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A COMBINATORIAL CENTRAL LIMIT THEOREM FOR RANDOMIZED ORTHOGONAL ARRAY SAMPLING DESIGNS[1]

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Let X be a random vector uniformly distributed on the unit cube and f: $[0,1]^3 \to \mathscr{R}$ be a measurable function. An objective of many computer experiments is to estimate $\mu = E(f \circ X)$ by computing f at a set of points in $[0,1]^3$. There is a design issue in choosing these points. Recently Owen and Tang independently suggested using randomized orthogonal arrays in the choice of such a set. This paper investigates the convergence rate to normality of the distribution of the average of a set of f values taken from one of these designs.

1. Introduction. Let d, n and t be positive integers with $t \le d$. An orthogonal array of strength t is a matrix of n rows and d columns with elements taken from the set $\{0, 1, \dots, q-1\}$ such that in any $n \times t$ submatrix, each of the q^t possible rows occurs the same number of times. The class of all such arrays is denoted by $\mathrm{OA}(n, d, q, t)$ and a more detailed description can be found in Raghavarao (1971).

Owen (1992, 1994) and Tang (1993) independently suggested the use of randomized orthogonal arrays in sampling designs for computer experiments on the d-dimensional unit hypercube $[0,1]^d$. The main attraction of these designs is that they, in contrast to simple random sampling, stratify on all t-variate margins simultaneously.

In this paper we shall be concerned with the following orthogonal array–based sampling design on the unit cube $[0,1]^3$. Let:

(a) π_1, π_2, π_3 be random permutations of $\{0, 1, \dots, q-1\}$, each uniformly distributed on all the $q!$ possible permutations;

(b) $U_{i_1, i_2, i_3, j}$, $0 \le i_1, i_2, i_3 \le q-1$, $1 \le j \le 3$, be $[0,1]$ uniform random variables; and

(c) $U_{i_1, i_2, i_3, j}$'s and π_k's be all stochastically independent.

An orthogonal array–based sample of size q^2 (taken from $[0,1]^3$) is defined to be $\{X(\pi_1(a_{i,1}), \pi_2(a_{i,2}), \pi_3(a_{i,3}))\colon 1 \le i \le q^2\}$, where, for all $0 \le i_1, i_2, i_3 \le$

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$q-1$,

$$X_j(i_1, i_2, i_3) = (i_j + U_{i_1, i_2, i_3, j})/q \qquad \forall\, 1 \le j \le 3,$$
$$X(i_1, i_2, i_3) = (X_1(i_1, i_2, i_3), X_2(i_1, i_2, i_3), X_3(i_1, i_2, i_3))',$$

and $a_{i,j}$ is the (i,j)th element of some arbitrary but fixed $A \in \mathrm{OA}(q^2, 3, q, 2)$.

REMARK. The above sampling design is a special case of those proposed by Owen (1992).

Let X be a random vector uniformly distributed on $[0,1]^3$ and f be a measurable function from $[0,1]^3$ to $\mathscr{R}$. An objective of many computer experiments [see, e.g., McKay, Conover and Beckman (1979), Stein (1987), Owen (1992) and Tang (1993)] is to estimate $\mu = E(f \circ X)$ by computing f at a fixed number of points. The estimator for μ that we are concerned with here is one based on an orthogonal array; namely

$$\hat{\mu} = q^{-2} \sum_{i=1}^{q^2} f \circ X(\pi_1(a_{i,1}), \pi_2(a_{i,2}), \pi_3(a_{i,3})), \tag{1}$$

where $\hat{\mu}$ is the usual average of the orthogonal array–based sample

$$\{f \circ X(\pi_1(a_{i,1}), \pi_2(a_{i,2}), \pi_3(a_{i,3})) \colon 1 \le i \le q^2\}$$

and is an unbiased estimator of μ.

In 1972, Stein introduced a powerful and general method for obtaining an explicit bound for the error in the normal approximation to the distribution of a sum of dependent random variables. Even though since then Stein's method has found considerable applications in combinatorics, probability and statistics [see, e.g., Stein (1986) and the references cited therein], it appears to have largely escaped the attention of researchers in the area of computer experiments. In Section 2 Stein's method is used to investigate the rate of convergence to normality of the distribution of $\hat{\mu}$. In particular, Theorem 2 shows that $\hat{\mu}$ is asymptotically normal (as $q \to \infty$) under the finiteness of rth moments with a corresponding error bound of the order $O(q^{-(r-2)/(2r-2)})$, whether r is an even integer greater than or equal to 4.

The Appendix contains a number of somewhat technical lemmas that are needed in the proof of Theorem 2.

Throughout this paper, Φ and ϕ denote the cumulative distribution function and probability density function of the standard normal distribution, respectively. Given any event B, $I(B)$ denotes its indicator function, and if $x \in \mathscr{R}^3$, then x' is the transpose of x. Finally, if $g \colon \mathscr{R} \to \mathscr{R}$ is a differentiable function, we write $g^{(1)}$ as its derivative.

REMARK. From an application viewpoint, it would be highly desirable to extend the results of this paper to the case of an orthogonal array

$\mathrm{OA}(q^2, d, q, 2)$ with d arbitrary. While we believe that such results should still hold true, the proof here does not appear to be easily adaptable to the case of an arbitrary d.

2. Stein's method. In this section, we shall use Stein's method to investigate the rate of convergence to normality of the distribution of $\hat{\mu}$, where $\hat{\mu}$ is defined as in (1) with $A \in \mathrm{OA}(q^2, 3, q, 2)$. Central to this normal approximation technique is the following lemma.

LEMMA 1 (Stein). *Let $z \in \mathscr{R}$. The unique bounded solution $g_z\colon \mathscr{R} \to \mathscr{R}$ of the differential equation*

$$g^{(1)}(w) - wg(w) = I(w \le z) - \Phi(z) \qquad \forall\, w \in \mathscr{R},$$

is given by

$$g_z(w) = \begin{cases} \Phi(w)[1 - \Phi(z)]/\phi(w), & \text{if } w \le z, \\ \Phi(z)[1 - \Phi(w)]/\phi(w), & \text{if } w > z, \end{cases}$$

with $0 \le g_z(w) \le 1$ and $|g_z^{(1)}(w)| \le 1$ for all $w \in \mathscr{R}$.

PROOF. Lemma 1 is due to Stein (1972), and we refer the reader to his paper for a proof. □

Next we state a simple expression for the asymptotic variance of $\hat{\mu}$ due to Owen (1992).

THEOREM 1. *Suppose $E(f \circ X)^2 < \infty$. Let $\sigma_{\mathrm{oas}}^2 = \mathrm{Var}(\hat{\mu})$ with $\hat{\mu}$ as in (1) for some $A \in OA(q^2, 3, q, 2)$. Then, as $q \to \infty$, we have*

$$q^2\sigma_{\mathrm{oas}}^2 = \int_{[0,1]^3} f_{\mathrm{rem}}^2(x)\,dx + o(1),$$

where, for all $x = (x_1, x_2, x_3)' \in [0,1]^3$,

$$f_j(x_j) = \int_{[0,1]^2} [f(x) - \mu] \prod_{k \ne j} dx_k \qquad \forall\, 1 \le j \le 3,$$

$$f_{k,l}(x_k, x_l) = \int_0^1 [f(x) - \mu - f_k(x_k) - f_l(x_l)] \prod_{j \ne k,l} dx_j \qquad \forall\, 1 \le k < l \le 3, \tag{2}$$

$$f_{\mathrm{rem}}(x) = f(x) - \mu - \sum_{j=1}^{3} f_j(x_j) - \sum_{1 \le k < l \le 3} f_{k,l}(x_k, x_l).$$

Assuming that $\mathrm{Var}(\hat{\mu}) = \sigma_{\mathrm{oas}}^2 > 0$, we define

$$W = \sigma_{\mathrm{oas}}^{-1}(\hat{\mu} - \mu).$$

For all $0 \le i_1, i_2, i_3 \le q-1$, we write

$$
\begin{aligned}
&Ef \circ X(i_1, i_2, i_3) = \mu(i_1, i_2, i_3),\\
&\mu_j(i_j) = q^{-2}\left\{\prod_{k \neq j} \sum_{i_k=0}^{q-1}\right\}[\mu(i_1, i_2, i_3) - \mu] \qquad \forall\, 1 \le j \le 3,\\
&\mu_{k,l}(i_k, i_l) = q^{-1}\left\{\prod_{j \neq k,l} \sum_{i_j=0}^{q-1}\right\}[\mu(i_1, i_2, i_3) - \mu - \mu_k(i_k) - \mu_l(i_l)] \qquad \forall\, 1 \le k < l \le 3,\\
&Y(i_1, i_2, i_3) = q^{-2}\sigma_{\mathrm{oas}}^{-1}\Big[f \circ X(i_1, i_2, i_3) - \mu - \sum_{j=1}^{3} \mu_j(i_j) - \sum_{1 \le k < l \le 3} \mu_{k,l}(i_k, i_l)\Big]
\end{aligned}
\tag{3}
$$

and

$$
\tilde{\mu}(i_1, i_2, i_3) = EY(i_1, i_2, i_3). \tag{4}
$$

A useful consequence of the above construction is that

$$
\sum_{i_j=0}^{q-1} \tilde{\mu}(i_1, i_2, i_3) = 0 \qquad \forall\, 1 \le j \le 3. \tag{5}
$$

Since the orthogonal array A has strength 2, we also observe that W can be rewritten as

$$
W = \sum_{i=1}^{q^2} Y(\pi_1(a_{i,1}), \pi_2(a_{i,2}), \pi_3(a_{i,3})). \tag{6}
$$

We shall now state and prove the main result of this paper.

THEOREM 2. *Let W be as in* (6) *for some $A \in OA(q^2, 3, q, 2)$. Suppose that $E(f \circ X)^r < \infty$ for some even integer $r \ge 4$, and*

$$
\int_{[0,1]^3} f_{\mathrm{rem}}^2(x)\, dx > 0, \tag{7}
$$

with $f_{\mathrm{rem}}(x)$ as in (2). *Then*

$$
\sup\{|P(W \le w) - \Phi(w)|: -\infty < w < \infty\} = O(q^{-(r-2)/(2r-2)})
$$

as $q \to \infty$.

PROOF. Theorem 1 and (7) ensure that $\sigma_{\mathrm{oas}} > 0$ and hence that W is well defined for sufficiently large q.

Let (J_1, J_2) be a random vector uniformly distributed over the set

$$
\{(j_1, j_2) \in \{0, 1, \ldots, q-1\}^2: j_1 \neq j_2\}.
$$

Also, we assume that they are independent of all other random quantities previously defined (e.g., W). Define

$$W^* = \sum_{i=1}^{q^2} Y\big(\tau_{J_1, J_2} \circ \pi_1(a_{i,1}), \pi_2(a_{i,2}), \pi_3(a_{i,3})\big),$$

where τ_{J_1, J_2} is a random permutation of $\{0, \dots, q-1\}$ which transposes J_1 and J_2, leaving all other elements fixed. We observe that (W, W^*) is an exchangeable pair of random variables in that (W, W^*) and (W^*, W) have the same joint distribution.

Since the orthogonal array A is of strength 2, we note that W can be rewritten as

$$W = \sum_{i_1=0}^{q-1} \sum_{i_2=0}^{q-1} Y(i_1, i_2, \rho_\pi(i_1, i_2)), \tag{8}$$

where ρ_π is a random function that maps $\{0, \dots, q-1\}^2$ to $\{0, \dots, q-1\}$ such that

$$\big(i_1, i_2, \rho_\pi(i_1, i_2)\big) = \big(\pi_1(a_{i,1}), \pi_2(a_{i,2}), \pi_3(a_{i,3})\big)$$

for some $1 \le i \le q^2$. Thus it follows from the definition of W^* and (8) that

$$\begin{aligned} W^* &= W - \sum_{i_2=0}^{q-1} Y(J_1, i_2, \rho_\pi(J_1, i_2)) - \sum_{i_2=0}^{q-1} Y(J_2, i_2, \rho_\pi(J_2, i_2)) \\ &\quad + \sum_{i_2=0}^{q-1} Y(J_2, i_2, \rho_\pi(J_1, i_2)) + \sum_{i_2=0}^{q-1} Y(J_1, i_2, \rho_\pi(J_2, i_2)) \\ &= W - S_1 - S_2 - S_3 - S_4, \end{aligned} \tag{9}$$

say, respectively. For convenience, we write

$$V = W - S_1 - S_2.$$

Let $\mathscr{W}$ be the σ-field generated by the random quantities

$$\big\{(\pi_1(a_{i,1}), \pi_2(a_{i,2}), \pi_3(a_{i,3})), U_{\pi_1(a_{i,1}), \pi_2(a_{i,2}), \pi_3(a_{i,3}), j} \colon 1 \le i \le q^2, 1 \le j \le 3\big\}.$$

We observe that W and ρ_π are both $\mathscr{W}$-measurable.

Next let $z \in \mathscr{R}$ and $g_z \colon \mathscr{R} \to \mathscr{R}$ be as in Lemma 1. For the exchangeability of (W, W^*), we have

$$\begin{aligned} 0 &= E(W^* - W)[g_z(W) + g_z(W^*)] \\ &= 2E[E(W^* - W \mid \mathscr{W}) g_z(W)] + E(W^* - W)[g_z(W^*) - g_z(W)]. \end{aligned}$$

Consequently, we observe from Lemma 3 (see the Appendix) that

$$\begin{aligned} EWg_z(W) &= (q/4)E(W^* - W)[g_z(W^*) - g_z(W)] - \Delta \\ &= E\int g_z^{(1)}(V + w)K(w)\,dw - \Delta, \end{aligned} \tag{10}$$

where

$$\Delta = \frac{1}{q-1} E\left[g_z(W) \sum_{i_1=0}^{q-1} \sum_{i_2=0}^{q-1} \tilde{\mu}(i_1, i_2, \rho_\pi(i_1, i_2)) \right],$$

and, for all $w \in \mathscr{R}$,

$$K(w) = \begin{cases} (q/4)(W^* - W), & \text{if } W - V < w \le W^* - V, \\ (q/4)(W - W^*), & \text{if } W^* - V < w \le W - V, \\ 0, & \text{otherwise.} \end{cases}$$

We further observe that

$$|\Delta| \le \frac{1}{q-1} \left[E g_z^2(w) \right]^{1/2} (EW^2)^{1/2} \le \frac{1}{q-1}, \tag{11}$$

since $0 \le g_z(w) \le 1$ for all $w \in \mathscr{R}$. Now we observe from Lemma 1 and (10) that

$$\begin{aligned} &|P(W \le z) - \Phi(z)| \\ &\quad = \left| E\{ g_z^{(1)}(W) - W g_z(W) \} \right| \\ &\quad \le \left| E \int \left[g_z^{(1)}(W) - g_z^{(1)}(V + w) \right] K(w)\, dw \right| \\ &\qquad + \left| E g_z^{(1)}(W) E \int K(w)\, dw - E\left[g_z^{(1)}(W) \int K(w)\, dw \right] \right| \\ &\qquad + \left| E g_z^{(1)}(W) \right| \left| 1 - E \int K(w)\, dw \right| + |\Delta|. \end{aligned} \tag{12}$$

Thus, to prove Theorem 2, it suffices to obtain appropriate bounds for the terms on the right-hand side of (12). This is achieved by (11) and Lemmas 4, 5, and 6 (see the Appendix). Hence we conclude that

$$\sup\{ |P(W \le w) - \Phi(w)| : -\infty < w < \infty \} = O\left(q^{-(r-2)/(2r-2)} \right)$$

as $q \to \infty$. This proves Theorem 2. □

REMARK. We would like to add that the first two terms on the right-hand side of (12) have been studied in some detail by Ho and Chen (1978) in the context of investigating the convergence rate of Hoeffding's combinatorial central limit theorem and our proof of Lemma 6 in this paper was motivated by their results.

APPENDIX

LEMMA 2. *Let W and S_1 be defined as in* (8) *and* (9), *respectively. Then*

$$E \frac{1}{q} \sum_{i_1=0}^{q-1} \sum_{i_2=0}^{q-1} \sum_{i_3=0}^{q-1} Y^2(i_1, i_2, i_3) = 1 + O\left(\frac{1}{q} \right), \tag{13}$$

and, if $E(f\circ X)^r<\infty$ *for some positive even integer* r, *then*

$$E(S_1^r)=O(q^{-r/2}) \tag{14}$$

as $q\to\infty$.

PROOF. We observe from (8) that

$$\begin{aligned}
1 &= EW^2\\
&= E\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1} Y^2(i_1,i_2,\rho_\pi(i_1,i_2))\\
&\quad + E\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{j_1\neq i_1}\sum_{j_2\neq i_2}\tilde\mu(i_1,i_2,\rho_\pi(i_1,i_2))\tilde\mu(j_1,j_2,\rho_\pi(j_1,j_2))\\
&\quad + E\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{j_2\neq i_2}\tilde\mu(i_1,i_2,\rho_\pi(i_1,i_2))\tilde\mu(i_1,j_2,\rho_\pi(i_1,j_2))\\
&\quad + E\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{j_1\neq i_1}\tilde\mu(i_1,i_2,\rho_\pi(i_1,i_2))\tilde\mu(j_1,i_1,\rho_\pi(j_1,i_2))\\
&= E\frac{1}{q}\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{i_3=0}^{q-1} Y^2(i_1,i_2,i_3)\\
&\quad + \frac{1}{q(q-1)}\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{i_3=0}^{q-1}\sum_{j_1\neq i_1}\sum_{j_2\neq i_2}\tilde\mu(i_1,i_2,i_3)\tilde\mu(j_1,j_2,i_3)\\
&\quad + \frac{q-2}{q(q-1)^2}\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{i_3=0}^{q-1}\sum_{j_1\neq i_1}\sum_{j_2\neq i_2}\sum_{j_3\neq i_3}\tilde\mu(i_1,i_2,i_3)\tilde\mu(j_1,j_2,j_3)\\
&\quad + \frac{1}{q(q-1)}\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{i_3=0}^{q-1}\sum_{j_2\neq i_2}\sum_{j_3\neq i_3}\tilde\mu(i_1,i_2,i_3)\tilde\mu(i_1,j_2,j_3)\\
&\quad + \frac{1}{q(q-1)}\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{i_3=0}^{q-1}\sum_{j_1\neq i_1}\sum_{j_3\neq i_3}\tilde\mu(i_1,i_2,i_3)\tilde\mu(j_1,i_2,j_3)\\
&= E\frac{1}{q}\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{i_3=0}^{q-1} Y^2(i_1,i_2,i_3)\\
&\quad + \frac{2q-1}{q(q-1)^2}\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{i_3=0}^{q-1}\tilde\mu^2(i_1,i_2,i_3)\\
&= \frac{1}{q}\left(1+O\left(\frac{1}{q}\right)\right)E\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{i_3=0}^{q-1} Y^2(i_1,i_2,i_3)
\end{aligned}$$

as $q\to\infty$. The second last equality follows from (5). This proves (13).

Next we observe from the definition of S_1 that

$$
\begin{aligned}
E(S_1^r) &= E\left[\sum_{i_2=0}^{q-1} Y(J_1, i_2, \rho_\pi(J_1, i_2))\right]^r \\
&= E\frac{1}{q}\sum_{i_1=0}^{q-1}\left[\sum_{i_2=0}^{q-1} Y(i_1, i_2, \rho_\pi(i_1, i_2))\right]^r.
\end{aligned}
\tag{15}
$$

We observe that on expansion, the right-hand side of (15) consists of a finite (independent of q) sum of terms each of the form

$$
\begin{aligned}
&E\frac{1}{q}\sum_{i_1=0}^{q-1}\sum_{(i_{2,1},\dots,i_{2,m})\in\mathscr{B}(m)}\left\{\prod_{j=1}^{l} Y^{r_j}(i_1, i_{2,j}, \rho_\pi(i_1, i_{2,j}))\right\} \\
&\qquad\times\left\{\prod_{k=l+1}^{m} Y(i_1, i_{2,k}, \rho_\pi(i_1, i_{2,k}))\right\} \\
&= E\frac{(q-m)!}{q(q!)} \\
&\quad\times\sum_{i_1=0}^{q-1}\sum_{(i_{2,1},\dots,i_{2,m})\in\mathscr{B}(m)}\sum_{(i_{3,1},\dots,i_{3,m})\in\mathscr{B}(m)}\left\{\prod_{j=1}^{l} Y^{r_j}(i_1, i_{2,j}, i_{3,j})\right\} \\
&\qquad\times\left\{\prod_{k=l+1}^{m}\tilde{\mu}(i_1, i_{2,k}, i_{3,k})\right\}
\end{aligned}
\tag{16}
$$

for some $0 \le l \le m \le r$, where $r_j \ge 2$ for all $1 \le j \le l$, $m - l + \sum_{j=1}^{l} r_j = r$ and $\mathscr{B}(m)$ is the subset of $\{0, \dots, q-1\}^m$ with all its coordinates distinct.

If $m - l \ge 1$, it follows from (4) and (5) that the number of summations on the right-hand side of (16) can be reduced by 2, namely the variables $i_{2,l+1}$ and $i_{3,l+1}$ can be eliminated. Proceeding in this way, we observe that the right-hand side of (16) can eventually be rewritten as a finite (which does not depend on q) sum of terms each of the form

$$
\begin{aligned}
&E\frac{(q-m)!}{q(q!)} \\
&\quad\times\sum_{i_1=0}^{q-1}\sum_{(i_{2,1},\dots,i_{2,m})\in\mathscr{C}(m,l)}\sum_{(i_{3,1},\dots,i_{3,m})\in\mathscr{C}(m,l)}\left\{\prod_{j=1}^{l} Y^{r_j}(i_1, i_2, i_{3,j})\right\} \\
&\qquad\times\left\{\prod_{k=l+1}^{m}\tilde{\mu}(i_1, i_{2,k}, i_{3,k})\right\}
\end{aligned}
\tag{17}
$$

for some $0 \le l \le m \le r$, where $\mathscr{C}(m,l) = \{(i_1, \dots, i_m) \in \{0, \dots, q-1\}^m\colon i_k = i_j$ for some $j \ne k$ of $l + 1 \le k \le m\}$. Since $r_j \ge 2$ for all $1 \le j \le l$, this implies that the number of distinct coordinates of each point in $\mathscr{C}(m,l)$ is at most

$r/2$. Hence, by Hölder's inequality, we conclude that the absolute value of (17) is bounded by

$$O(q^{(r-6)/2})\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{i_3=0}^{q-1}E[Y^r(i_1,i_2,i_3)] = O(q^{-r/2})$$

as $q\to\infty$. The last equality follows from Theorem 1 and (3) and this proves (14). □

LEMMA 3. *With the notation and assumptions of Theorem* 2,

$$E(W^*-W|\mathscr{W}) = -\frac{2}{q}W - \frac{2}{q(q-1)}\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\tilde{\mu}(i_1,i_2,\rho_\pi(i_1,i_2)).$$

PROOF. We observe from (9) that

$$\begin{aligned} &E(W^*-W|\mathscr{W})\\ &\quad= \frac{2}{q(q-1)}E\left[\sum_{j_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{j_2\neq j_1}Y(j_1,i_2,\rho_\pi(j_2,i_2))|\mathscr{W}\right] - \frac{2}{q}W\\ &\quad= \frac{2}{q(q-1)}E\left[\sum_{j_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{j_2\neq j_1}\tilde{\mu}(j_1,i_2,\rho_\pi(j_2,i_2))|\mathscr{W}\right] - \frac{2}{q}W.\end{aligned} \tag{18}$$

The last equality follows from the observation that, given $\mathscr{W}$, $U_{j_1,i_2,\rho_\pi(j_3,i_2),k}$ is still a uniform [0, 1] random variable whenever $j_1\neq j_3$ and $1\le k\le 3$. Lemma 3 follows from (5) and (18). □

LEMMA 4. *With the notation and assumptions of Theorem* 2, *we have*

$$|Eg_z^{(1)}(W)|\left|E\int K(w)\,dw-1\right|\le\frac{1}{q-1}.$$

PROOF. Since $|g_z^{(1)}(w)|\le 1$ for all $w\in\mathscr{R}$, it suffices only to prove

$$\left|E\int K(w)\,dw-1\right|\le\frac{1}{q-1}.$$

By replacing $g_z(W)$ by W in (10), we have

$$1=E(W^2)=E\int K(w)\,dw-\frac{1}{q-1}E\left[W\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\tilde{\mu}(i_1,i_2,\rho_\pi(i_1,i_2))\right].$$

Lemma 4 follows since, as in (11), we observe that the last term of the above equation is bounded by $1/(q-1)$. □

LEMMA 5. *With the notation and assumptions of Theorem* 2, *we have*

$$\left| Eg_z^{(1)}(W) E\int K(w)\,dw - E\left[g_z^{(1)}(W) \int K(w)\,dw \right] \right| = O(q^{-1/2})$$

as $q \to \infty$ *uniformly over* $z \in \mathscr{R}$.

PROOF. We observe from (9), Lemma 4 and the definition of $K(w)$ that

$$\begin{aligned}
&\left| Eg_z^{(1)}(W) E\int K(w)\,dw - E\left[g_z^{(1)}(W) \int K(w)\,dw \right] \right| \\
&\quad \le E\left| E\left[\int K(w)\,dw - 1 | \mathscr{W} \right] \right| + \frac{1}{q-1} \\
&\quad = \frac{1}{4} E\left| E\left[q(W^* - W)^2 - 4 | \mathscr{W} \right] \right| + \frac{1}{q-1} \\
&\quad \le \frac{1}{4} \sum_{k=1}^{4} E\left| E\left(qS_k^2 - 1 | \mathscr{W} \right) \right| \\
&\qquad + \frac{q}{2} \sum_{1 \le j < k \le 4} E\left| E\left(S_j S_k | \mathscr{W} \right) \right| + \frac{1}{q-1}.
\end{aligned} \tag{19}$$

To prove the lemma, it suffices to find appropriate bounds for the terms on the right-hand side of (19). For the sake of clarity, we shall break the proof down into five steps.

Step 1. From the Cauchy–Schwarz inequality, we observe that

$$\begin{aligned}
\left\{ E \left| E\left(qS_1^2 - 1 | \mathscr{W} \right) \right| \right\}^2 &= \left\{ E \left| E\left[q\left(\sum_{i_2=0}^{q-1} Y(J_1, i_2, \rho_\pi(J_1, i_2)) \right)^2 - 1 \,|\, \mathscr{W} \right] \right| \right\}^2 \\
&\le E\left\{ \sum_{i_1=0}^{q-1} \left[\sum_{i_2=0}^{q-1} Y(i_1, i_2, \rho_\pi(i_1, i_2)) \right]^2 \right\}^2 \\
&\quad - 2E \sum_{i_1=0}^{q-1} \left[\sum_{i_2=0}^{q-1} Y(i_1, i_2, \rho_\pi(i_1, i_2)) \right]^2 + 1.
\end{aligned} \tag{20}$$

We note from (5) and (13) that

$$\begin{aligned}
E \sum_{i_1=0}^{q-1} \left[\sum_{i_2=0}^{q-1} Y(i_1, i_2, \rho_\pi(i_1, i_2)) \right]^2 &= E\frac{1}{q} \sum_{i_1=0}^{q-1} \sum_{i_2=0}^{q-1} \sum_{i_3=0}^{q-1} Y^2(i_1, i_2, i_3) \\
&\quad + \frac{1}{q(q-1)} \sum_{i_1=0}^{q-1} \sum_{i_2=0}^{q-1} \sum_{i_3=0}^{q-1} \tilde{\mu}^2(i_1, i_2, i_3) \\
&= 1 + O\left(\frac{1}{q} \right),
\end{aligned} \tag{21}$$

and similarly (though more tediously),

$$E\left\{\sum_{i_1=0}^{q-1}\left[\sum_{i_2=0}^{q-1} Y(i_1, i_2, \rho_\pi(i_1, i_2))\right]^2\right\}^2 = 1 + O\left(\frac{1}{q}\right)$$

as $q \to \infty$. Thus we conclude from (20) and the symmetry of S_k, $1 \le k \le 2$, that

$$E|E(qS_1^2 - 1|\mathscr{W})| + E|E(qS_2^2 - 1|\mathscr{W})| = O(q^{-1/2}) \tag{22}$$

as $q \to \infty$.

Step 2. Next, we have

$$\begin{aligned}
E(qS_3^2|\mathscr{W}) &= E\left[q\left(\sum_{i_2=0}^{q-1} Y(J_2, i_2, \rho_\pi(J_1, i_2))\right)^2\middle|\mathscr{W}\right] \\
&= E\left[\frac{1}{q-1}\sum_{i_1=0}^{q-1}\sum_{j_1\neq i_1}\sum_{i_2=0}^{q-1} Y^2(i_1, i_2, \rho_\pi(j_1, i_2))\middle|\mathscr{W}\right] \\
&\quad + E\left[\frac{1}{q-1}\sum_{i_1=0}^{q-1}\sum_{j_1\neq i_1}\sum_{i_2=0}^{q-1}\sum_{j_2\neq i_2} \tilde{\mu}(i_1, i_2, \rho_\pi(j_1, i_2))\right. \\
&\qquad\qquad \left.\times\tilde{\mu}(i_1, j_2, \rho_\pi(j_1, j_2))|\mathscr{W}\right] \\
&= E(S_{3,1}|\mathscr{W}) + E(S_{3,2}|\mathscr{W}),
\end{aligned} \tag{23}$$

say, respectively. We observe from (13) that

$$\begin{aligned}
&\{E|E(S_{3,1} - 1|\mathscr{W})|\}^2 \\
&\quad \le E(S_{3,1} - 1)^2 \\
&\quad = E\left\{1 - \frac{2}{q-1}\sum_{i_1=0}^{q-1}\sum_{j_1\neq i_1}\sum_{i_2=0}^{q-1} Y^2(i_1, i_2, \rho_\pi(j_1, i_2))\right. \\
&\qquad + \left(\frac{1}{q-1}\right)^2\sum_{i_1=0}^{q-1}\sum_{j_1\neq i_1}\sum_{i_2=0}^{q-1}\sum_{a_1=0}^{q-1}\sum_{b_1\neq a_1}\sum_{a_2=0}^{q-1} Y^2(i_1, i_2, \rho_\pi(j_1, i_2)) \\
&\qquad\qquad \left.\times Y^2(a_1, a_2, \rho_\pi(b_1, a_2))\right\} \\
&\quad = O\left(\frac{1}{q}\right)
\end{aligned} \tag{24}$$

as $q \to \infty$. Also we note from (5) that

$$
\begin{aligned}
&E(S_{3,2}^2)\\
&\quad = E\frac{1}{(q-1)^2}\sum_{i_1=0}^{q-1}\sum_{j_1\neq i_1}\sum_{i_2=0}^{q-1}\sum_{j_2\neq i_2}\tilde{\mu}(i_1,i_2,\rho_\pi(j_1,i_2))\tilde{\mu}(i_1,j_2,\rho_\pi(j_1,j_2))\\
&\qquad \times\sum_{a_1=0}^{q-1}\sum_{b_1\neq a_1}\sum_{a_2=0}^{q-1}\sum_{b_2\neq a_2}\tilde{\mu}(a_1,a_2,\rho_\pi(b_1,a_2))\tilde{\mu}(a_1,b_2,\rho_\pi(b_1,b_2))\\
&\quad = O\left(\frac{1}{q}\right)
\end{aligned}
\tag{25}
$$

as $q \to \infty$. Thus we conclude from (23), (24), (25) and the symmetry between S_3 and S_4 that

$$
E|E(qS_3^2 - 1|\mathscr{W})| + E|E(qS_4^2 - 1|\mathscr{W})| = O(q^{-1/2})
\tag{26}
$$

as $q \to \infty$.

Step 3. We observe that

$$
\begin{aligned}
&qE|E(S_1S_2|\mathscr{W})|\\
&\quad = \frac{1}{q-1}E\left|\sum_{i_1=0}^{q-1}\sum_{j_1\neq i_1}\sum_{i_2=0}^{q-1}\sum_{j_2=0}^{q-1}Y(i_1,i_2,\rho_\pi(i_1,i_2))Y(j_1,j_2,\rho_\pi(j_1,j_2))\right|\\
&\quad \le \frac{1}{q-1}E(W^2) + \frac{1}{q-1}E\sum_{i_1=0}^{q-1}\left[\sum_{i_2=0}^{q-1}Y(i_1,i_2,\rho_\pi(i_1,i_2))\right]^2\\
&\quad = O\left(\frac{1}{q}\right)
\end{aligned}
\tag{27}
$$

as $q \to \infty$ from (21).

Step 4. Now we note from (9) that

$$
\begin{aligned}
&qE|E(S_3S_4|\mathscr{W})|\\
&\quad = \frac{1}{q-1}E\\
&\qquad \times\left|E\left(\sum_{i_1=0}^{q-1}\sum_{j_1\neq i_1}\sum_{i_2=0}^{q-1}\sum_{j_2=0}^{q-1}\tilde{\mu}(i_1,i_2,\rho_\pi(j_1,i_2))\tilde{\mu}(j_1,j_2,\rho_\pi(i_1,j_2))|\mathscr{W}\right)\right|\\
&\quad \le \frac{1}{q(q-1)^2}\sum_{i_3=0}^{q-1}\sum_{j_3\neq i_3}\left|\sum_{i_1=0}^{q-1}\sum_{j_1\neq i_1}\sum_{i_2=0}^{q-1}\tilde{\mu}(i_1,i_2,i_3)\tilde{\mu}(j_1,i_2,j_3)\right|\\
&\qquad + \frac{q-2}{q(q-1)^3}\sum_{i_3=0}^{q-1}\sum_{j_3\neq i_3}\left|\sum_{i_1=0}^{q-1}\sum_{j_1\neq i_1}\sum_{i_2=0}^{q-1}\sum_{j_2\neq i_2}\tilde{\mu}(i_1,i_2,i_3)\tilde{\mu}(j_1,j_2,j_3)\right|\\
&\qquad + \frac{1}{q(q-1)^2}\sum_{i_3=0}^{q-1}\left|\sum_{i_1=0}^{q-1}\sum_{j_1\neq i_1}\sum_{i_2=0}^{q-1}\sum_{j_2\neq i_2}\tilde{\mu}(i_1,i_2,i_3)\tilde{\mu}(j_1,j_2,i_3)\right|.
\end{aligned}
\tag{28}
$$

Thus it follows from (5) and (13) that the right-hand side of (28) is bounded by

$$\frac{2q-3}{q(q-1)^3}\sum_{i_3=0}^{q-1}\sum_{j_3\neq i_3}\left|\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\tilde{\mu}(i_1,i_2,i_3)\tilde{\mu}(i_1,i_2,j_3)\right| \tag{29}$$
$$+\frac{1}{q(q-1)^2}\sum_{i_3=0}^{q-1}\left|\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\tilde{\mu}^2(i_1,i_2,i_3)\right| = O\left(\frac{1}{q}\right)$$

as $q \to \infty$.

Step 5. We observe from (5) and (13) that

$$\begin{aligned} &qE|E(S_1S_3|\mathscr{W})| \\ &\quad = \frac{1}{q-1}E\left|\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{j_2=0}^{q-1}Y(i_1,i_2,\rho_\pi(i_1,i_2))\tilde{\mu}(i_1,j_2,\rho_\pi(i_1,j_2))\right| \\ &\quad \le \frac{1}{q(q-1)^2}E\sum_{i_3=0}^{q-1}\sum_{j_3\neq i_3}\left|\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}Y(i_1,i_2,i_3)\tilde{\mu}(i_1,i_2,j_3)\right| \\ &\qquad + \frac{1}{q(q-1)}E\sum_{i_3=0}^{q-1}\left|\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}Y(i_1,i_2,i_3)\tilde{\mu}(i_1,i_2,i_3)\right| \\ &\quad = O\left(\frac{1}{q}\right)\end{aligned}$$

as $q \to \infty$. Thus it follows by symmetry that

$$qE|E(S_1S_3|\mathscr{W})| + qE|E(S_1S_4|\mathscr{W})| = O\left(\frac{1}{q}\right) \tag{30}$$

and

$$qE|E(S_2S_3|\mathscr{W})| + qE|E(S_2S_4|\mathscr{W})| = O\left(\frac{1}{q}\right) \tag{31}$$

as $q \to \infty$.

Now we conclude from (19) and the results of the above five steps, namely (22), (26), (27), (29), (30), and (31), that

$$\left|Eg_z^{(1)}(W)E\int K(w)\,dw - E\left[g_z^{(1)}(W)\int K(w)\,dw\right]\right| = O(q^{-1/2})$$

as $q \to \infty$ uniformly over $z \in \mathscr{R}$. This proves Lemma 5. □

LEMMA 6. *With the notation and assumptions of Theorem* 2, *we have*

$$\left|E\int\left[g_z^{(1)}(W) - g_z^{(1)}(V+w)\right]K(w)\,dw\right| = O(q^{-(r-2)/(2r-2)})$$

as $q \to \infty$ *uniformly over* $z \in \mathscr{R}$.

PROOF. Let $\varepsilon > 0$. We observe as in Ho and Chen (1978), page 247, that

$$
\begin{aligned}
E\int \big| g_z^{(1)}(W) - g_z^{(1)}(V+w)\big| K(w)\,dw & \\
\le 2E\int_{|w|>2\varepsilon} K(w)\,dw + 2E\int_{|w|\le 2\varepsilon} I(|S_1+S_2| > 2\varepsilon)K(w)\,dw & \\
+ 4\varepsilon E\int_{|w|\le 2\varepsilon} (|W|+1)K(w)\,dw & \qquad (32)\\
+ E\int_{|w|\le 2\varepsilon} I(z-2\varepsilon \le V \le z+3\varepsilon)K(w)\,dw. &
\end{aligned}
$$

Now, to prove Lemma 6, it suffices to get appropriate bounds for the terms on the right-hand side of (32). To do so, we shall divide the remainder of this proof into four steps.

Step 1. First, we observe as in Lemma 4.6 of Ho and Chen (1978) that

$$
\begin{aligned}
E\int_{|w|>2\varepsilon} K(w)\,dw &\le 2q\sum_{k=1}^{4} E\big[S_k^2 I(|S_k| > \varepsilon)\big] \\
&\le 8qE\big[S_1^2 I(|S_1| > \varepsilon)\big],
\end{aligned} \qquad (33)
$$

since S_k, $1 \le k \le 4$, all share the same marginal probability distribution. Hence, using Hölder and Markov inequalities and (14), the right-hand side of (33) is bounded by

$$
8q\big[E(S_1^r)\big]^{2/r}\big[P(|S_1| > \varepsilon)\big]^{(r-2)/r} \le 8qE(S_1^r)/\varepsilon^{r-2} = \varepsilon^{-(r-2)}O\big(q^{-(r-2)/2}\big)
$$

as $q \to \infty$ uniformly over $\varepsilon > 0$.

Step 2. Also by Hölder and Markov inequalities, we have

$$
\begin{aligned}
E\int_{|w|\le 2\varepsilon} & I(|S_1+S_2| > 2\varepsilon)K(w)\,dw \\
&\le \frac{q}{2}\varepsilon E\left\{\left(\sum_{j=1}^{4}|S_j|\right)\big[I(|S_1| > \varepsilon) + I(|S_2| > \varepsilon)\big]\right\} \\
&\le 4q\varepsilon\big[E(S_1^r)\big]^{1/r}\big[P(|S_1| > \varepsilon)\big]^{(r-1)/r} \\
&\le 4qE(S_1^r)/\varepsilon^{r-2} \\
&= \varepsilon^{-(r-2)}O\big(q^{-(r-2)/2}\big)
\end{aligned}
$$

as $q \to \infty$ uniformly over $\varepsilon > 0$.

Step 3. Next, we observe from (14) that

$$\begin{aligned}\varepsilon E\int_{|w|\le 2\varepsilon}(|W|+1)K(w)\,dw &\le q\varepsilon E(|W|+1)\left(\sum_{k=1}^{4} S_k\right)^2\\ &= O(q\varepsilon)\left[E\left(S_1^4\right)\right]^{1/2}\\ &= O(\varepsilon)\end{aligned}$$

as $q \to \infty$ uniformly over $\varepsilon > 0$.

Step 4. Define

$$h_z(w) = \begin{cases} -4\varepsilon, & \text{if } w \le z - 4\varepsilon,\\ w - z, & \text{if } z - 4\varepsilon \le w \le z + 4\varepsilon,\\ 4\varepsilon, & \text{if } z + 4\varepsilon \le w.\end{cases}$$

Consequently,

$$E\int h_z^{(1)}(V+w)K(w)\,dw \ge E\int_{|w|\le 2\varepsilon} I(z - 2\varepsilon \le V \le z + 2\varepsilon)K(w)\,dw.$$

Thus we observe as in (10) that

$$\begin{aligned}&E\int_{|w|\le 2\varepsilon} I(z-2\varepsilon \le V \le z+2\varepsilon)K(w)\,dw\\ &\quad\le EWh_z(W) + \frac{1}{q-1}E\left[h_z(W)\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\tilde{\mu}(i_1, i_2, \rho_\pi(i_1, i_2))\right]\\ &\quad\le 4\varepsilon E\left|W + \frac{1}{q-1}\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\tilde{\mu}(i_1, i_2, \rho_\pi(i_1, i_2))\right|\\ &\quad= O(\varepsilon)\end{aligned}$$

as $q \to \infty$ uniformly over $\varepsilon > 0$ and $z \in \mathscr{R}$.

Now Lemma 6 follows from (32) and Steps 1 to 4, by taking $\varepsilon = q^{-(r-2)/(2r-2)}$. □

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