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A COMPARISON OF CONTINUITY CONDITIONS FOR GAUSSIAN PROCESSES

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Three sufficient conditions for continuity of real-valued, separable, Gaussian processes on $\mathbb{R}^1$ are compared. They are: (1) Fernique's (1964) integral condition, (2) the Kahane (1960)-Nisio (1969) condition on the spectrum of stationary processes and (3) Dudley's (1967) condition involving ε-entropy. Let $S_1 \equiv$ set of stationary, separable, Gaussian processes that can be proven continuous by condition $i = 1, 2, 3$. Dudley (1967) has shown that $S_1 \subseteq S_3$. It is shown here that $S_2 \subset S_1 \subset S_3$, that is, the inclusion is strict. These results extend to non-stationary processes where appropriate.

The Kahane-Nisio condition is strengthened and the best possible integral condition for continuity involving the spectrum is given. A result on the ε-entropy of blocks in a separable Hilbert space is also of independent interest.

1. Discussion of results. There are several sufficient conditions for the continuity of real-valued, separable, Gaussian processes (which we shall henceforth refer to simply as Gaussian processes). Let $X(t)$, $t \in [0, 1]$ be a Gaussian process; Fernique's (1964) condition involves a monotone majorant for the increments variance of the process. Let

$$E(X(t) - X(s))^2 \leqq \phi^2(|t - s|) , \qquad \phi \uparrow t, s \in [0, 1] . \tag{1.1}$$

Fernique's condition is that $X(t)$ is continuous if

$$\int^\infty \phi(e^{-x^2})\, dx < \infty . \tag{1.2}$$

Nisio (1969) has given the following condition; for $X(t)$ also stationary let $\rho(t) = EX(t + s)X(s) = \int_0^\infty \cos t\lambda\, dF(\lambda)$ where F is a distribution function. Let $s_n = F(2^n, 2^{n+1}] \equiv F(2^{n+1}) - F(2^n)$, $n = 0, 1, \cdots$. If a decreasing sequence $\{M_n\}$ can be obtained such that $s_n \leqq M_n$ and

$$\sum_{n=0}^\infty M_n^{\frac{1}{2}} < \infty \tag{1.3}$$

then $X(t)$ is continuous. Nisio's condition includes an earlier result of Kahane (1960). Kahane considers random Fourier series. His result, in the case that the series is a stationary Gaussian process, i.e.

$$X(t) = \sum_{n=0}^\infty a_n[\xi_n \cos nt + \xi_n' \sin nt] \tag{1.4}$$

where ξ_n and ξ_n' are independent Gaussian random variables with mean zero and variance 1, is the following: Define

$$s_n = \sum_{j+2^n+1}^{2^{n+1}} a_n^2 ;$$

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a sufficient condition for the continuity of (1.4) is

$$s_n \text{ non-increasing}, \qquad \sum s_n^{\frac{1}{2}} < \infty . \tag{1.5}$$

Dudley (1967) has given sufficient conditions for $X(t)$ in terms of ε-entropy. Let H be a real infinite dimensional Hilbert space. An isonormal Gaussian process L on H is a linear map from H into real Gaussian random variables with $EL(x) = 0$, $EL(x)L(y) = (x, y)$ for all $x, y \in H$. A GC-set is a subset $C \subset H$ on which the isonormal L has continuous sample functions. One measure of the size of a subset S of H is the minimal number $N(S, \varepsilon)$ of sets of diameter $\leqq 2\varepsilon$ which cover it. The ε-entropy $H(S, \varepsilon)$ of S is defined as $\log N(S, \varepsilon)$. Dudley's condition is that if S is a subset of a Hilbert space and

$$\int_0^\delta H^{\frac{1}{2}}(S, \varepsilon)\, d\varepsilon < \infty \tag{1.6}$$

then S is a GC-set.

Let $X(t)$, $t \in [0, 1]$ be a Gaussian process. A concrete way to realize $X(t)$ as a linear map from some Hilbert space is to use the Karhunen-Loève expansion of $X(t)$ (see Garsia, Rodemich and Rumsey (1970) for a more detailed treatment of the following). Let $\Gamma(t, s) = EX(t)X(s)$. Then

$$\Gamma(s, t) = \sum_{n=1}^\infty \lambda_n \phi_n(s)\phi_n(t)$$

where the λ_n are eigenvalues and the ϕ_n, which are orthonormal with respect to Lebesgue measure, are eigenfunctions of $\Gamma(s, t)$. The function

$$y_s(t) = \sum_{n=1}^\infty \lambda_n^{\frac{1}{2}} \phi_n(s)\phi_n(t)$$

is an element of $L^2[[0, 1], \mu]$, where μ is Lebesgue measure, since $\int_0^1 y_s^2(t)\, dt = \sum_{n=1}^\infty \lambda_n \phi_n(s)\phi_n(s) = \Gamma(s, s)$. The isonormal linear map L is defined as follows:

$$L : y_s(\cdot) \to \sum_{n=1}^\infty \lambda_n^{\frac{1}{2}} \phi_n(s)\eta_n(\omega)$$

where η_n are independent $N(0, 1)$. It is obvious that

$$\int_0^1 y_s(u)y_t(u)\, du = E(Ly_s)(Ly_t) = \Gamma(t, s) .$$

Suppose that $\Gamma(t + h, t + h) + \Gamma(t, t) - 2\Gamma(t, t + h) \leqq \phi^2(h)$, $\phi \uparrow t \in [0, 1]$. For any $\varepsilon > 0$ let $\delta \equiv \sup\{t : \phi(t) < \varepsilon\}$. Let $S = \{y_s(\cdot) : s \in [0, 1]\}$, $S \subset L^2[[0, 1], \mu]$ then $N(S, \varepsilon) < C/\delta$ for some constant C and $H(S, \varepsilon) \leqq \log C + \log 1/\delta$. Dudley (1967, Theorem 7.1) shows that if (1.2) holds so does (1.6) or, equivalently, those processes that can be shown to be continuous by the integral test (1.2) can be shown to be continuous by the ε-entropy condition (1.6).

Marcus and Shepp (1970, Section 5) have shown that neither (1.2) nor (1.3) is necessary for sample continuity on $\mathbb{R}^1$ (in dealing with (1.3) we must further restrict ourselves to stationary processes). Even though Qualls and Watanabe (1971) have given examples in which (1.2) and (1.3) are equivalent it has appeared that these conditions, one dealing with the spectrum and the other with the covariance function, are not the same. However, we prove that (1.3) can be derived from (1.2), in fact, (1.2) implies a result that is stronger than (1.3). Specifically, we prove the following theorem.

THEOREM 1. *Let $X(t)$ be a stationary Gaussian process, $EX^2(t) = 1$, $\rho(t) = EX(t+s)X(s) = \int_0^\infty \cos \lambda t \, dF(\lambda)$. Define $F_n = 1 - F(2^n)$, and let $\phi(t)$ be the least monotone majorant for $2(1 - p(t))$. Then*

$$(1.7) \qquad \sum_{n=1}^\infty \left(\frac{F_n}{n}\right)^{\frac{1}{2}} < \infty \Rightarrow \int^\infty \phi(e^{-x^2})\, dx < \infty .$$

To clarify the relationship between (1.7) and (1.3) set $s_j = F(2^j, 2^{j+1})$ so that $F_n = \sum_{j=n}^\infty s_j \leqq \sum_{j=n}^\infty M_j$. It follows from Boas' lemma (1960) [see also Marcus, Shepp (1970) page 389] that if $M_j \downarrow$ then

$$(1.8) \qquad \sum_{n=1}^\infty \left(\frac{F_n}{n}\right)^{\frac{1}{2}} \leqq 2 \sum_{n=1}^\infty M_n^{\frac{1}{2}} .$$

One can readily obtain examples of processes satisfying the left side of (1.7) and not (1.3) so (1.2) is strictly stronger than (1.3). We do not take up the question of whether the implication in (1.7) can be reversed.

Since the random Fourier series (1.4) are of interest in their own right we note that applying Theorem 1 we obtain that a sufficient condition for continuity of these series is

$$(1.9) \qquad \sum_{n=1}^\infty \left(\frac{1}{n} \sum_{j=n}^\infty s_j\right)^{\frac{1}{2}} < \infty .$$

Convergence of the sum in (1.7) is equivalent to

$$(1.10) \qquad \int^\infty \frac{(1 - F(t))^{\frac{1}{2}}}{t(\log t)^{\frac{1}{2}}}\, dt < \infty .$$

This is the best possible integral condition on the spectrum of the process since we know from Marcus, Shepp (1970) and the Tauberian theorems of Pitman (1961) that if $1 - F(t) = C[\log t(\log\log t)^2 \cdots (\log\log \cdots \log t)^2]^{-1}$, for large t, the corresponding Gaussian process is discontinuous. (Note: The integral (1.10) converges for $1 - F(t) = C_1[\log t(\log\log t)^2 \cdots (\log\log \cdots \log t)^{2+\delta}]^{-1}$, t sufficiently large, $\delta > 0$.)

Let η_n be an increasing sequence of positive numbers and $a_n > 0$, $\sum_{n=1}^\infty a_n < \infty$; ξ_n, ξ_n' are independent $N(0, 1)$. Consider

$$(1.11) \qquad X(t) = \sum_{n=1}^\infty a_n[\xi_n \cos \eta_n t + \xi_n' \sin \eta_n t] .$$

An obvious deficiency of (1.2) is that it cannot be used to prove the continuity of these Gaussian processes, a result which follows easily from the three-series theorem. In Marcus, Shepp (1970, Section 5) an example of a continuous Gaussian process of this type, with $\eta_n = 2^n$, is given for which the least monotone majorant of the characteristic function does not satisfy (1.2). The second result of this paper can be used to show that (1.6) implies continuity of the processes (1.11).

Let H be a separable Hilbert space and $\{\phi_n\}$ an orthonormal basis for H. A "block" in this space is a subset of the form

$$(1.12) \qquad B(\{a_n\}) = \{\sum_{n=1}^\infty x_n \phi_n : |x_n| \leqq a_n, n = 1, 2, \cdots\} .$$

We assume that a_n is non-increasing and that $\sum a_n^2 < \infty$. The following theorem is proved:

THEOREM 2. *Let $B(\{a_n\})$ be a block in a separable Hilbert space. Let $H(B\{a_n\}, \varepsilon)$ be the ε-entropy of $B(\{a_n\})$. Then*

$$(1.13) \qquad \int_0^\delta H^{\frac{1}{2}}(B\{a_n\}, x)\, dx < \infty \Leftrightarrow \sum_{n=1}^\infty a_n < \infty .$$

(Note that the convergence of the integral implies the convergence of the sum by (1.6) and the three-series theorem; the other direction is the new result.)

The random series (1.11) can be rearranged so that the a_n are non-increasing. They are then isonormal Gaussian processes on subsets of blocks with $\sum a_n < \infty$. Using Theorem 2, (1.6) implies the continuity of these random series. Therefore, we have shown that (1.6) is strictly stronger than (1.2).

This result leads us to question whether (1.6) is necessary and sufficient for continuity of stationary Gaussian processes on $\mathbb{R}^1$. We will relate the condition

$$(1.14) \qquad \int^\infty H^{\frac{1}{2}}(S, \varepsilon)\, d\varepsilon = \infty$$

to the result of Marcus and Shepp (1970, 1971) on a sufficient condition for a Gaussian process on $\mathbb{R}^1$ to be discontinuous. Their result is: Let $X(t)$ be a Gaussian process on $\mathbb{R}^1$ and suppose that $E(X(t+s) - X(s))^2 \geqq \rho^2(t)$, where for some $\delta > 0$, $\rho(h)\uparrow$, $h \in [0, \delta]$. Then

$$(1.15) \qquad \int^\infty \rho(e^{-x^2})\, dx = \infty$$

is sufficient for $X(t)$ to be discontinuous. Let $X(t)$ be a process satisfying (1.15) and $\rho(t)$ the monotone minorant such that $E(X(t+s) - X(s))^2 \geqq \rho^2(t)$. Recall the discussion showing how the Karhunen–Loève expansion gives rise to a linear map from some Hilbert space to $X(t)$; following that discussion let $S = \{y_s(\cdot) : s \in [0, 1]\}$. Define $\delta \equiv \sup\{t : \rho(t) < \varepsilon\}$; by an argument similar to Dudley's (1967, Theorem 7.1) with the inequalities reversed (1.15) implies (1.14).

The reader is remined that Dudley (1967) and Sudakov (1969) have shown that (1.6) is not necessary and sufficient for continuity of GC-sets in general, nevertheless, it is still not known whether (1.6) is necessary and sufficient for continuity of stationary Gaussian processes on $\mathbb{R}^n$. (The result for $\mathbb{R}^n$ should be no more difficult than the result for $\mathbb{R}^1$.)

Fernique (1971) has a new sufficient condition for continuity of Gaussian processes that is stronger than (1.2); however it is not known whether it implies the continuity of the examples in (1.11).

2. Proofs.

(a) PROOF OF THEOREM 1. Let $\rho(t) = EX(t+s)X(s) = \int_0^\infty \cos t\lambda\, dF(\lambda)$.

$$E(X(t+h) - X(t))^2 = 2\int_0^1 \sin^2 \frac{\lambda h}{2}\, dF(\lambda) + 2\sum_{j=0}^\infty \int_{2^j}^{2^{j+1}} \sin^2 \frac{\lambda h}{2}\, dF(\lambda) .$$

For $1/2^{n+2} < h \leqq 1/2^{n+1}$,

$$E(X(t+h) - X(t))^2 \leqq \frac{h^2}{2} + \sum_{j=0}^{n} \int_{2j}^{2j+1} \lambda^2 h^2 \, dF(\lambda) + \sum_{j=n+1}^{\infty} \int_{2j}^{2j+1} dF(\lambda)$$
$$\leqq \frac{1}{8.2^{2n}} + \sum_{j=0}^{n} \frac{2^{2j}}{2^{2n}} F(2^j, 2^{j+1}] + \sum_{j=n+1}^{\infty} F(2^j, 2^{j+1}] .$$

For $1/2^{n+2} < h \leqq 1/2^{n+1}$ define

$$\phi^2(h) = \frac{1}{8.2^{2n}} + \sum_{j=0}^{n} \frac{2^{2j}}{2^{2n}} F(2^j, 2^{j+1}] + [1 - F(2^n)]$$

and observe that $\phi^2(h)$ is a monotone majorant for $E(X(t+h) - X(t))^2$. By a change of variables (1.2) is equivalent to

$$\sum_{n=1}^{\infty} \frac{\phi(2^{-n})}{n^{\frac{1}{2}}} < \infty . \tag{2.1}$$

Using $(|x| + |y|)^{\frac{1}{2}} \leqq |x|^{\frac{1}{2}} + |y|^{\frac{1}{2}}$ we see that (2.1) will converge if both

$$\sum_{n=1}^{\infty} \left(\frac{4^{-n}}{n} \sum_{j=0}^{n} 2^{2j} F(2^j, 2^{j+1}]\right)^{\frac{1}{2}} < \infty \tag{2.2}$$

and

$$\sum_{n=1}^{\infty} \left(\frac{F_n}{n}\right)^{\frac{1}{2}} < \infty . \tag{2.3}$$

That (2.3) implies (2.1) is what we want to prove; therefore, we need only to show that (2.3) implies (2.2).

The left side of inequality (2.2) is bounded above by

$$\sum_{n=1}^{\infty} 2^{-n} \sum_{j=0}^{n} 2^j F^{\frac{1}{2}}(2^j, 2^{j+1}] \leqq \sum_{j=0}^{\infty} \sum_{n=j}^{\infty} 2^{-n} 2^j F^{\frac{1}{2}}(2^j, 2^{j+1}]$$
$$= 2 \sum_{j=0}^{\infty} F^{\frac{1}{2}}(2^j, 2^{j+1}] .$$

By Hardy, Littlewood and Pólya (1934, Theorem 345)

$$\sum_{j=0}^{\infty} F^{\frac{1}{2}}(2^j, 2^{j+1}) \leqq 2^{\frac{1}{2}} \sum_{n=1}^{\infty} \left(\frac{F_n}{n}\right)^{\frac{1}{2}} .$$

(b) In order to prove Theorem 2 we need the following lemma.

LEMMA 1.

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=n}^{\infty} a_n^{\,2}\right)^{\frac{1}{2}} < \infty \Rightarrow \sum_{n=1}^{\infty} \frac{a_n^{\,2}}{(1/n \sum_{k=n}^{\infty} a_k^{\,2})^{\frac{1}{2}}} < \infty . \tag{2.4}$$

PROOF. If $\sum_{n=1}^{\infty} (n^{-1} \sum_{k=n}^{\infty} a_k^{\,2})^{\frac{1}{2}}$ converges so does

$$\sum_{n=1}^{\infty} n\left[\left(\frac{1}{n} \sum_{k=n}^{\infty} a_k^{\,2}\right)^{\frac{1}{2}} - \left(\frac{1}{n+1} \sum_{k=n+1}^{\infty} a_k^{\,2}\right)^{\frac{1}{2}}\right]$$
$$\geqq \sum_{n=1}^{\infty} n^{\frac{1}{2}}[(\sum_{k=n}^{\infty} a_k^{\,2})^{\frac{1}{2}} - (\sum_{k=n+1}^{\infty} a_k^{\,2})]^{\frac{1}{2}} \cdot \frac{(\sum_{k=n}^{\infty} a_k^{\,2})^{\frac{1}{2}} + (\sum_{k=n+1}^{\infty} a_k^{\,2})^{\frac{1}{2}}}{(\sum_{k=n}^{\infty} a_k^{\,2})^{\frac{1}{2}} + (\sum_{k=n+1}^{\infty} a_k^{\,2})^{\frac{1}{2}}}$$
$$\geqq \frac{1}{2} \sum_{n=1}^{\infty} n^{\frac{1}{2}} \frac{a_n^{\,2}}{(\sum_{k=n}^{\infty} a_k^{\,2})^{\frac{1}{2}}} .$$

(c) PROOF OF THEOREM 2. Given a block $B(\{a_n\})$ we will show that $\sum a_n < \infty \Rightarrow \int_0^\delta H^{\frac{1}{2}}(B\{a_n\}, x)\,dx < \infty$. It is convenient to eliminate large gaps in the sequence $\{a_n\}$. Construct a sequence $\{b_n\}$ so that $\{a_n\} \subset \{b_n\}$ and so that

$$b_n^2 \leqq 4 \textstyle\sum_{k=n+1}^\infty b_k^2 . \tag{2.5}$$

This can be done in such a way that $\{b_n\}$ is non-increasing and $\sum_{n=1}^\infty b_n < \infty$. The block $B\{(a_n\})$ is a proper subset of $B(\{b_n\})$. We will show that $\int_0^\delta H^{\frac{1}{2}}(B\{b_n\}, x)\,dx < \infty$ which implies that $\int_0^\delta H^{\frac{1}{2}}(B\{a_n\}, x)\,dx < \infty$.

Let $\delta_n = ((n+1)^{-1} \sum_{k=n+1}^\infty b_k^2)^{\frac{1}{2}}$; since the δ_n are decreasing, if $b_n/\delta_n \geqq 1$ then $b_n/\delta_{n+1} \geqq 1$. Let

$$M(n) = \min \left\{n,\ \max\left\{k : \frac{b_k}{\delta_n} \geqq 1\right\}\right\} . \tag{2.6}$$

We shall now find an upper bound for the minimal number of sets of diameter $2\varepsilon_n = 2(2 \sum_{k=n+1}^\infty b_k^2)^{\frac{1}{2}}$ that covers $B(\{b_n\})$.

For n fixed consider the following element in $B(\{b_n\})$:

$$\{j_1\delta_n, \cdots, j_{M(n)}\delta_n\} \quad \text{where the } j_k \text{ are integers,} \tag{2.7}$$

$$0 \leqq |j_k| \leqq \left[\frac{b_k}{\delta_n}\right],\ k = 1, \cdots, M(n), \qquad \text{([ ] denotes integral part).}$$

Let $x \in B(\{b_n\})$. There is an element in (2.7), call it $\bar{x}$ for which

$$||x - \bar{x}||^2 \leqq M(n)\delta_n^2 + \textstyle\sum_{k=M(n)+1}^n b_k^2 + \sum_{k=n+1}^\infty b_k^2 . \tag{2.8}$$

By (2.6)

$$||x - \bar{x}||^2 \leqq n\delta_n^2 + \textstyle\sum_{k=n+1}^\infty b_k^2 \leqq \varepsilon_n^2 .$$

The covering sets are of the form $\bar{x} + S$, where $\bar{x}$ is an element of (2.7) and S is the following set:

$$S = \{x : |x_i| \leqq \delta_n, i = 1, \cdots, n;\ |x_i| \leqq b_i, i > n\} .$$

Therefore $N(B\{b_n\}, \varepsilon_n)$ is less than the number of elements in (2.7).

$$N(B\{b_n\}, \varepsilon_n) \leqq \textstyle\prod_{k=1}^{M(n)} 2\left(\dfrac{b_k}{\delta_n} + 1\right) \leqq \dfrac{(n+1)^{M(n)/2} 4^n \prod_{k=1}^{M(n)} 2b_k}{\varepsilon_n^{M(n)}} .$$

The additional factor 2^n is introduced since $((b_k/\delta_n) + 1) < 2(b_k/\delta_n)$; although 2^n seems excessively large, using it simplifies the proof. Let

$$\bar{N}(\varepsilon_n) = \frac{8^n (n+1)^{M(n)/2} \prod_{k=1}^{M(n)} b_k}{\varepsilon_n^{M(n)}} \quad \text{and} \quad \bar{H}(\varepsilon_n) = \log \bar{N}(\varepsilon_n) .$$

For any δ there is an integer N for which

$$\int_0^\delta H^{\frac{1}{2}}(x)\,dx \leqq \textstyle\sum_{n=N}^\infty [\varepsilon_{n-1} - \varepsilon_n]\bar{H}^{\frac{1}{2}}(\varepsilon_n) .$$

$$\textstyle\sum_{n=N}^\infty [\varepsilon_{n-1} - \varepsilon_n]\bar{H}^{\frac{1}{2}}(\varepsilon_n) \leqq \sum_{n=N}^\infty \varepsilon_{n-1}[\bar{H}^{\frac{1}{2}}(\varepsilon_n) - \bar{H}^{\frac{1}{2}}(\varepsilon_{n-1})] + \varepsilon_{N-1}\bar{H}^{\frac{1}{2}}({}_{N-1}) . \tag{2.9}$$

$$\textstyle\sum_{n=N}^\infty \varepsilon_{n-1}[\bar{H}^{\frac{1}{2}}(\varepsilon_n) - \bar{H}^{\frac{1}{2}}(\varepsilon_{n-1})] = \sum_{n=N}^\infty \varepsilon_{n-1} \dfrac{\bar{H}(\varepsilon_n) - \bar{H}(\varepsilon_{n-1})}{\bar{H}^{\frac{1}{2}}(\varepsilon_n) + \bar{H}^{\frac{1}{2}}(\varepsilon_{n-1})}$$

$$\leqq \textstyle\sum_{n=N}^\infty \dfrac{\varepsilon_{n-1}}{n^{\frac{1}{2}}} [\bar{H}(\varepsilon_n) - \bar{H}(\varepsilon_{n-1})] .$$

The last inequality follows from the fact that $\bar{H}(\varepsilon_n) > n$. Suppose that $M(n-1) < M(n)$, then

$$(2.10)\quad \bar{H}(\varepsilon_n) - \bar{H}(\varepsilon_{n-1}) = \log\left[8 \prod_{k=M(n-1)+1}^{M(n)} b_k\left(\frac{n+1}{\varepsilon_n^{\,2}}\right)^{M(n)/2}\left(\frac{n}{\varepsilon_{n-1}^2}\right)^{-M(n-1)/2}\right]$$
$$\leqq \frac{M(n-1)}{2}\log\frac{\varepsilon_{n-1}^2}{\varepsilon_n^{\,2}} + \frac{M(n)}{2}\log\left(1+\frac{1}{n}\right)$$
$$+ \frac{M(n)-M(n-1)}{2}\log\frac{nb_{M(n-1)+1}^2}{\varepsilon_n^{\,2}} + \log 8\,.$$

Since $b_{M(n-1)+1}/\delta_{n-1} < 1$, $nb^2_{M(n-1)+1} < n\delta^2_{n-1} < \varepsilon^2_{n-1}$. Therefore

$$(2.11)\quad \bar{H}(\varepsilon_n) - \bar{H}(\varepsilon_{n-1}) \leqq \frac{M(n)}{2}\log\frac{\varepsilon_{n-1}^2}{\varepsilon_n^{\,2}} + \frac{M(n)}{2n} + \log 8 \leqq \frac{n}{2}\log\frac{\varepsilon_{n-1}^2}{\varepsilon_n^{\,2}} + \tfrac{1}{2}\,.$$

When $M(n) = M(n-1)$ the product term in (2.10) is replaced by 1, (2.11) follows easily. We now have

$$\int_0^\delta H^{\frac{1}{2}}(x)\,dx \leqq \varepsilon_{N-1}\bar{H}^{\frac{1}{2}}(\varepsilon_{N-1}) + \sum_{n=N}^\infty \frac{\varepsilon_{N-1}}{n^{\frac{1}{2}}}\left[\frac{n}{2}\log\frac{\varepsilon_{n-1}^2}{\varepsilon_n^{\,2}} + \tfrac{1}{2}\right].$$

Note that $\sum_{n=N}^\infty \varepsilon_{n-1}/n^{\frac{1}{2}} = \sum_{n=N}^\infty (2n^{-1}\sum_{k=n}^\infty b_k^{\,2})^{\frac{1}{2}} \leqq 2^{\frac{3}{2}}\sum_{n=N}^\infty b_k$ when $b_k \downarrow$ by Boas' lemma (see 1.8).

$$\sum_{n=N}^\infty n^{\frac{1}{2}}\varepsilon_{n-1}\log\frac{\varepsilon_{n-1}^2}{\varepsilon_n^{\,2}} \leqq \sum_{n=N}^\infty (2n)^{\frac{1}{2}}\left(\sum_{k=n}^\infty b_k^{\,2}\right)^{\frac{1}{2}}\frac{b_n^{\,2}}{\sum_{k=n+1}^\infty b_k^{\,2}}$$
$$< 8\sum_{n=N}^\infty \frac{n^{\frac{1}{2}}b_n^{\,2}}{(\sum_{k=n}^\infty b_k^{\,2})^{\frac{1}{2}}} \quad \text{by (2.5)}.$$

The proof follows from Lemma 1 and Boas' lemma.

For further results on the ε-entropy of compact subsets of l^p spaces see Marcus (1972).

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