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CONDITIONS FOR RECURRENCE AND TRANSIENCE OF A MARKOV CHAIN ON $\mathbb{Z}^+$ AND ESTIMATION OF A GEOMETRIC SUCCESS PROBABILITY

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Let Z be a discrete random variable with support $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$. We consider a Markov chain $Y = (Y_n)_{n=0}^\infty$ with state space $\mathbb{Z}^+$ and transition probabilities given by $P(Y_{n+1} = j|Y_n = i) = P(Z = i + j)/P(Z \geq i)$. We prove that convergence of $\sum_{n=1}^\infty 1/[n^3 P(Z = n)]$ is sufficient for transience of Y while divergence of $\sum_{n=1}^\infty 1/[n^2 P(Z \geq n)]$ is sufficient for recurrence. Let X be a Geometric(p) random variable; that is, $P(X = x) = p(1 - p)^x$ for $x \in \mathbb{Z}^+$. We use our results in conjunction with those of M. L. Eaton [*Ann. Statist.* **20** (1992) 1147–1179] and J. P. Hobert and C. P. Robert [*Ann. Statist.* **27** (1999) 361–373] to establish a sufficient condition for $\mathcal{P}$-admissibility of improper priors on p. As an illustration of this result, we prove that all prior densities of the form $p^{-1}(1 - p)^{b-1}$ with $b > 0$ are $\mathcal{P}$-admissible.

1. Introduction. We begin with the statistical problem. Suppose that X is a Geometric(p) random variable; that is, $P(X = x) = p\,(1 - p)^x$ for $x \in \mathbb{Z}^+ = \{0, 1, 2, \ldots\}$. Set $\mathbb{R}^+ = (0, \infty)$ and let $\nu : (0, 1) \to \mathbb{R}^+$ be such that $\int_0^1 \nu(p)\,dp = \infty$ and $\int_0^1 p\,\nu(p)\,dp < \infty$. Under these conditions, $\nu(p)$ can be viewed as an *improper* prior density for the parameter p which yields a *proper* posterior density given by

$$\pi(p|x) = \frac{p\,(1 - p)^x\,\nu(p)}{m_\nu(x)},$$

where, of course, $m_\nu(x) := \int_0^1 p\,(1 - p)^x\,\nu(p)\,dp$.

We associate with each such ν an irreducible, aperiodic Markov chain $\Phi^\nu = (\Phi_n^\nu)_{n=0}^\infty$ with state space $\mathbb{Z}^+$ and transition probabilities given by

$$P(\Phi_{n+1}^\nu = j|\Phi_n^\nu = i) = \frac{\int_0^1 p^2\,(1 - p)^{i+j}\,\nu(p)\,dp}{\int_0^1 p\,(1 - p)^i\,\nu(p)\,dp}$$

for $i, j \in \mathbb{Z}^+$. It follows from results of Eaton (1992) and Hobert and Robert (1999) that if Φ^ν is recurrent, then the prior ν is $\mathcal{P}$-admissible under squared error loss.

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[Roughly speaking, an improper prior is $\mathcal{P}$-admissible if the generalized Bayes estimates it generates are admissible; see Eaton (1997) for a detailed introduction to these ideas.] In this paper, we analyze a family of Markov chains that includes all the Φ^ν's. Our main result is a sufficient condition for recurrence and a sufficient condition for transience. A corollary of this result is a simple sufficient condition for the $\mathcal{P}$-admissibility of ν. We now describe the family of chains that we will study.

Suppose Z is a discrete random variable with support $\mathbb{Z}^+$. Let $Y = (Y_n)_{n=0}^\infty$ be a Markov chain with state space $\mathbb{Z}^+$ and transition probabilities given by

$$P(Y_{n+1} = j | Y_n = i) = p_{ij} = \frac{P(Z = i + j)}{P(Z \geq i)} \tag{1}$$

for all $i, j \in \mathbb{Z}^+$. The fact that $P(Z = i + j) > 0$ for all $i, j \in \mathbb{Z}^+$ implies that Y is irreducible and aperiodic. Let $\pi_i = P(Z \geq i)$ and note that $\pi_i \, p_{ij} = \pi_j \, p_{ji}$ for all $i, j \in \mathbb{Z}^+$. Thus, Y is *reversible* and the sequence $(\pi_i)_{i=0}^\infty$ is an invariant sequence for Y since

$$\sum_{i=0}^\infty \pi_i \, p_{ij} = \sum_{i=0}^\infty \pi_j \, p_{ji} = \pi_j$$

for all $j \in \mathbb{Z}^+$. It follows [see, e.g., Durrett (1996), Chapter 5] that if $\sum_{i=0}^\infty \pi_i < \infty$, then the chain is positive recurrent, and if $\sum_{i=0}^\infty \pi_i = \infty$, then the chain is either null recurrent or transient. Moreover, since $\sum_{i=0}^\infty \pi_i = 1 + \mathrm{E}[Z]$, the Markov chain Y is positive recurrent if and only if $\mathrm{E}[Z] < \infty$.

In this paper, we focus on differentiating between null recurrence and transience of Y when Z has infinite expectation. [Note that Φ^ν is never positive recurrent since $m_\nu(x)$ is an invariant sequence.] Standard results for establishing recurrence and transience of Markov chains on $\mathbb{Z}^+$ [e.g., Lamperti (1960)] involve relationships between $\mathrm{E}(Y_{n+1} | Y_n = i)$ and $\mathrm{E}(Y_{n+1}^2 | Y_n = i)$. However, when $\mathrm{E}[Z] = \infty$, $\mathrm{E}(Y_{n+1} | Y_n = i) = \infty$ for all $i \in \mathbb{Z}^+$. Thus, these results are of no use for analyzing Y. We prove a result which can often be used to determine whether Y is null recurrent or transient when Z has infinite expectation. Our main result is as follows.

THEOREM 1. *If*

$$\sum_{n=1}^\infty \frac{1}{n^3 P(Z = n)} < \infty, \tag{2}$$

then the Markov chain Y is transient. If

$$\sum_{n=1}^\infty \frac{1}{n^2 P(Z \geq n)} = \infty, \tag{3}$$

then Y is recurrent.

REMARK. Since $(\pi_i)_{i=0}^{\infty}$ is a decreasing sequence, $\sum_{i=0}^{\infty} \pi_i < \infty$ implies $i\pi_i \to 0$ [Knopp (1990), page 124] and this in turn implies that (3) holds. Thus, our sufficient condition for recurrence is satisfied for every positive recurrent chain.

We now give three examples. Examples 1 and 2 demonstrate the application of Theorem 1. Example 3 shows that it is possible for neither (2) nor (3) to hold.

EXAMPLE 1. Suppose for all $z \geq N$, we have $P(Z = z) = Cz^{\alpha}$ for some $C > 0$ and $\alpha < -1$. Then $1/[n^3 P(Z = n)] = 1/Cn^{3+\alpha}$ for $n \geq N$. Therefore, if $\alpha > -2$, then (2) holds, so the chain is transient. For $n \geq N$, we have $P(Z \geq n) \leq \int_{n-1}^{\infty} Cx^{\alpha}\, dx = -C(n-1)^{\alpha+1}/(\alpha+1)$, and so $1/[n^2 P(Z \geq n)] \geq -(\alpha+1)/[Cn^2(n-1)^{\alpha+1}]$. Therefore, if $\alpha \leq -2$, then (3) holds, so the chain is recurrent. Note that when $\alpha < -2$, we have $\mathrm{E}[Z] < \infty$, so the chain is positive recurrent. It follows that the chain is null recurrent if and only if $\alpha = -2$.

EXAMPLE 2. Suppose for all $z \geq N$, we have $P(Z = z) = Cz^{-2}(\log z)^{\alpha}$ for some $C > 0$ and $\alpha > 0$. We have $1/[n^3 P(Z = n)] = 1/[Cn(\log n)^{\alpha}]$ for $n \geq N$. Therefore, if $\alpha > 1$, then (2) holds and the chain is transient. Also, for sufficiently large n, we have

$$P(Z \geq n) = \sum_{k=n}^{\infty} \frac{C(\log k)^{\alpha}}{k^{1/2}} \frac{1}{k^{3/2}} \leq \frac{C(\log n)^{\alpha}}{n^{1/2}} \sum_{k=n}^{\infty} \frac{1}{k^{3/2}} \leq \frac{A(\log n)^{\alpha}}{n}$$

for some constant $A > 0$. Therefore, $1/[n^2 P(Z \geq n)] \geq 1/[An(\log n)^{\alpha}]$ for sufficiently large n. It follows that when $\alpha \leq 1$, (3) holds and the chain is recurrent. In this example, the chain is never positive recurrent since $\mathrm{E}[Z] = \infty$ for all $\alpha > 0$. Thus, the chain is null recurrent for $\alpha \leq 1$.

EXAMPLE 3. Suppose, for some $C > 0$, we have $P(Z = z) = Cz^{-3}$ when z is odd and $P(Z = z) = Cz^{-3/2}$ when $z > 0$ and z is even. Then $1/[n^3 P(Z = n)] = C^{-1}$ when n is odd, so (2) is false. If $n > 0$ and n is even, then $P(Z \geq n) \geq \frac{1}{2}\sum_{k=n}^{\infty} Cn^{-3/2} \geq Cn^{-1/2}$. Therefore, $1/[n^2 P(Z \geq n)] \leq 1/Cn^{3/2}$. It follows that (3) also does not hold.

We now return to our statistical problem. Any Markov chain on $\mathbb{Z}^+$ whose transition probabilities take the form $p_{ij} = c_i\, d_{i+j}$, where $(c_k)_{k=0}^{\infty}$ and $(d_k)_{k=0}^{\infty}$ are sequences of positive numbers, is a member of the family described above. This follows by taking Z such that $P(Z = z) = c_0\, d_z$ for $z \in \mathbb{Z}^+$. Thus, Φ^{ν} is a member of this family and the corresponding Z, call it Z_{ν}, has distribution

$$P(Z_{\nu} = z) = \frac{\int_0^1 p^2 (1-p)^z\, \nu(p)\, dp}{\int_0^1 p\, \nu(p)\, dp}. \tag{4}$$

Combining Theorem 1 with (4) yields the following result.

COROLLARY 1. *If*

$$\sum_{n=1}^{\infty} \frac{1}{n^2\, m_\nu(n)} = \infty\,,$$

then Φ^ν *is recurrent and* ν *is* $\mathcal{P}$*-admissible. If*

$$\sum_{n=1}^{\infty} \frac{1}{n^3 \int_0^1 p^2\,(1-p)^n\,\nu(p)\,dp} < \infty,$$

then Φ^ν *is transient.*

The rest of this paper is organized as follows. In Section 2, we will use a theorem of Lyons to prove that (2) implies that Y is transient. In Section 3, we will apply a result of McGuinness to show why (3) implies that the chain is recurrent. Finally, in Section 4, we generalize a result of Hobert and Robert (1999) by applying Corollary 1 in the special case where $\nu(p) \propto p^{a-1}\,(1-p)^{b-1}$.

2. Proving transience. The results that we will use to prove Theorem 1 are related to a well-known connection between reversible Markov chains and electrical networks. For a summary of this connection, and how the connection can be used to prove the transience or recurrence of reversible Markov chains, see Doyle and Snell (1984) or Sections 8–10 of Peres (1999). A *network* is a pair $N = [G, c]$, where G is a connected graph with countable vertex set $V(G)$ and edge set $E(G)$, and c is a function from $E(G)$ to the positive real numbers. If $e \in E(G)$, then $c(e)$ is called the *conductance* of the edge e. If v and w are vertices of G which are connected by an edge, then we write $v \sim w$ and denote the edge connecting v and w by e_{vw}. For $v \in V(G)$, let $c(v) = \sum_{w\,:\,v\sim w} c(e_{vw})$. A *weighted random walk* on N is a Markov chain $S = (S_n)_{n=0}^{\infty}$ with state space $V(G)$ whose transition probabilities are given by $P(S_{n+1} = w | S_n = v) = c(e_{vw})/c(v)$ if $v \sim w$ and $P(S_{n+1} = w | S_n = v) = 0$ otherwise.

If $a \in V(G)$, a *flow* from a to ∞ is a real-valued function θ defined on $V(G) \times V(G)$ such that $\theta(v, w) = 0$ unless $v \sim w$, $\theta(v, w) = -\theta(w, v)$ for all $v, w \in V(G)$, and $\sum_{w\in V(G)} \theta(v, w) = 0$ if $v \neq a$. We call the flow a *unit flow* if $\sum_{w\in V(G)} \theta(a, w) = 1$. The *energy* of the flow is defined by $\mathcal{E}(\theta) = \frac{1}{2} \sum_{(v,w):v\sim w} \theta(v, w)^2/c(e_{vw})$. The following theorem is due to Lyons (1983).

THEOREM 2. *The weighted random walk on a network* $N = [G, c]$ *is transient if and only if, for some* $a \in V(G)$, *there exists a unit flow from* a *to* ∞ *having finite energy.*

We will now apply Theorem 2 to the Markov chain Y defined in Section 1 to show that (2) implies that the chain is transient. First, we must show how to interpret this chain as a weighted random walk on a network. Let G be the graph in which $V(G) = \mathbb{Z}^+$ and there is an edge between any two distinct vertices in G. Define $c(e_{ij}) = \pi_i p_{ij} = P(Z = i + j)$. Then, for the network $N = [G, c]$, the transition probabilities of the weighted random walk S are given by

$$P(S_{n+1} = j | S_n = i) = \frac{c(e_{ij})}{c(i)} = \frac{\pi_i p_{ij}}{\sum_{j\,:\,j \neq i} \pi_i p_{ij}} = \frac{p_{ij}}{1 - p_{ii}}$$

for all $i \neq j$. It is easily verified that these are also the transition probabilities of the chain $\tilde{Y} = (\tilde{Y}_n)_{n=0}^{\infty}$ obtained from the chain Y by removing repeated values. That is, we define $\tilde{Y}_n = Y_{T_n}$, where $T_0 = 0$ and $T_n = \inf\{k > T_{n-1} : Y_k \neq Y_{T_{n-1}}\}$ for all $n \in \mathbb{N} = \{1, 2, \ldots\}$. Note that Y is transient if and only if $\tilde{Y}$ is transient, so Y is transient if and only if S is transient. Therefore, to show that (2) implies the transience of the chain Y, it suffices to find a unit flow from some vertex to infinity on the network N which has finite energy whenever (2) holds.

We now define a real-valued function θ on $V(G) \times V(G)$. Let $B_0 = \{0\}$. For $k \in \mathbb{N}$, let $B_k = \{2^{k-1}, 2^{k-1} + 1, \ldots, 2^k - 1\}$. The sets B_k are disjoint, and $\mathbb{Z}^+ = \bigcup_{k=0}^{\infty} B_k$. Suppose $i \in B_k$ and $j \in B_l$, where we assume $l \geq k$ without loss of generality. If $l = k$ or $l \geq k + 2$, define $\theta(i, j) = \theta(j, i) = 0$. If $l = k + 1$, define $\theta(i, j) = 2^{-2k+1}$ and $\theta(j, i) = -2^{-2k+1}$, unless $k = 0$ in which case we define $\theta(0, 1) = 1$ and $\theta(1, 0) = -1$. Note that $\sum_{j=0}^{\infty} \theta(1, j) = \theta(1, 0) + \theta(1, 2) + \theta(1, 3) = -1 + 1/2 + 1/2 = 0$. Also, if $i \in B_k$ for $k \geq 2$, then

$$\sum_{j=0}^{\infty} \theta(i, j) = \sum_{j \in B_{k-1}} \theta(i, j) + \sum_{j \in B_{k+1}} \theta(i, j) = -2^{k-2} 2^{-2(k-1)+1} + 2^k 2^{-2k+1} = 0.$$

Therefore, θ is a flow from 0 to ∞. Since $\sum_{j=0}^{\infty} \theta(0, j) = \theta(0, 1) = 1$, this flow is a unit flow.

We now obtain an upper bound for the energy of θ. We have

$$\mathcal{E}(\theta) = \frac{1}{2} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{\theta(i, j)^2}{P(Z = i + j)} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{P(Z = n)} \sum_{i=0}^{n} \theta(i, n - i)^2.$$

Note that $\theta(i, n - i) = 0$ if $n - i \geq 4i$, unless $i = 0$ and $n = 1$, so

$$\mathcal{E}(\theta) = \frac{1}{P(Z = 1)} + \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{P(Z = n)} \sum_{i=\lceil n/5 \rceil}^{n} \theta(i, n - i)^2.$$

If $i \in B_k$ and $k \geq 2$, then $\theta(i, j)^2 \leq (2^{-2(k-1)+1})^2 = 64(2^k)^{-4} \leq 64 i^{-4}$. It follows that

$$\mathcal{E}(\theta) \leq \frac{1}{P(Z = 1)} + \sum_{n=2}^{\infty} \frac{32n}{P(Z = n)} \left(\frac{n}{5}\right)^{-4} = \frac{1}{P(Z = 1)} + \sum_{n=2}^{\infty} \frac{20{,}000}{n^3 P(Z = n)}.$$

Thus, $\mathcal{E}(\theta) < \infty$ whenever (2) holds.

3. Proving recurrence. In this section we will prove that (3) implies that the Markov chain Y defined in Section 1 is recurrent. Given a graph G, we can obtain a new graph by subdividing an edge of G. That is, we can add vertices $u_1, \dots, u_{n-1}$ to the graph and then replace an edge e in G connecting the vertices v and w with edges $e_1, \dots, e_n$, where e_1 connects v to u_1, e_k connects u_{k-1} to u_k for $2 \le k \le n-1$, and e_n connects u_{n-1} to w. A network $M = [H, d]$ is said to be a *refinement* of the network $N = [G, c]$ if the graph H can be obtained by subdividing some of the edges of G and if, whenever $e \in E(G)$ is replaced by edges $e_1, \dots, e_n \in E(H)$, we have

$$\sum_{i=1}^{n} d(e_i)^{-1} = c(e)^{-1}. \tag{5}$$

McGuinness (1991) observes that if M is a refinement of N, then the weighted random walk on M is recurrent if and only if the weighted random walk on N is recurrent. One explanation for this result is that the condition for recurrence of the weighted random walk on a network can be expressed in terms of the effective conductances between vertices of the network [see Peres (1999)] and the series law for electrical conductances states that replacing an edge from v to w by the edges $e_1, \dots, e_n$ does not change the effective conductance between the vertices v and w when (5) holds.

Given any network $N = [G, c]$, let $\mathcal{U} = \{U_n\}_{n=0}^{\infty}$ be a partition of $V(G)$. If, whenever $|m - n| \ge 2$, there is no edge between a vertex in U_m and a vertex in U_n, we call $\mathcal{U}$ an *N-constriction*. Let $\tau_a^N(U_n)$ denote the probability that the weighted random walk on N starting at a eventually reaches a vertex in the set U_n. Let E_n be the set of edges connecting a vertex in U_{n-1} to a vertex in U_n. We will use the following theorem due to McGuinness (1991).

THEOREM 3. *Let $N = [G, c]$ be a network, and let $a \in V(G)$. Then the weighted random walk on N is recurrent if and only if there exists a refinement $M = [H, d]$ of N having an M-constriction $\mathcal{U} = \{U_n\}_{n=0}^{\infty}$ such that $a \in U_0$, $\tau_a^M(U_n) = 1$ for all $n \in \mathbb{N}$, and*

$$\sum_{n=1}^{\infty} \left(\sum_{e \in E_n} d(e) \right)^{-1} = \infty. \tag{6}$$

Theorem 3 is a generalization of a result of Nash-Williams (1959), which established a necessary and sufficient condition for recurrence in locally finite networks (i.e., networks in which each vertex is connected to only finitely many other vertices).

Let $N = [G, c]$ be the network described in Section 2. We will construct a refinement $M = [H, d]$ as follows. For all $i, j \in \mathbb{Z}^+$ such that $i < j$, we add vertices v_{ij}^n for $n = i + 1, i + 2, \dots, j - 1$ and replace the edge e_{ij} with edges

e_{ij}^n for $n = i+1, i+2, \dots, j$. When $j = i+1$, the edge e_{ij}^j connects i to j. Otherwise, e_{ij}^{i+1} connects i to v_{ij}^{i+1}, e_{ij}^j connects v_{ij}^{j-1} to j, and e_{ij}^n connects v_{ij}^{n-1} to v_{ij}^n for $n = i+2, i+3, \dots, j-1$. Define

$$d(e_{ij}^n) = P(Z = i+j)\left(\sum_{m=i+1}^{j} m^{-3/2}\right) n^{3/2}.$$

Note that

$$\begin{aligned}\sum_{n=i+1}^{j} d(e_{ij}^n)^{-1} &= \frac{1}{P(Z=i+j)}\left(\sum_{m=i+1}^{j} m^{-3/2}\right)^{-1} \sum_{n=i+1}^{j} n^{-3/2} \\ &= \frac{1}{P(Z=i+j)} = c(e_{ij})^{-1},\end{aligned}$$

which verifies (5).

For all $n \in \mathbb{Z}^+$, let $U_n = \{n\} \cup \{v_{ij}^n : i < n < j\}$. It follows from the definition of H that every edge in $E(H)$ with one end in U_n has its other end in U_{n-1} or U_{n+1}. Therefore, $\mathcal{U} = \{U_n\}_{n=0}^{\infty}$ is an M-constriction. For $n \in \mathbb{N}$, let $E_n = \{e_{ij}^n : i < n \le j\}$ be the set of edges connecting a vertex in U_{n-1} to a vertex in U_n. Clearly $0 \in U_0$. We claim that $\tau_0^M(U_n) = 1$ for all $n \in \mathbb{Z}^+$. To prove the claim, let $S = (S_k)_{k=0}^{\infty}$ be the weighted random walk on M starting at 0. Let $T_0 = 0$, and for $m \in \mathbb{N}$, let $T_m = \inf\{k > T_{m-1} : S_k \ne S_{T_{m-1}} \text{ and } S_k \in \mathbb{Z}^+\}$, which is almost surely finite. Then define a Markov chain $\tilde{S} = (\tilde{S}_k)_{k=0}^{\infty}$ with state space $\mathbb{Z}^+$ by $\tilde{S}_k = S_{T_k}$. Note that if $\tilde{S}_k \ge n$, then $S_j \in U_n$ for some $j \le T_k$. Since it is easily verified that the chain $\tilde{S}$ is irreducible, we have $P(\tilde{S}_k \ge n \text{ for some } k) = 1$, so $\tau_0^M(U_n) = 1$.

Now, assume that (3) holds. By Theorem 3, if we can show that (6) holds, it will follow that the Markov chain Y described in Section 1 is recurrent, which will complete the proof of Theorem 1. For any $n \in \mathbb{N}$, we have

$$\begin{aligned}\sum_{e \in E_n} d(e) &= \sum_{i=0}^{n-1} \sum_{j=n}^{\infty} d(e_{ij}^n) = \sum_{k=n}^{\infty} \sum_{i=0}^{\min\{n-1,k-n\}} d(e_{i,k-i}^n) \\ &\le \sum_{k=n}^{\infty} \sum_{i=0}^{n-1} P(Z=k)\left(\sum_{m=i+1}^{k-i} m^{-3/2}\right) n^{3/2} \\ &= n^{3/2} \sum_{k=n}^{\infty} P(Z=k)\left(\sum_{i=0}^{n-1} \sum_{m=i+1}^{k-i} m^{-3/2}\right).\end{aligned}$$

Note that

$$\sum_{i=0}^{n-1}\sum_{m=i+1}^{k-i} m^{-3/2} \le \sum_{m=1}^{n} m^{-1/2} + n\sum_{m=n+1}^{k} m^{-3/2}$$
$$\le 1 + \int_1^n x^{-1/2}\,dx + n\int_n^k x^{-3/2}\,dx$$
$$= 1 + (2n^{1/2}-2) + 2n(n^{-1/2}-k^{-1/2}) \le 4n^{1/2}.$$

Therefore,

$$\sum_{e\in E_n} d(e) \le 4n^2 \sum_{k=n}^{\infty} P(Z=k) = 4n^2 P(Z\ge n).$$

It follows that

$$\sum_{n=1}^{\infty}\biggl(\sum_{e\in E_n} d(e)\biggr)^{-1} \ge \sum_{n=1}^{\infty} \frac{1}{4n^2 P(Z\ge n)},$$

which is infinite by (3).

4. $\mathcal{P}$-admissibility of improper conjugate priors. Consider the following family of prior densities for p:

$$\nu(p;a,b) = \begin{cases} p^{a-1}(1-p)^{b-1}, & 0<p<1,\\ 0, & \text{otherwise.}\end{cases}$$

Each (a,b) pair corresponds to a particular prior and the set

$$Q = \{(a,b): a\in(-1,0] \text{ and } b>0\}$$

contains all the pairs for which $\int_0^1 \nu(p;a,b)\,dp = \infty$ and $\int_0^1 p\,\nu(p;a,b)\,dp < \infty$. For this class, the transition probabilities of Φ^ν take the form

$$P(\Phi_{n+1}^\nu = j|\Phi_n^\nu = i) = \frac{(a+1)\,\Gamma(i+a+b+1)\,\Gamma(j+i+b)}{\Gamma(i+b)\,\Gamma(j+i+a+b+2)}$$

for $i, j\in\mathbb{Z}^+$. Hobert and Robert (1999) showed that Φ^ν is null recurrent on $Q_r = \{(a,b): a=0 \text{ and } b\ge 1\}$ and transient on $Q_t = \{(a,b): a\in(-1,0) \text{ and } b\ge 1\}$ but the stability of Φ^ν on the set $Q\backslash(Q_r\cup Q_t) = \{(a,b): a\in(-1,0] \text{ and } b\in(0,1)\}$ remained an open question. We now prove a result which completely characterizes the stability of Φ^ν on Q.

THEOREM 4. *The chain Φ^ν is null recurrent when $a=0$ and $b>0$ and is transient when $a\in(-1,0)$ and $b>0$. Hence, the prior ν is $\mathcal{P}$-admissible when $a=0$ and $b>0$.*

PROOF. We simply apply Corollary 1. First, when $a = 0$, we have

$$\sum_{n=1}^{\infty} \frac{1}{n^2 m_\nu(n)} = \sum_{n=1}^{\infty} \frac{n+b}{n^2} = \infty,$$

and Φ^ν is null recurrent. Now,

$$\sum_{n=1}^{\infty} \frac{1}{n^3 \int_0^1 p^2 (1-p)^n \nu(p)\, dp} = C \sum_{n=1}^{\infty} \frac{\Gamma(n+a+b+2)}{n^3 \Gamma(n+b)} \tag{7}$$

where $C > 0$ is a constant. According to Abramowitz and Stegun [(1972), page 257],

$$\lim_{n\to\infty} n^{d-c} \frac{\Gamma(n+c)}{\Gamma(n+d)} = 1.$$

Therefore, when $a \in (-1, 0)$, (7) converges and Φ^ν is transient. $\square$

REMARK. It is worth pointing out that Hobert and Robert (1999) did not actually analyze Φ^ν. These authors proved results about Φ^ν *indirectly* by analyzing a different Markov chain and appealing to a duality result relating the two chains.

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REFERENCES

ABRAMOWITZ, M. and STEGUN, I. A. (1972). *Handbook of Mathematical Functions*. Dover, New York.

DOYLE, P. G. and SNELL, J. L. (1984). *Random Walks and Electric Networks*. Math. Assoc. Amer., Washington, DC.

DURRETT, R. (1996). *Probability*: *Theory and Examples*, 2nd ed. Duxbury Press, Belmont, CA.

EATON, M. L. (1992). A statistical diptych: Admissible inferences–recurrence of symmetric Markov chains. *Ann. Statist.* **20** 1147–1179.

EATON, M. L. (1997). Admissibility in quadratically regular problems and recurrence of symmetric Markov chains: Why the connection? *J. Statist. Plann. Inference* **64** 231–247.

HOBERT, J. P. and ROBERT, C. P. (1999). Eaton's Markov chain, its conjugate partner and $\mathcal{P}$-admissibility. *Ann. Statist.* **27** 361–373.

KNOPP, K. (1990). *Theory and Application of Infinite Series*. Dover, New York.

LAMPERTI, J. (1960). Criteria for the recurrence or transience of stochastic processes, I. *J. Math. Anal. Appl.* **1** 314–330.

LYONS, T. (1983). A simple criterion for transience of a reversible Markov chain. *Ann. Probab.* **11** 393–402.

MCGUINNESS, S. (1991). Recurrent networks and a theorem of Nash-Williams. *J. Theoret. Probab.* **4** 87–100.

NASH-WILLIAMS, C. ST. J. A. (1959). Random walk and electric currents in networks. *Proc. Cambridge Philosophical Soc.* **55** 181–194.

PERES, Y. (1999). Probability on trees: An introductory climb. *Lectures on Probability Theory and Statistics. Lecture Notes in Math.* **1717** 193–280. Springer, Berlin.

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