2^J \sum_k \int f(\phi_{Jk})^2 \left(\sum_{A_k} (\beta_{L'})^2\right)^2 = \frac{A_n^5}{\big(A_n^3\big)^2} 2^{3J/2} \sum_k \alpha_{Jk} \left(\sum_{A_k} (\beta_{L'})^2\right)^2.$$

As previously, we used the support properties of the ϕ_L. We have, using (2) and the definition of A_k,

$$\sum_{A_k} (\beta_{L'})^2 \le C\|f\|_{s\infty\infty}^2 \sum_{J}^{J_*-1} 2^{(j-J)} 2^{-j(2s+1)} \le c\|f\|_{s\infty\infty}^2 2^{-J(2s+1)}. \tag{20}$$

Using (3) and (4), we obtain

$$|e_{1,1}| \le \frac{C 2^{-4Js}}{n} \|f\|_\infty \|f\|_{s\infty\infty}^4.$$

(ii) Bound for $e_{1,2} = [A_n^5/(A_n^3)^2]EG(X,Y,Z)G(U,Y,T)$:

$$\begin{aligned} e_{1,2} &= \frac{A_n^5}{\big(A_n^3\big)^2} 2^J \sum_{kk_1} \alpha_{Jk}\alpha_{Jk_1} \sum_{A_k, A_{k_1}} \beta_{L'}\beta_{L'_1} \int f \psi_{L'} \psi_{L'_1} \\ &= \frac{A_n^5}{\big(A_n^3\big)^2} 2^J \sum_k (\alpha_{Jk})^2 \int f\left(\sum_{A_k} \beta_{L'}\psi_{L'}\right)^2. \end{aligned}$$

As usual, we used here the support properties of the ψ_L. Using (3) and (4) now, we get

$$|e_{1,2}| = \frac{C}{n}\|f\|_\infty^2 \sum_k \int f\Big(\sum_{A_k} \beta_{L'}\psi_{L'}\Big)^2$$

$$\le \frac{C}{n}\|f\|_\infty^3 \sum_k \sum_{A_k} (\beta_{L'})^2 \le \frac{C2^{-2Js}}{n}\|f\|_\infty^3 \|f\|_{s2\infty}^2.$$

(iii) Bound for $e_{1,3} = [A_n^5/(A_n^3)^2]EG(X,Y,Z)G(T,X,U)$:

$$e_{1,3} = \frac{A_n^5}{(A_n^3)^2} 2^J \sum_{kk_1} \alpha_{Jk_1} \sum_{A_k, A_{k_1}} (\beta_{L'})^2 \beta_{L'_1} \int f\phi_{Jk}\psi_{L'_1}$$

$$= \frac{A_n^5}{(A_n^3)^2} 2^{3J/2} \sum_k \sum_{A_k} \sum_{A_k} (\beta_{L'})^2 (\beta_{L'_1})^2 \alpha_{Jk}$$

$$= \frac{A_n^5}{(A_n^3)^2} 2^{3J/2} \sum_k \alpha_{Jk} \Big(\sum_{A_k} (\beta_{L'})^2\Big)^2 = e_{1,1}.$$

A.2.3. *Bound for* e_2.

(i) Bound for $e_{2,1} = [A_n^4/(A_n^3)^2]EG(X,Y,Z)G(T,Y,Z)$:

$$e_{2,1} = \frac{A_n^4}{(A_n^3)^2} 2^J \sum_{kk_1} \alpha_{Jk}\alpha_{Jk_1} \sum_{A_k, A_{k_1}} \Big(\int f\psi_{L'}\psi_{L'_1}\Big)^2$$

$$= \frac{A_n^4}{(A_n^3)^2} 2^J \sum_k (\alpha_{Jk})^2 \sum_{A_k}\sum_{A_k} \Big(\int f\psi_{L'}\psi_{L'_1}\Big)^2,$$

using again the support property of the ψ_L. Now using (4), we get

$$|e_{2,1}| \le \frac{C\|f\|_\infty^2}{n^2} \sum_k \sum_{A_k}\sum_{A_k} \Big(\int f\psi_{L'}\psi_{L'_1}\Big)^2$$

$$\le \frac{C\|f\|_\infty^2}{n^2} \sum_k \sum_{A_k} \int (f\psi_{L'})^2$$

$$= \frac{C\|f\|_\infty^2}{n^2} \sum_{L'} \int (f\psi_{L'})^2$$

$$\le \frac{C2^{J_*}\|f\|_\infty^2\|f\|_2^2}{n^2} \qquad [\text{cf. } (18)].$$

(ii) Bound for $e_{2,2} = [A_n^4/(A_n^3)^2] EG(X,Y,Z)G(X,T,Z)$:

$$e_{2,2} = \frac{A_n^4}{(A_n^3)^2} 2^J \sum_{kk_1} \int f\phi_{Jk}\phi_{Jk_1} \sum_{A_k, A_{k_1}} \int f\beta_{L'}\beta_{L'_1}\psi_{L'}\psi_{L'_1}$$
$$= \frac{A_n^4}{(A_n^3)^2} 2^J \sum_k 2^{J/2}\alpha_{Jk} \int f\Big(\sum_{A_k} \beta_{L'}\psi_{L'}\Big)^2.$$

Then, using the same arguments as for $e_{1,2}$,

$$|e_{2,2}| \le \frac{C2^J\|f\|_\infty^2}{n^2} \sum_k \int \Big(\sum_{A_k} \beta_{L'}\psi_{L'}\Big)^2 \le \frac{C2^J\|f\|_\infty^2}{n^2} 2^{-2Js}\|f\|_{s2\infty}^2.$$

(iii) Bound for $e_{2,3} = [A_n^4/(A_n^3)^2] EG(X,Y,Z)G(Y,T,Z)$:

$$e_{2,3} = \frac{A_n^4}{(A_n^3)^2} 2^J \sum_{kk_1} \sum_{A_k, A_{k_1}} \alpha_{Jk}\beta_{L'_1} \int f\psi_{L'}\phi_{Jk_1} \int f\psi_{L'}\psi_{L'_1}$$
$$= \frac{A_n^4}{(A_n^3)^2} 2^J \sum_k \sum_{A_k, A_k} \alpha_{Jk}\beta_{L'_1} 2^{J/2}\beta_{L'} \int f\psi_{L'}\psi_{L'_1}$$
$$= \frac{A_n^4}{(A_n^3)^2} 2^J \sum_k 2^{J/2}\alpha_{Jk} \int f\Big(\sum_{A_k} \beta_{L'}\psi_{L'}\Big)^2 = e_{2,2}.$$

(iv) Bound for $e_{2,4} = [A_n^4/(A_n^3)^2] EG(X,Y,Z)G(Z,T,X)$:

$$e_{2,4} = \frac{A_n^4}{(A_n^3)^2} 2^J \sum_{kk_1} \sum_{A_k, A_{k_1}} \beta_{L'}\beta_{L'_1} \int f\psi_{L'}\phi_{Jk_1} \int f\phi_{Jk}\psi_{L'_1}$$
$$= \frac{A_n^4}{(A_n^3)^2} 2^{2J} \sum_k \sum_{A_k, A_k} (\beta_{L'}\beta_{L'_1})^2$$
$$= \frac{A_n^4}{(A_n^3)^2} 2^{2J} \sum_k \Big(\sum_{A_k} (\beta_{L'})^2\Big)^2.$$

Using (20), we have

$$\sum_k \Big(\sum_{A_k} (\beta_{L'})^2\Big)^2 \le \sum_k \sum_{A_k} (\beta_{L'})^2 \|f\|_{s\infty\infty}^2 2^{-J(2s+1)},$$

so we get

$$|e_{2,4}| \le \frac{C2^{-4Js+J}}{n^2} \|f\|_{s\infty\infty}^2 \|f\|_{s2\infty}^2.$$

A.2.4. *Bound for* e_3.

(i) Bound for $e_{3,1} = [A_n^3/(A_n^3)^2]E(G(X,Y,Z))^2$:

$$e_{3,1} = \frac{1}{(A_n^3)} 2^J \sum_{kk_1} \int f\phi_{Jk}\phi_{Jk_1} \sum_{A_k, A_{k_1}} \left(\int f\psi_{L'}\psi_{L'_1}\right)^2$$

$$= \frac{1}{(A_n^3)} 2^J \sum_k 2^{J/2}\alpha_{Jk} \sum_{A_k, A_k} \left(\int f\psi_{L'}\psi_{L'_1}\right)^2.$$

Using an earlier bound for $e_{2,1}$, $|e_{3,1}| \le \|f\|_\infty (C/n^3) 2^{J+J_*}\|f\|_2^2$.

(ii) Bound for $e_{3,2} = [A_n^3/(A_n^3)^2]EG(X,Y,Z)G(Y,X,Z)$:

$$e_{3,2} = \frac{1}{(A_n^3)} 2^J \sum_{kk_1} \sum_{A_k, A_{k_1}} \int f\psi_{L'}\phi_{Jk_1} \int f\phi_{Jk}\psi_{L'_1} \int f\psi_{L'}\psi_{L'_1}$$

$$= \frac{1}{(A_n^3)} 2^{2J} \sum_k \sum_{A_k, A_k} \beta_{L'}\beta_{L'_1} \int f\psi_{L'}\psi_{L'_1}$$

$$= \frac{1}{(A_n^3)} 2^{2J} \sum_k \int f\left(\sum_{A_k} \beta_{L'}\psi_{L'}\right)^2.$$

Hence,

$$|e_{3,2}| \le \frac{C}{n^3} 2^{2J} 2^{-2Js} \|f\|_{s2\infty}^2 \|f\|_\infty.$$

Putting together all the previous evaluations, we obtain the following result.

LEMMA 2. *As soon as* f *is such that* $\|f\|_{s2\infty} \le M$, $\|f\|_{s\infty\infty} \le M$, *then there exists a constant* C *depending on* M *so that*

$$E\left(M^3(J, J_*) - \int (D_{J,J_*} f)^2 E_J f\right)^2 \le C\left\{\frac{2^{-2Js}}{n} + \frac{2^{J_*}}{n^2} + \frac{2^{J+J_*}}{n^3}\right\}.$$

So if $n \le 2^J$,

$$E\left(M^3(J, J_*) - \int (D_{J,J_*} f)^2 E_J f\right)^2 \le C\left\{\frac{2^{J+J_*}}{n^3}\right\}.$$

A.3. Estimation of the low-frequency term. As $\int (E_j f)^3 = 2^{j/2}\sum_k \alpha_{jk}^3$, $B^3(j)$ is an unbiased estimator of $\int (E_j f)^3$, and we are going to evaluate $E(B^3(j))^2 - (\int (E_j f)^3)^2$. However, as in the previous sections (we omit in G the indices B, j),

$$E(B^3(j))^2 = \frac{1}{(C_n^3)^2} \sum_{I\times I} EG(X_{i_1}, X_{i_2}, X_{i_3}) G(X_{i'_1}, X_{i'_2}, X_{i'_3}).$$

And now we have to split $I \times I$ into four subsets in the following way:

$$I_0 = \{((i_1, i_2, i_3), (i'_1, i'_2, i'_3)) \in I \times I : \{i_1, i_2, i_3\} \cap \{i'_1, i'_2, i'_3\} = \emptyset\},$$
$$\operatorname{card} I_0 = C_n^3 C_{n-3}^3,$$
$$I_1 = \{((i_1, i_2, i_3), (i'_1, i'_2, i'_3)) \in I \times I : \operatorname{card}\{i_1, i_2, i_3\} \cap \{i'_1, i'_2, i'_3\} = 1\},$$
$$\operatorname{card} I_1 = n C_{n-1}^2 C_{n-3}^2,$$
$$I_2 = \{((i_1, i_2, i_3), (i'_1, i'_2, i'_3)) \in I \times I : \operatorname{card}\{i_1, i_2, i_3\} \cap \{i'_1, i'_2, i'_3\} = 2\},$$
$$\operatorname{card} I_2 = C_n^2 A_{n-2}^2,$$
$$I_3 = \{((i_1, i_2, i_3), (i'_1, i'_2, i'_3)) \in I \times I : \operatorname{card}\{i_1, i_2, i_3\} \cap \{i'_1, i'_2, i'_3\} = 3\},$$
$$\operatorname{card} I_3 = C_n^3.$$

Thus, we can decompose again the variance in $e_0 + e_1 + e_2 + e_3$, where

$$e_0 = \frac{1}{(C_n^3)^2} \sum_{I_0} EG(X_{i_1}, X_{i_2}, X_{i_3}) G(X_{i'_1}, X_{i'_2}, X_{i'_3}) - \left(\int (E_j f)^3\right)^2,$$

and the other e_j:

$$e_j = \frac{1}{(C_n^3)^2} \sum_{I_j} EG(X_{i_1}, X_{i_2}, X_{i_3}) G(X_{i'_1}, X_{i'_2}, X_{i'_3}).$$

A.3.1. *Bound for* $e_0 = [C_{n-3}^3 / C_n^3](EG(X, Y, Z))^2 - (\int (E_j f)^3)^2$.

$$|e_0| = \left| \frac{C_{n-3}^3}{C_n^3} - 1 \right| \left(\int (E_j f)^3\right)^2 \le \frac{C}{n} \|f\|_3^6.$$

We used (4)

A.3.2. *Bound for* $e_1 = [n C_{n-1}^2 C_{n-3}^2 / (C_n^3)^2] EG(X, Y, Z) G(X, T, U)$.

$$\begin{aligned} e_1 &= \frac{n C_{n-1}^2 C_{n-3}^2}{(C_n^3)^2} 2^j \sum_{kk'} \int f \phi_{jk} \phi_{jk'} (\alpha_{jk'} \alpha_{jk})^2 \\ &= \frac{n C_{n-1}^2 C_{n-3}^2}{(C_n^3)^2} 2^j \sum_k 2^{j/2} \alpha_{jk}^5 \\ &\le \frac{C}{n} \|f\|_5^5. \end{aligned}$$

A.3.3. *Bound for* $e_2 = [C_n^2(n-2)(n-3)/(C_n^3)^2]EG(X,Y,Z)G(T,Y,Z)$.

$$\begin{aligned} e_2 &= \frac{C_n^2(n-2)(n-3)}{\left(C_n^3\right)^2} 2^j \sum_k \left(\int \phi_{jk}^2 f\right)^2 \alpha_{jk}^2 \\ &= \frac{C_n^2(n-2)(n-3)}{\left(C_n^3\right)^2} 2^{2j} \sum_k \alpha_{jk}^4 \\ &\le \|f\|_4^4 \frac{C}{n^2} 2^j. \end{aligned}$$

A.3.4. *Bound for* $e_3 = [1/C_n^3]E(G(X,Y,Z))^2$.

$$e_3 = \frac{1}{C_n^3} 2^j \sum_k \alpha_{jk}^3 2^{3j/2} \le \frac{C 2^{2j}}{n^3} \|f\|_3^3.$$

Hence, putting together all the previous evaluations, we obtain the following result.

LEMMA 3. *As soon as the density f belongs to a bounded subset of L_5 (which is implied by the hypothesis f belongs to a bounded subset of $B_{\infty\infty}^2$), there exists a constant C depending on this bound such that*

$$E\left(B^3(j) - \int (E_j f)^3\right)^2 \le C\left\{\frac{1}{n} + \frac{2^{2j}}{n^3}\right\}. \tag{21}$$

So when $n \le 2^j$,

$$E\left(B^3(j) - \int (E_j f)^3\right)^2 \le C\left\{\frac{2^{2j}}{n^3}\right\}.$$

So when $2^j \le n$,

$$E\left(B^3(j) - \int (E_j f)^3\right)^2 \le \left\{\frac{C}{n}\right\}.$$

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REFERENCES

BERGH, J. and LÖFSTRÖM, J. (1976). *Interpolation Spaces: An Introduction*. Springer, Berlin.

BICKEL, P. and RITOV, Y. (1988). Estimating integrated squares density derivatives: sharp best order of convergence estimates. *Sankhyā Ser. A* **50** 381–393.

BIRGÉ, L. and MASSART, P. (1995). Estimation of integral functionals of a density. *Ann. Statist.* **23** 11–29.

DONOHO, D. L., JOHNSTONE, I. M., KERKYACHARIAN, G. and PICARD, D. (1995). Wavelet shrinkage: asymptopia? (with discussion). *J. Roy. Statist. Soc. Ser. B* **57** 301–369.

DONOHO, D. and NUSSBAUM, M. (1990). Minimax quadratic estimation of a quadratic functional. *J. Complexity* **6** 290–323.

GOLDSTEIN, L. and MESSER, K. (1992). Optimal plug-in estimate for nonparametric functional estimation. *Ann. Statist.* **20** 1306–1328.

HALL, P. and MARRON, S. (1987). Estimation of integrated squared density derivatives. *Statist. Probab. Lett.* **6** 109–115.

HASMINSKII, R. and IBRAGIMOV, I. (1979). On the non parametric estimation of functionals. In *Proceedings of the Second Prague Symposium in Asymptotic Statistics* (P. Mandl and M. Hushkova, eds.) 41–55. North-Holland, Amsterdam.

JOHNSTONE, I., KERKYACHARIAN, G. and PICARD, D. (1992). Estimation d'une densité de probabilité par méthode d'ondelettes. *C. R. Acad. Sci. Paris Sér. I Math.* **315** 211–216.

KERKYACHARIAN, G. and PICARD, D. (1992a). Density estimation in Besov spaces. *Statist. Probab. Lett.* **13** 15–24.

KERKYACHARIAN, G. and PICARD, D. (1992b). Density estimation by kernel and wavelet method: link between kernel geometry and regularity constraints. *C. R. Acad. Sci. Paris Sér. I Math.* **315** 79–84.

KERKYACHARIAN, G. and PICARD, D. (1993). Density estimation by kernel and wavelet methods: optimality of Besov spaces. *Statist. Probab. Lett.* **18** 327–336.

LAURENT, B. (1996). Efficient estimation of integral functionals of a density. *Ann. Statist.* **24** 659–681.

LEVIT, B. (1979). Asymptotically efficient estimation of non linear functionals. *Problems Inform. Transmission* **14** 65–72.

MEYER, Y. (1990). *Ondelettes et Opérateurs I*. Hermann, Paris.

NEMIROVSKII, A. S. (1990). On necessary conditions for efficient estimation of functional of a non parametric signal in white noise model. *Theory Probab. Appl.* **35** 94–103.

PEETRE, J. (1975). *New Thoughts on Besov Spaces*. Dept. Mathematics, Duke Univ., Durham, NC.

RITOV, Y. and BICKEL, P. (1990). Achieving information bounds in non and semi parametric models. *Ann. Statist.* **18** 925–938.

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