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SOME THEORETICAL RESULTS FOR FRACTIONAL FACTORIAL SPLIT-PLOT DESIGNS[1]

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Fractional factorial (FF) designs are commonly used in industrial experiments to identify factors affecting a process. When it is expensive or difficult to change the levels of some of the factors, fractional factorial split-plot (FFSP) designs represent a practical design option. Though FFSP design matrices correspond to FF design matrices, the randomization structure of the FFSP design is different. In this paper, we discuss the impact of randomization restrictions on the choice of FFSP designs and develop theoretical results. Some of these results are very closely related to those available for FF designs while others are more specific to FFSP designs and are more useful in practice. We pay particular attention to the minimum aberration criterion (MA) and emphasize the differences between FFSP and FF designs.

1. Introduction. Suppose we wish to run an experiment with n factors, each at two levels in 2^{n-k} runs. Typically, we would perform a 2^{n-k} fractional factorial design. Further suppose that it is very expensive or difficult to change the levels for some of the factors, say n_1 of them. To reduce costs, we could instead randomly choose one of the factor level settings of these n_1 hard-to-change factors and then run all of the level combinations of the remaining n_2 factors in a random order, while holding the n_1 factors fixed. This is repeated for each level combination of the n_1 factors. If the design matrix for this experimental setup is identical to a 2^{n-k} FF design, where $n = n_1 + n_2$ and $k = k_1 + k_2$, then it is said to be a $2^{(n_1+n_2)-(k_1+k_2)}$ FFSP design [Huang, Chen and Voelkel (1997), Bingham and Sitter (1999)]. The n_1 and n_2 factors are called *whole-plot* (WP) and *subplot* (SP) factors, respectively, and there are k_1 and k_2 WP and SP fractional generators, respectively. While a FFSP design matrix corresponds to a FF design matrix, the randomization of the experiment is different. For a discussion on split-plot designs in an industrial setting, see Box and Jones (1992).

The choice of factor level settings to be performed is determined by *k fractional generators*. For example, consider a $2^{(3+3)-(1+1)}$ 16-run FFSP design. It is easy to write down the 2^4 full factorial design matrix of 0's and 1's. To construct a $2^{(3+3)-(1+1)}$ design, we must assign one WP factor and one SP factor to interactions involving the remaining factors. A possible assignment of these

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factors is

$$3 = 12 \quad \text{and} \quad 6 = 1245,$$

where factors 1, 2 and 3 are WP factors and factors 4, 5 and 6 are SP factors. In this case, the settings of factor 3 for each run are determined by the sum of the level combinations of factors 1 and 2 modulo 2. Similarly, the settings of factor 6 are determined by the sum of the level combinations of factors 1, 2, 4 and 5 modulo 2. Letting I be the identity column of 0's, the fractional generators for this design are $I = 123$ and $I = 12456$. The two generators imply a third relation, $I = 3456$, and together, the three relations form the *defining contrast subgroup,*

$$I = 123 = 12456 = 3456.$$

Let A_i denote the number of words of length i in the defining contrast subgroup, and

$$W = (A_1, A_2, \dots)$$

be the word-length pattern of the design. The *resolution* of a design is the smallest i such that $A_i \neq 0$. So the above $2^{(3+3)-(1+1)}$ FFSP design has resolution III, and word-length pattern

$$W = (0, 0, 1, 1, 1, 0, 0).$$

Designs with larger resolution are typically said to be better than designs with smaller resolution. However, designs with equal resolution may have different word-length patterns and therefore are not all the same. A refinement of the resolution criterion that sorts through designs with equal resolution is the MA criterion [Fries and Hunter (1980)]. Applied to FFSP designs [Huang, Chen and Voelkel (1998), Bingham and Sitter (1999)] it is written as in the following.

DEFINITION (Minimum aberration fractional factorial split-plot). Let A_i denote the number of words of length i in the defining contrast subgroup of a FFSP design, and $W = (A_1, A_2, \dots, A_{n_1+n_2})$ be the word-length pattern for the FFSP design. Suppose that $D1$ and $D2$ are $2^{(n_1+n_2)-(k_1+k_2)}$ FFSP designs. Let r be the smallest i such that $A_i(D1) \neq A_i(D2)$. Then $D1$ is said to have less aberration than $D2$ if $A_r(D1) < A_r(D2)$. If no such i exists, then $D1$ and $D2$ have equal aberration. A design is said to be *MA* if no other design has less aberration.

While the MA criterion has no statistical meaning, it does provide a good general rule for comparing designs, particularly those designs with equal resolution.

It turns out that the above $2^{(3+3)-(1+1)}$ FFSP design is the MA $2^{(3+3)-(1+1)}$ FFSP design [Bingham and Sitter (1999)]. It is not, however, the MA 2^{6-2} FF design. This is because the restrictions on the randomization of a FFSP design have implications on the way the design is constructed. For instance, when viewing the WP factors alone, they must form a 2^{3-1} FF design. Consequently, the WP generators may contain only WP factors. On the other hand, SP factors must be assigned to interactions involving at least one other SP factor. That is, SP generators must contain at least 2 SP factors. If a SP generator contains only one SP factor, then the SP factor level settings will be fixed when the WP factors are fixed. This amounts to moving the SP factor to the WP level of the design. Thus, not all 2^{n-k} FF designs correspond to a $2^{(n_1+n_2)-(k_1+k_2)}$ FFSP design for fixed n_1, n_2, k_1 and k_2.

In the next section, we generalize the representation of FF designs, due to Franklin (1984) and Chen and Wu (1991), to FFSP designs. We introduce this representation first through an example and then via a more detailed development. In Section 3, we develop results for FFSP designs, some of which are related to those that apply to FF designs [Chen and Wu (1991), Chen (1992)] while others are more specialized to FFSP designs and are more useful in practice.

2. Representation of FFSP designs.

2.1. *Development of notation through an example.* Franklin (1984) and Chen and Wu (1991) introduce a matrix representation for the defining contrast subgroup of FF designs. This alternate representation allows for some theoretical development, with emphasis on the MA criterion. In the following discussion, we modify their representation so that it applies to FFSP designs. We then use this new representation to develop some theoretical results for FFSP designs and to highlight some of the differences between FF and FFSP designs. We introduce the new representation through an example and then by a more detailed discussion in the next section.

EXAMPLE 1. Suppose an experimenter wishes to run a $2^{(7+3)-(2+2)}$ FFSP design. The experimenter selects a design with the following fractional generators:

$$I = 126, \qquad I = 2347, \qquad I = 12589 \quad \text{and} \quad I = 2348t_{10},$$

where factors 1 to 7 are WP factors and factors 8, 9 and t_{10} are SP factors. The defining contrast subgroup for this design is

$$\begin{aligned} I &= 126 &&= 2347 &&= 13467 &&= 12589 \\ &= 5689 &&= 1345789 &&= 23456789 &&= 2348t_{10} \\ &= 13468t_{10} &&= 78t_{10} &&= 12678t_{10} &&= 13459t_{10} \\ &= 23459t_{10} &&= 12579t_{10} &&= 5679t_{10}. \end{aligned} \tag{2.1}$$

Ignoring the fact that there are WP and SP factors for the moment, the defining contrast subgroup for the 2^{10-4} FF design in Example 1 can be written as a matrix (see Table 1). In this representation, w_i denotes the ith word in the defining contrast subgroup and is obtained by looking at the ith row and identifying those factors which have a 1 in their column. For example, in

Table 1, $w_1 = 126$, since there are 1's in columns 1, 2 and 6. Using this table, it is easy to reconstruct the defining contrast subgroup given in (2.1).

In Table 1 we see that columns 3 and 4 are identical, as are columns 5 and 9. So writing these columns more than once is redundant. For FF designs, Chen and Wu (1991) noted that we could describe the FF design by a matrix containing only the unique columns and a frequency vector $\mathbf{f}$. For example, if the defining contrast subgroup in (2.1) is viewed as a FF design, then the design is represented by the reduced matrix,

$$\mathbf{M}_1 = \begin{pmatrix} 1&1&0&0&1&0&0&0\\ 0&1&1&0&0&1&0&0\\ 1&0&1&0&1&1&0&0\\ 1&1&0&1&0&0&1&0\\ 0&0&0&1&1&0&1&0\\ 1&0&1&1&0&1&1&0\\ 0&1&1&1&1&1&1&0\\ 0&1&1&0&0&0&1&1\\ 1&0&1&0&1&0&1&1\\ 0&0&0&0&0&1&1&1\\ 1&1&0&0&1&1&1&1\\ 1&0&1&1&0&0&0&1\\ 0&1&1&1&0&0&0&1\\ 1&1&0&1&0&1&0&1\\ 0&0&0&1&1&1&0&1 \end{pmatrix}$$

and the frequency vector $\mathbf{f} = (1, 1, 2, 2, 1, 1, 1, 1)$, where f_i is the number of factors associated with the ith column of $\mathbf{M}_1$.

For FFSP designs, we must indicate which factors are WP factors and which are SP factors so that we completely specify the design with this representation. So, rather than using a frequency vector, we use a split-plot frequency matrix. For the FFSP design in (2.1),

$$\mathbf{f} = \begin{pmatrix} 1&1&2&1&1&1&0&0\\ 0&0&0&1&0&0&1&1 \end{pmatrix}.$$

In this representation, $(f_{1,i}, f_{2,i})'$ indicates that column i of $\mathbf{M}_1$ has a frequency of $f_{1,i} + f_{2,i}$, of which $f_{1,i}$ is due to WP factors and $f_{2,i}$ is due to SP factors. So, given a split-plot frequency matrix, $\mathbf{f}$, and a matrix $\mathbf{M}_1$ as above, we can reconstruct the defining contrast subgroup for a FFSP design.

TABLE 1

	1	2	3	4	5	6	7	8	9	10
w_1	1	1	0	0	0	1	0	0	0	0
w_2	0	1	1	1	0	0	1	0	0	0
w_3	1	0	1	1	0	1	1	0	0	0
w_4	1	1	0	0	1	0	0	1	1	0
w_5	0	0	0	0	1	1	0	1	1	0
w_6	1	0	1	1	1	0	1	1	1	0
w_7	0	1	1	1	1	1	1	1	1	0
w_8	0	1	1	1	0	0	0	1	0	1
w_9	1	0	1	1	0	1	0	1	0	1
w_{10}	0	0	0	0	0	0	1	1	0	1
w_{11}	1	1	0	0	0	1	1	1	0	1
w_{12}	1	0	1	1	1	0	0	0	1	1
w_{13}	0	1	1	1	1	0	0	0	1	1
w_{14}	1	1	0	0	1	0	1	0	1	1
w_{15}	0	0	0	0	1	1	1	0	1	1

2.2. *A more detailed development.* Let $\mathbf{M}_2$ be a $(2^k - 1) \times (2^k - 1)$ matrix such that

$$\mathbf{M}_2 = \begin{pmatrix} I_k & B \\ B' & B'B \end{pmatrix}, \tag{2.2}$$

where I_k is the $k \times k$ identity matrix and the columns of (I_k, B) form the vector space spanned by the columns of I_k over the finite field $GF(2)$, excluding the identity column of 0's. For the remainder of this paper, all operations on a vector space or subspace are assumed to be over the finite field $GF(2)$. Similarly, the rows of M_2 form the vector space spanned by the rows of (I_k, B), excluding the identity row of 0's. A Hadamard matrix can be derived by replacing the 0's and 1's in $\mathbf{M}_2$ with $+1$'s and -1's, respectively, and adding a row and column of $+1$'s.

Chen and Wu (1991) showed that one can identify a 2^{n-k} FF design by assigning the n factors to the $2^k - 1$ columns of $\mathbf{M}_2$. The fact that more than one factor may be assigned to a column is captured by the associated frequency vector, $\mathbf{f}$. Note that $\mathbf{M}_2$ is in a slightly different form than $\mathbf{M}_1$ in the previous section. However, $\mathbf{M}_1$ is simply a subset of columns from $\mathbf{M}_2$ with a permutation of the rows and columns of $\mathbf{M}_2$, which does not affect the design.

To specify a FFSP design, we must indicate which factors are WP factors and which factors are SP factors. To assign WP and SP factors to the columns of $\mathbf{M}_2$, we partition the first k rows of (2.2) into the following form

$$\begin{pmatrix} I_{k_1} & 0 & B_1 & 0 & C_1 \\ 0 & I_{k_2} & 0 & B_2 & C_2 \end{pmatrix},$$

where I_{k_1} is the $k_1 \times k_1$ identity matrix and I_{k_2} is the $k_2 \times k_2$ identity matrix. See that the columns of (I_{k_1}, B_1) form the subspace spanned by the columns of I_{k_1}, and the columns of (I_{k_2}, B_2) form the subspace spanned by the columns of I_{k_2}, excluding the identity column in both cases. Therefore, B_1 and B_2 have $2^{k_1} - k_1 - 1$ and $2^{k_2} - k_2 - 1$ columns, respectively. $\mathbf{M}_2$ can now be written as

$$\mathbf{M}_2 = \begin{pmatrix} I_{k_1} & 0 & B_1 & 0 & C_1 \\ 0 & I_{k_2} & 0 & B_2 & C_2 \\ B_1' & 0 & B_1'B_1 & 0 & B_1'C_1 \\ 0 & B_2' & 0 & B_2'B_2 & B_2'C_2 \\ C_1' & C_2' & C_1'B_1 & C_2'B_2 & C_1'C_1 + C_2'C_2 \end{pmatrix}.$$

For simplicity of presentation, we reorder the columns of $\mathbf{M}_2$ to get $\mathbf{M}= (A_1 \quad A_2)$ where

$$A_1 = \begin{pmatrix} 0 & 0 \\ I_{k_2} & B_2 \\ 0 & 0 \\ B_2' & B_2'B_2 \\ C_2' & C_2'B_2 \end{pmatrix} \tag{2.3}$$

and

$$A_2 = \begin{pmatrix} I_{k_1} & B_1 & C_1 \\ 0 & 0 & C_2 \\ B_1' & B_1'B_1 & B_1'C_1 \\ 0 & 0 & B_2'C_2 \\ C_1' & C_1'B_1 & C_1'C_1 + C_2'C_2 \end{pmatrix}.$$

A FFSP design can now be represented by the matrix $\mathbf{M}$ and a split-plot frequency matrix,

$$\mathbf{f} = \begin{pmatrix} f_{1,1}, f_{1,2}, \dots, f_{1,2^{k_2}-1}, & f_{1,2^{k_2}}, \dots, f_{1,2^{k-1}} \\ f_{2,1}, f_{2,2}, \dots, f_{2,2^{k_2}-1}, & 0, \dots, 0 \end{pmatrix},$$

with $2^k - 1$ columns $(f_{1,i}, f_{2,i})'$. In this representation, the frequencies in the ith column of $\mathbf{f}$ correspond to the number of factors assigned to the ith column of $\mathbf{M}$. Then it is obvious that $\sum f_{1,i} = n_1$ and $\sum f_{2,i} = n_2$.

SP generators may contain both WP and SP factors. For that reason we can assign the WP factors to any of the 2^{k-1} columns of $\mathbf{M}$ [see Huang, Chen and Voelkel (1997) or Bingham and Sitter (1999) for discussion on the selection of fractional generators for FFSP designs]. However, WP generators cannot contain SP factors; therefore we may not assign the n_2 SP factors to all of the columns of $\mathbf{M}$. This implies that $f_{2,i}$ may only be nonzero for columns in A_1.

3. Main results.

3.1. *Results analogous to those for FF designs.* Once the representation of the previous section for a FFSP design is formulated, we can generalize some of the results of Chen and Wu (1991) and Chen (1992) for FF designs to FFSP designs, using similar techniques. In doing so, we must pay particular attention to the restrictions on the assignment of SP factors. These results enable us to search for FFSP designs with a large number of factors from sets of smaller FFSP designs and reveal some of the differences between FF and FFSP designs.

THEOREM 1. *Let* $D(n_1, n_2, k_1, k_2)$ *be a* $2^{(n_1+n_2)-(k_1+k_2)}$ *FFSP design with word-length pattern* W, *and let* $\text{lag}(W, m) = (0, 0, \ldots, 0, W)$ *be the* lag *vector of the word-length pattern W with m leading zeroes. For* $0 \le r \le 2^{k_2} - 1$, *there exists a* $D(n_1+2^k-r-1, n_2+r, k_1, k_2)$ *with word-length pattern* $\text{lag}(W, 2^{k-1})$.

PROOF. Suppose D_1 is a $D(n_1, n_2, k_1, k_2)$ FFSP design represented by $(\mathbf{M}, \mathbf{f})$. Let $\mathbf{f}^*$ be a split-plot frequency matrix with $2^k - 1$ columns $(f^*_{1,i}, f^*_{2,i})'$ such that there are $2^k - r - 1$ columns of $\mathbf{f}^*$ of the form $(1, 0)'$ and r columns of the form $(0, 1)'$. Furthermore, columns of the form $(0, 1)'$ are assigned only to columns in (2.3). The $2^k - r - 1$ columns of the form $(1, 0)'$ are assigned to the remaining columns of $\mathbf{M}$. It is obvious that $\sum(f_{1,i} + f^*_{1,i}) = n_1 + 2^k - r - 1$ and $\sum(f_{2,i} + f^*_{2,i}) = n_2 + r$.

Let D_2 be the FFSP design corresponding to $(\mathbf{M}, \mathbf{f} + \mathbf{f}^*)$. This amounts to adding r new SP factors and $2^k - r - 1$ WP factors to D_1. Thus, D_2 is a $D(n_1 + 2^k - r - 1, n_2 + r, k_1, k_2)$ FFSP design. Since $\mathbf{M}$ is analogous to a Hadamard matrix, there are 2^{k-1} 1's in each row of $\mathbf{M}$. Therefore, since we have assigned each of the $2^k - 1$ new factors to separate columns of $\mathbf{M}$, each word in the defining contrast subgroup of D_2 is 2^{k-1} longer than the corresponding word in D_1. Consequently, the word-length pattern of D_2 is $\text{lag}(W, 2^{k-1})$. □

In Theorem 1, we use the properties of the Hadamard matrix to show that, by assigning new WP and SP factors to each of the columns, we can guarantee the existence of larger designs and predict their word-length patterns. This is similar to the FF case. However, there are some distinguishing features for FFSP designs that should be noted. Firstly, we notice the restriction, r, placed on the number of SP factors being added. Since we can only add factors to columns in (2.3), we can add at most $2^{k_2} - 1$ SP factors by this procedure. This is different from FF designs where all factors are treated the same and r has no meaning. Furthermore, because the number of SP factors being added ranges from 0 to $2^{k_2} - 1$, starting with a MA $2^{(n_1+n_2)-(k_1+k_2)}$ FFSP design D with word-length pattern W, we guarantee the existence of 2^{k_2} different larger designs and their word-length patterns are known. For example, letting $r = 2$

we can guarantee the existence of a $2^{\{(n_1+2^k-3)+(n_2+2)\}-(k_1+k_2)}$ FFSP design with word-length pattern $\text{lag}(W, 2^{k-1})$.

Theorem 1 allows us to find larger designs by only adding WP factors (i.e., $r = 0$) with the word-length pattern of the larger design known. This is useful since we may be interested in searching for designs with a large number of WP factors and only a few SP factors. The same cannot be said for SP factors because we cannot add SP factors to each column of $\mathbf{M}$. We will delay consideration of adding only SP factors to Section 3.2 as this case is less apparent and of more practical interest and thus warrants detailed discussion.

As a result of Theorem 1, we know the word-length pattern of larger designs based on the properties of designs with fewer factors. Therefore, we can establish a lower bound on the maximum resolution of the set of larger designs. This is quite useful if we are searching for MA designs. For example, we could begin with a MA design with word-length pattern W for a smaller design and find a design with more factors by applying Theorem 1. In searching for the MA design for the larger design we discard any designs that have smaller resolution than the design with word-length pattern $\text{lag}(W_1, 2^{k-1})$. The following theorem illustrates this more clearly.

THEOREM 2. *Let $R(n_1, n_2, k_1, k_2)$ be the maximum resolution for a $D(n_1, n_2, k_1, k_2)$ FFSP design. For $0 \le r \le 2^{k_2} - 1$, $R(n_1 + 2^k - r - 1, n_2 + r, k_1, k_2) \ge 2^{k-1} + R(n_1, n_2, k_1, k_2)$.*

PROOF. Suppose D_1 is a maximum resolution $D(n_1, n_2, k_1, k_2)$ FFSP design represented by $(\mathbf{M}, \mathbf{f})$ and has word-length pattern W. Let D_2 be the FFSP design corresponding to $(\mathbf{M}, \mathbf{f} + \mathbf{f}^*)$, where $\mathbf{f}^*$ is defined in the proof of Theorem 1. Then by Theorem 1, D_2 is a $D(n_1 + 2^k - r - 1, n_2 + r, k_1, k_2)$ FFSP design with word-length pattern $\text{lag}(W, 2^{k-1})$ and therefore has resolution $R(n_1, n_2, k_1, k_2) + 2^{k-1}$. Thus, the maximum resolution $D(n_1 + 2^k - 1, n_2, k_1, k_2)$ design must have resolution at least $R(n_1, n_2, k_1, k_2) + 2^{k-1}$. $\square$

Theorem 1 guarantees the existence of larger FFSP designs, and Theorem 2 shows that we have a lower bound on the maximum resolution of the larger design. According to this result, we can find optimal designs of a larger size through a smaller search. That is, we need not consider designs with resolution less than those that result from application of Theorem 2.

Theorem 2 allows us to reduce the number of designs considered when searching for MA FFSP designs. The search for optimal designs could be reduced further if we had an upper bound on the resolution. We do not have a solution to this problem, but we can show that there is a limit to the upper bound. Chen and Wu (1991) show that the maximum resolution of a FF design is periodic. Using similar techniques, we show in Theorem 3 below that the maximum resolution of a FFSP design is also periodic. In the FFSP case, however, the maximum resolution need not correspond to the maximum resolution FF design.

THEOREM 3. *For any fixed n_1, n_2, k_1, k_2 and $0 \le r \le 2^{k_2} - 1$ there exists L_1 such that*

$$R\big(n_1 + l(2^k - r - 1), n_2 + lr, k_1, k_2\big) = 2^{k-1} + R\big(n_1 + (l-1)(2^k - r - 1), n_2 + (l-1)r, k_1, k_2\big)$$

$\forall\, l > L_1$. That is, there exists L_1 such that $\forall\, l > L_1$ the maximum resolution of a FFSP design is periodic.

PROOF. We prove the theorem by contradiction. That is, we assume that there are infinitely many $l_{1,i}$ such that

$$R\big(n_1 + l_{1,i}(2^k - r - 1), n_2 + l_{1,i}r, k_1, k_2\big) \ge 2^{k-1} + R\big(n_1 + (l_{1,i} - 1)(2^k - r - 1), n_2 + (l_{1,i} - 1)r, k_1, k_2\big) + 1. \tag{3.1}$$

Therefore, there exists an infinite sequence of l_1's, $\{l_{1,i}\}_{i=0}^{\infty}$, such that (3.1) is satisfied. Applying Theorem 2 and (3.1) again and again, we see that

$$R\big(n_1 + l_{1,i}(2^k - r - 1), n_2 + l_{1,i}r, k_1, k_2\big) \ge l_{1,i}2^{k-1} + R(n_1, n_2, k_1, k_2) + i. \tag{3.2}$$

Because of a result due to Plotkin (1960), we know that there is an upper bound on the maximum resolution for a FF design. That is, if $R(n, k)$ is the maximum resolution for the 2^{n-k} FF design, then

$$R(n, k) \le \frac{2^{k-1}}{2^k - 1} n.$$

It is obvious that $R(n_1, n_2, k_1, k_2) \le R(n, k)$ and, consequently,

$$R(n_1, n_2, k_1, k_2) \le \frac{2^{k-1}}{2^k - 1}(n_1 + n_2). \tag{3.3}$$

It follows from (3.3) that

$$\begin{aligned} R\big(n_1 + l_{1,i}(2^k - r - 1), n_2 + l_{1,i}r, k_1, k_2\big) &\le \frac{2^{k-1}}{2^k - 1}\big\{n_1 + l_{1,i}(2^k - r - 1) + n_2 + l_{1,i}r\big\} \\ &= \frac{2^{k-1}}{2^k - 1}\big\{n_1 + n_2 + l_{1,i}(2^k - 1)\big\} \\ &= \frac{2^{k-1}}{2^k - 1}(n_1 + n_2) + l_{1,i}2^{k-1}. \end{aligned} \tag{3.4}$$

From (3.2) and (3.4) we get

$$\frac{2^{k-1}}{2^k - 1}(n_1 + n_2) \ge R(n_1, n_2, k_1, k_2) + i,$$

which is clearly not true as $i \to \infty$. □

In the FF case, only the number of fractional generators, k, need be fixed. In the FFSP case with fixed k_1 and k_2, the value of L_1 may vary for different values of n_1, n_2 and r. This is due to the restrictions on the allocation of SP factors to the columns of **M**.

It turns out that not only is the maximum resolution property periodic, but so is the MA criterion. This is useful, since if we can identify smaller MA designs, we can more easily identify larger MA designs without searching through the set of larger designs.

THEOREM 4. *For any fixed n_1, n_2, k_1, k_2 and $0 \le r \le 2^{k_2} - 1$, there exists Q_1 such that $\forall\, q > Q_1$, if the MA $2^{\{[n_1+(q-1)(2^k-r-1)]+(n_2+(q-1)r)\}-(k_1+k_2)}$ FFSP design has word-length pattern W, then the MA $2^{\{[n_1+q(2^k-r-1)]+[n_2+qr]\}-(k_1+k_2)}$ FFSP design has word-length pattern $\mathrm{lag}(W, 2^{k-1})$.*

PROOF. The proof parallels that of Theorem 2 in Chen and Wu (1991) and therefore, we provide only a sketch to make apparent the connection. For the FFSP case, let $v_{q_{1,i}}$ be the number of shortest length words in the defining contrast subgroup of the MA $2^{\{[n_1+q_{1,i}(2^k-r-1)]+(n_2+q_{1,i}r)\}-(k_1+k_2)}$ FFSP design, with $q_{1,i} < q_{1,j}$ $\forall\, i < j$. Using Theorems 1, 2 and 3 and the definition of the MA property for FFSP designs, one can show that for large enough i, there exists a positive integer v_1 such that $v_{q_{1,i}} = v_1$. That is, there exists a limit to the number of words of shortest length, and by the periodicity of maximum resolution this limit is nonzero. One constructs a similar sequence for the number of second shortest length words, third shortest and so on. It remains only to show that there are only finitely many such sequences, which is done by showing that the word lengths lie in an interval of finite length for any $q_{1,i}$. □

Theorem 4 allows us to determine large MA FFSP designs from smaller MA FFSP designs. Finding MA designs in the FF case can be quite difficult when there are many factors, and in the FFSP case these difficulties are compounded by the fact that we must consider the two different types of factors. The periodicity property of the MA criterion is quite useful in the sense that it makes it easier to find larger MA designs, since we can find the MA design for a smaller design and appeal to the periodicity properties. Application of Theorems 2, 3, or 4 imply that we must add $2^k - 1$ new factors to a small MA design to find a larger one. So for instance, if we begin with the MA $2^{(3+3)-(1+2)}$ 8 run FFSP design with word-length pattern W and let $r = 2$, then, by application of Theorem 1, we know that there exists a $2^{[(3+8-2-1)+(3+2)]-(1+2)} = 2^{(8+5)-(1+2)}$ FFSP design with word-length pattern $\mathrm{lag}(W, 2^k - 1)$. In this case, it turns out that for this choice of n_1, n_2, k_1, k_2 and r that Theorem 4 applies and the MA design is immediately periodic [Bingham (1998)]. However, the uses of such a large design are limited at best. In the next section we will consider the case where only SP factors are added and demonstrate that using the results therein one can obtain much smaller and thus more practical designs.

3.2. *Adding SP factors only.* As previously noted, letting $r = 0$ in Theorem 1 amounts to adding $2^k - 1$ WP factors and no SP factors. The same cannot be done with SP factors because we cannot add SP factors to each column of $\mathbf{M}$. However, despite the restriction on the number of SP factors being added, we can guarantee the existence of larger SP designs by assigning SP factors to all of the columns of (2.3) and not adding any additional WP factors. The existence of the larger design is guaranteed, but the length of the words containing only WP factors will remain unchanged. On the other hand, because of the properties of the Hadamard matrix, we can be sure that the lengths of all words containing SP factors will increase by 2^{k_2-1}. We summarize this result in the following theorem.

THEOREM 5. *Let $D(n_1, n_2, k_1, k_2)$ be a $2^{(n_1+n_2)-(k_1+k_2)}$ FFSP design with word-length pattern W. Let W^{WP} be the word-length pattern of D for words containing only WP factors and W^{SP} be the word-length pattern of D for words containing at least one SP factor. There exists a $D(n_1, n_2+2^{k_2}-1, k_1, k_2)$ FFSP design with word-length pattern $W^{WP} + \mathrm{lag}(W^{SP}, 2^{k_2-1})$.*

PROOF. The proof of this theorem is similar to the proof of Theorem 1. However, in this case, we only assign factors to the columns in (2.3) and not the entire matrix $\mathbf{M}$. Suppose D_1 is a $D(n_1, n_2, k_1, k_2)$ FFSP design, then D_1 can be represented by $(\mathbf{M}, \mathbf{f})$. Let $\mathbf{f}^*$ be a split-plot frequency matrix with $2^{k_2} - 1$ columns of the form $(0, 1)'$ and $2^k - 2^{k_2}$ columns of the form $(0, 0)'$. The $2^{k_2} - 1$ columns of the form $(0, 1)'$ are assigned to each of the $2^{k_2} - 1$ columns of (2.3). Let D_2 be the FFSP design corresponding to $(\mathbf{M}, \mathbf{f} + \mathbf{f}^*)$. Since $\sum(f_{1,i} + f^*_{1,i}) = n_1$ and $\sum(f_{2,i} + f^*_{2,i}) = n_2 + 2^{k_2} - 1$, then D_2 is a $D(n_1, n_2 + 2^{k_2} - 1, k_1, k_2)$ FFSP design.

To complete the proof, we must now prove that the word-length pattern of the $D(n_1, n_2 + 2^{k_2} - 1, k_1, k_2)$ FFSP design is $W^{WP} + \mathrm{lag}(W^{SP}, 2^{k_2-1})$. In the same manner as the matrix $\mathbf{M}$,

$$\mathbf{M}_3 = \begin{pmatrix} I_{k_2} & B_2 \\ B_2' & B_2' B_2 \end{pmatrix}$$

can be viewed as similar to a Hadamard matrix. Therefore, there are 2^{k_2-1} 1's in each row of $\mathbf{M}_3$. Furthermore, the rows of (2.3) corresponding to $(C_2', C_2' B_2)$ are simply nonzero linear combinations of the rows of $\mathbf{M}_3$ and also have 2^{k_2-1} 1's in each row. Therefore, by assigning new SP factors to each of the columns in (2.3), each word in the defining contrast subgroup containing a SP factor is increased by 2^{k_2-1}. The lengths of the words with only WP factors remain unchanged. □

Theorem 5 is not simply an extension of the results for FF designs. Instead, it provides a compromise between finding larger designs and the restrictions imposed by the split-plot structure of the design. Using Theorem 5, we can obtain parallel results to Theorems 2, 3 and 4 for the case where we want to add only SP factors. We will state and prove these and then discuss how these can be far more useful in practice.

THEOREM 6. *Let $R(n_1, n_2, k_1, k_2)$ be the maximum resolution for a $2^{(n_1+n_2)-(k_1+k_2)}$ FFSP design, and let $R^{SP}(n_1, n_2, k_1, k_2)$ be the maximum resolution for the words containing at least one SP factor. Then $R^{SP}(n_1, n_2+2^{k_2}-1, k_1, k_2) \geq 2^{k_2-1} + R^{SP}(n_1, n_2, k_1, k_2)$.*

The result follows directly from Theorem 5 using the same argument as in the proof of Theorem 2.

THEOREM 7. *For any fixed n_1, n_2, k_1 and k_2 there exists L_2 such that $\forall\, l > L_2$,*

$$R^{SP}\big(n_1, n_2 + l(2^{k_2} - 1), k_1, k_2\big) = 2^{k_2-1} + R^{SP}\big(n_1, n_2 + (l-1)r, k_1, k_2\big).$$

That is, there exists L_2 such that $\forall\, l > L_2$ the maximum resolution for words in the defining contrast subgroup of a FFSP design containing SP factors is periodic.

PROOF. Like Theorem 3, we prove Theorem 7 by contradiction. So, we assume the contrary for infinitely many l. That is, there exists an infinite sequence of l's, $\{l_{2,i}\}_{i=0}^{\infty}$, such that the following relation is satisfied:

$$\begin{aligned} R^{SP}\big(n_1, n_2 + l_{2,i}(2^{k_2} - 1), k_1, k_2\big) \\ \geq 2^{k_2-1} + R^{SP}\big(n_1, n_2 + (l_{2,i} - 1)(2^{k_2} - 1), k_1, k_2\big) + 1. \end{aligned} \tag{3.5}$$

It follows that by repeated application of Theorem 6 and (3.5), we get

$$\begin{aligned} R^{SP}\big(n_1, n_2 + l_{2,i}(2^{k_2} - 1), k_1, k_2\big) \\ \geq R^{SP}(n_1, n_2, k_1, k_2) + l_{2,i}2^{k_2-1} + i. \end{aligned} \tag{3.6}$$

We can view a FFSP as a combination of two designs, a $2^{n_1-k_1}$ FF and a $2^{(n_1-k_1+n_2)-k_2}$ FF. The defining contrast subgroup of the FFSP consists of the defining contrast subgroup of the two separate FF designs and words generated by crossing the two subgroups. See that the 2^{k_2-1} words in the defining contrast subgroup of the second FF design are the result of assigning k_2 SP factors to interactions of some of the $n_1 - k_1$ WP factors and $n_2 - k_2$ SP factors. Therefore, since we can view the words in this defining contrast subgroup as words from a $2^{(n_1-k_1+n_2)-k_2}$ FF design, and because of the restriction on the assignment of SP factors to the columns of $\mathbf{M}$ and the existence of k_1 other generators in the FFSP, $R^{SP}(n_1, n_2, k_1, k_2) \leq R(n_1 - k_1 + n_2, k_2)$. In addition, it is obvious that $R(n_1 - k_1 + n_2, k_2) \leq R(n_1 + n_2, k_2)$. Then, applying the result of Plotkin (1960),

$$R^{SP}(n_1, n_2, k_1, k_2) \leq \frac{2^{k_2-1}}{2^{k_2} - 1}(n_1 + n_2).$$

Therefore, following the same argument as before we see that

$$\frac{2^{k_2-1}}{2^{k_2}-1}(n_1+n_2) \geq R^{SP}(n_1, n_2, k_1, k_2) + i,$$

which is clearly untrue as $i \to \infty$. □

THEOREM 8. *Let D be a $2^{(n_1+n_2)-(k_1+k_2)}$ FFSP design with word-length pattern W. Let W^{WP} be the word-length pattern of D for words containing only WP factors and W^{SP} be the word-length pattern of D for words containing at least one SP factor. For any fixed n_1, n_2, k_1, k_2, there exists Q_2 such that $\forall\, q > Q_2$, if the MA $2^{\{n_1+[n_2+(q-1)(2^{k_2}-1)]\}-(k_1+k_2)}$ FFSP design has word-length pattern W then the MA $2^{\{n_1+[n_2+q(2^{k_2}-1)]\}-(k_1+k_2)}$ FFSP design has word-length pattern $W^{WP} + \operatorname{lag}(W^{SP}, 2^{k_2-1})$.*

PROOF. Consider a $2^{\{n_1+[n_2+q_{2,i}(2^{k_2}-1)]\}-(k_1+k_2)}$ FFSP design. We begin this proof by noting that each time we increase q by 1, we increase the length of words with at least one SP factor by 2^{k_2-1}. Therefore, for large enough q, the $2^{k_1}-1$ shortest words in the defining contrast subgroup will correspond only to the words containing only WP factors. Therefore for large enough q, to be a MA FFSP design the WP factors must be arranged as a MA $2^{n_1-k_1}$ FF design.

The remainder of the proof is similar to that of Theorem 2 in Chen and Wu (1991), except that we begin with the WP factors arranged as a MA $2^{n_1-k_1}$ FF design. Let $v_{q_{2,i}}$ be the number of shortest length words containing at least one SP factor in the defining contrast subgroup of the MA $2^{\{n_1+[n_2+q_{2,i}(2^{k_2}-1)]\}-(k_1+k_2)}$ FFSP design, and let $q_{2,i} < q_{2,j}\ \forall\, i < j$. By Theorems 5 and 6 we see that $v_{q_{2,i}} \geq v_{q_{2,j}}\ \forall\, i \leq j$. Then there exists, for large enough i, a positive integer v_2 such that $v_{q_{2,i}} = v_2$. That is, there exists a limit to the number of shortest length words containing at least one SP factor. Unlike Theorem 4, we are not guaranteed that this limit is nonzero, but we are guaranteed that it exists and naturally is periodic. In light of this periodicity property, the number of shortest words is also periodic. We can construct a similar sequence for the number of second shortest words, third shortest and so on. If there are finitely many such sequences, then the result follows.

Again, we note that the length of the shortest word in the defining contrast subgroup containing at least one SP factor is bounded below by 1. Therefore, since we have added $q_{2,i}(2^{k_2}-1)$ SP factors to the design, by Theorem 5, the length of the shortest word is bounded below by $1 + q_{2,i}(2^{k_2-1})$.

To find an upper bound on the longest word containing SP factors, we begin by noting an identity due to Brownlee, Kelly and Loraine (1948). The result is modified slightly to incorporate both WP and SP factors and general $q_{2,i}$:

$$\sum iA_i = \{n_1 + [n_2 + q_{2,i}(2^{k_2}-1)]\}2^{k-1}. \tag{3.7}$$

Because the WP design is a $2^{n_1-k_1}$ FF design, we can modify (3.7) for the words containing only WP factors,

$$\sum iA_i = n_1 2^{k_1-1}. \tag{3.8}$$

By (3.7) and (3.8), and summing over words containing at least one SP factor,

$$\sum iA_i = \{n_1 + [n_2 + q_{2,i}(2^{k_2} - 1)]\}2^{k-1} - n_1 2^{k_1 - 1}. \quad (3.9)$$

So, if the resolution of words containing SP factors is R, the longest possible word length has an upper bound. That is, letting U be the length of the longest word,

$$U \le \{n_1 + [n_2 + q_{2,i}(2^{k_2} - 1)]\}2^{k-1} - n_1 2^{k_1 - 1} - (2^k - 2^{k_1} - 1)R. \quad (3.10)$$

Let R_0 be the maximum resolution for the design with $q_{2,i} = 0$. Then by Theorem 6 and (3.10),

$$U \le \{n_1 + [n_2 + q_{2,i}(2^{k_2} - 1)]\}2^{k-1} \\ - n_1 2^{k_1 - 1} - (2^k - 2^{k_1} - 1)(R_0 + q_{2,i}2^{k_2 - 1}).$$

This reduces to

$$U \le (n_1 + n_2)2^{k-1} - n_1 2^{k_1 - 1} - (2^k - 2^{k_1})R_0 + q_{2,i}2^{k_2 - 1}.$$

Consequently, the word lengths of words containing at least one SP factor for the MA FFSP design is bounded by $[1+q_{2,i}(2^{k_2-1}), (n_1+n_2)2^{k-1} - n_1 2^{k_1-1} - (2^k - 2^{k_1})R_0 + q_{2,i}2^{k_2-1}]$, which is of finite length for any $q_{2,i}$. □

Theorems 6, 7 and 8 reveal a separate strategy from the one discussed in the previous section that allows us to add only SP factors to the design. This is useful when we are interested in a design with quite a few SP factors and relatively few WP factors. In addition, the run sizes of such designs do not become large as fast as following the previous procedure or that of Chen and Wu (1991) and thus is of far more practical use.

EXAMPLE 2. Again consider a $2^{(3+3)-(1+2)}$ 8 run FFSP design. Letting 1, 2 and 3 represent the WP factors and 4, 5 and 6 be the SP factors, the MA FFSP design [Bingham and Sitter (1999)] has fractional generators $g_1 = 123$, $g_2 = 145$ and $g_3 = 246$ with word-length pattern $W_1 = (0, 0, 3, 4, 0, 0, 0)$. Suppose that a $2^{(3+6)-(1+2)}$ 64 run FFSP design is desired. Theorem 8 demonstrates that a $2^{(3+6)-(1+2)}$ FFSP design can be constructed from the MA $2^{(3+3)-(1+2)}$ design by assigning the three additional SP factors, 7, 8 and 9, to the three columns in (2.3). Therefore, the generators become $g_1 = 123$, $g_2 = 14579$ and $g_3 = 24689$ and the design has word-length pattern $W_2 = (0, 0, 1, 0, 3, 3, 0, 0, 0)$. It turns out that this design is also the MA $2^{(3+6)-(1+2)}$ FFSP design and that the MA criterion for this choice of n_1, n_2, k_1 and k_2 is immediately periodic [Bingham (1998)]. Unlike the large experiment created by adding factors to each of the columns of $\mathbf{M}$, this 64 run FFSP is a reasonable size design for many experimental situations.

The theorems presented in this section also have applications beyond the scope of FFSP designs. For example consider a situation where an experimenter has more interest in a subset of the factors under investigation. This is often the case in robust parameter design where the experimenter is less interested in estimating main effects and interactions between "noise" factors, even if the design was run in a completely randomized fashion. Theorem 8 provides the experimenter with a technique by which the resolution of the more important factors can be made large at the expense of the remaining factors.

EXAMPLE 3. Suppose an experimenter wishes to run a 2^{9-3} 64 run FF design. In addition, suppose that the experimenter has particular interest in all second order interactions of six of the factors. The MA 2^{9-3} FF design, with word-length pattern $W = (0, 0, 0, 1, 4, 2, 0, 0, 0)$ [Chen, Sun and Wu (1993)], will not allow estimation of all second-order interactions of a chosen six factors. In fact, the ten best 2^{9-3} FF designs in terms of aberration [Chen, Sun and Wu (1993)] do not satisfy the experimenter's needs. However, there does exist a design that will estimate all second-order interactions involving six chosen factors which can be found in the same manner as in Example 2. Begin with the MA $2^{(3+3)-(1+2)}$ 8 run FFSP design of Example 2. For the purpose of design construction, treat the six factors for which all second-order interactions are of interest as SP factors, and the remaining three factors as WP factors. Thus, $n_1 = 3$ is the number of factors for which we are not interested in all second-order interactions, and $n_2 = 3$ represents three of the six factors for which we wish to estimate all second-order interactions. By definition of a FFSP design, one of the three 3-letter words involves all three of the WP factors. Applying Theorem 8 as in Example 2, we get a $2^{(3+6)-(1+2)}$ design with word-length pattern $W_1^{WP} + \mathrm{lag}(W_1^{SP}, 2^{3-1})$, where the SP factors represent the factors for which we wish to estimate all second-order interactions. Therefore, in our case, we know that there exists a 2^{9-3} FF design with word-length pattern $W = (0, 0, 1, 0, 3, 3, 0, 0, 0)$, and it can be found from the MA $2^{(3+3)-(1+2)}$ FFSP design. Note that the one remaining three-letter word contains all three factors that are treated as WP factors, so that the design is resolution V in the six chosen factors (the SP factors) as desired.

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